WHAT CAN TEACHERS LEARN FROM RESEARCH IN MATHEMATICS EDUCATION?

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When asked to give a talk at a mathematics teacher conference that focused on research in mathematics education, I decided not to report, as usual, on a specific piece of research, nor to summarize research on learning in a specific mathematical area Rather, I viewed the invitation as an opportunity to address the meta-question: what can teachers learn from research in mathematics education? The ground for this decision was my feeling, as a researcher and teacher educator, that the relevance of research in mathematics education deserves explicit discussion. Although research in mathematics education has flourished in the last decades and improving mathematics teaching and learning has always been the overarching goal of researchers in the field, mathematics education research is often accused of having only a minor impact on practice (if it has any effect at all). For research to have an impact on practice there is a need for it to be relevant for practitioners and for practitioners to have access to it and be able to use it.

Research findings are generally published in academic journals, but teachers usually do not read these journals, perhaps, for some of them, because of the language used There is also a common belief among practitioners that academic studies are irrelevant for educational practice (e g. Bromme and Tillema, 1995). This feeling of irrelevance seems to stem from an unfulfilled expectation that:

> research should provide reliable and relevant rules for action, rules that can be put to immediate use (Kennedy, 1997, p. 10)

Consequently, many teachers are not familiar with research in mathematics education, nor wish to become so My experiences of work with teacher-leaders (Even, 1999) taught me that research in mathematics education could become meaningful and relevant for practitioners, even if it does not provide them with clear rules for action. Thus, I decided to focus my talk on this issue

I started to work on my talk by asking myself what ideas in current research in mathematics education are relevant for practitioners in mathematics education My aim was to focus on ideas that both influenced my own thinking about mathematics learning and teaching during my professional life and have the potential to contribute to the development of *knowledge, courage,* and *modesty* in teaching. In her influential paper, Lampert (1990) followed Polya (1954) in considering courage and modesty to be essential for doing mathematics, since making a mathematical conjecture requires the courage to take a risk and the modesty to admit that one's conclusion may have been inappropriate My claim is that courage and modesty are essential not only for doing mathematics, but also for teaching mathematics Teaching involves, among other things, interpreting what students are saying and doing, examining taken for granted assumptions about what it means to know, learn and teach mathematics and making teaching conjectures (e g. making an on-the-spot change in a lesson plan in response to students' talk and action, trying new instructional methods and experimenting with an innovative curriculum). Making a teaching conjecture requires the modesty to admit that one's conclusion may have been inappropriate and the courage to take a risk

In my talk, I focused on the following four ideas, derived from a synthesis of current research in mathematics education.

1. Mathematical knowledge is constructed in ways that do not necessarily mirror instruction.
2. Mathematical meaning is both subjective and sociocultural.
3. Knowledge and practices of learning and knowing are inseparable.
4. Knowing is a 'slippery' notion

I believe that these ideas have the potential to empower teachers so that they are more knowledgeable about what it means to know, learn, and teach mathematics and can afford to be courageous and modest in the sense described above. Each of the following four sections focuses on one of the four ideas, highlighting the importance for teachers to be knowledgeable about them The concluding section suggests ways for teachers to learn and explore these ideas

Ways of constructing mathematical knowledge

Students' mathematical learning has been the center of attention of researchers in mathematics education for several decades This research provides important information regarding the processes involved in learning specific mathematical topics It also presents concrete illustrations of what it might mean for students to construct and develop their own knowledge and ideas about the mathematics they learn, and how students' ideas are not necessarily identical to the structure of the discipline, nor to what was intended by instruction In other words, research in mathematics education in the last decades informed the community about how students learn various mathematical concepts, ideas, and procedures. It also supported the adaptation of constructivism as a theoretical perspective on learning.

Consider, for instance, the following examples of students' conjoining algebraic expressions:

$10 + 2b = 12b$
$5t + 3t + t + 2 = 11t$

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$3m + 2 + 2m = 5m + 2 = 7m$
$3 + 4x = 7x = 7$

No teacher of algebra would be surprised by such responses They are all too familiar. Most teachers also know how unsuccessful attempts to avoid or get rid of this phenomenon are However, research indicates that teachers are not as knowledgeable about possible reasons for students' tendency to conjoin or 'finish' open expressions (Tirosh *et al.*, 1998) Research in mathematics education has a contribution to make here.

> *Conventions in natural language* (Tall and Thomas, 1991). For instance, in natural language 'and' and 'plus' have similar meanings. Thus, the symbol 'ab' is read as 'a and b' and interpreted as 'a + b'.
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> *Previous learning from other areas that do not differentiate between conjoining and adding* (Stacey and MacGregor, 1994). For example, in chemistry adding oxygen to carbon produces CO_2
>
> *Previous learning in mathematics: the 'behavior' of algebraic expressions is expected to be similar to that of arithmetic expressions* (Booth, 1988; Davis, 1975; Tall and Thomas, 1991). For example, students expect a final, single-termed answer or interpret symbols such as '+' only in terms of actions to be performed
>
> *The dual nature of mathematical notations: process and object* (Davis, 1975; Sfard, 1991; Tall and Thomas, 1991) For instance, the symbol $5x + 8$ stands both for the process 'add five times x and eight' and for an object that can be manipulated. Conceiving the same expression both operationally, as a process, and structurally, as an object, is problematic.

Such research results indicate that:

- learning is not a straightforward, faithful image of instruction
- knowledge cannot simply be transferred from the teacher to the student
- learning involves the student's own construction of knowledge.

The general idea of constructivism, by itself, may not be such great news for a teacher However, the contribution of research in mathematics education is in making this theoretical idea more specific, fine-detailed and concrete – in other words, more usable for teachers. Such research results as those listed above, showing reasons for students' tendency to conjoin open expressions, highlight complex learning processes in specific domains, topics, and concepts from the school mathematics curriculum and can therefore be of direct relevance for the mathematics teacher.

Mathematical meaning is both subjective and sociocultural

Related to the above is the conclusion that mathematical meaning is both subjective and sociocultural rather than objective. Although related to constructivism, this idea is not as accepted For illustration, many teachers try to deal with the problem of students' tendency to conjoin open expressions by using the 'fruit salad' approach (Pimm, 1987). They refer to 2a + 5b, for instance, as 2 apples and 5 bananas, and explain that, because one cannot add apples and bananas, 2a + 5b cannot be simplified further.

Let us ignore the problematic nature of such an explanation that treats 'a' as 'apples' instead of 'the number of apples' and focus on what research tells us about the use of such an approach. Results of a study by Booth (1988) indicate that students do not necessarily conceive the situation the way the teacher does. Booth found that the same number of students simplified 2a + 5b to 7ab basing this on the argument that 2 apples plus 5 bananas is 7 apples-and-bananas, as those who claimed that 2a + 5b could not be simplified further because it is impossible to add apples and bananas. This example illustrates that meaning is 'in the eye of the conceiver' and that students may understand an explanation very differently that looks 'clear' to the teacher.

Furthermore, it is rarely acknowledged how heavily mathematical meanings depend on sociocultural conventions. For example, imagine how strange it would be, to someone not familiar with our conventions, that:

> 1.5 meters is 1 meter and 50 centimeters
>
> 1.5 feet is 1 foot and 6 inches and
>
> 1.5 hours is 1 hour and 30 minutes

Work with number systems, different from the one with which we are familiar, can illustrate this. For example, if in Chinese:

> 1 is –
>
> 10 is +
>
> 2 is =

What is 20 in Chinese? Many people expect 20 in Chinese to be ‡, as this would fit our convention But actually 20 in Chinese is = +.

We often assume that objects 'speak for themselves' and that the other person sees what we see, but shared meaning requires, among other things, common experiences. Students are, in a way, newcomers to the mathematical community Consequently, they do not share the same mathematical experiences with their teachers They may look at a sliced pizza and see food, not fractions! *red* apples instead of *three* apples (or is it we who see *three* apples instead of *red* apples?). Teachers need to be aware of this lack of common experiences when using models, illustrations, and analogies. This is not an easy task, of course.

Knowledge and practices of learning and knowing are inseparable

Another conclusion from research in mathematics education, mainly from studies that belong to the situated learning perspective (Lave and Wenger, 1991), is that what people know is defined by ways of learning and knowing. In other words, the product or the forms of mathematical knowledge are closely connected to, and inseparable from, the processes that produced it through classroom practices or in the world. Boaler (1997, 2002) has clearly documented this relationship between knowledge and classroom practice. She conducted a comprehensive, longitudinal study aiming to understand the relationship between two different approaches to the teaching of mathematics and the beliefs and understandings that students developed. One approach was traditional 'chalk and talk', and the other was open-ended and project-based. Boaler's findings indicate that students who experienced mathematics at school as working through textbook exercises had difficulties using their mathematics in new situations. Students who experienced mathematics at school as discussing, interpreting, and using mathematical ideas in open group-based projects, on the other hand, were more able to use mathematics in new situations.

To illustrate further the idea that knowledge and classroom practices are inseparable, suppose that students are told that $15.24 \times 4.5 = 6858$ and they are asked to locate the decimal point (a task designed by Markovits, 1988). The students' answers would not depend on cognitive factors only, but rather be interrelated with the formal and informal ways in which they learn mathematics, and with the social and sociomathematical norms developed in the classroom (Yackel and Cobb, 1996). In classrooms that practice mathematics problem solving by following rules presented by the teacher, it is likely that students would count three places and then locate the decimal point after the 6. However, in classrooms that emphasize estimation and common sense, we would expect students to estimate the answer to be around 60 and then locate the decimal point after the 68. In classrooms where students are expected to look for different solutions to the same problem, a student who counted 3 places and located the decimal point after the 6 may also look for another way to solve the problem. The new way may include attention to the ones digit. Noticing that $4 \times 5 = 20$ and that the 0 has disappeared, it becomes clear that it is impossible to use the rule.

Awareness of the inseparability of classroom practices and students' mathematical knowledge could assist teachers with the development of insights into students' mathematical behavior, often interpreted by teachers as nonsensical. For example, many students give answers which do not make sense (Markovits, 1988) when asked questions such as:

> The height of a 10 year-old boy is 5 feet. What do you think his height will be when he is 20?
>
> There were 40 cars in the parking lot at noon, 30 cars at 2pm, and 20 at 4pm. How many cars will be in the parking lot at 6pm? at 8pm? at 10pm?

Students often claim that the twenty-year-old man will be 10 feet high or that there will be a negative number of cars in the parking lot at night. Markovits' study suggests that children do not necessarily think that there are people who are 10 feet high or that it is possible to have a negative number of cars in a parking lot (or even that the number of cars follows such a nice pattern). Rather, children believe that there is an answer in mathematics and an answer in real-life, and that these two answers do not necessarily coincide. Examination of such answers in the light of common classroom practices suggests that the seemingly insensible answers are actually quite sensible. Such knowledge about the nature of mathematical answers is related to the way mathematics is commonly taught at school and is a natural outcome of many years of participation in mathematics classes solving '(non-)real life' (word) problems. For many children, for example, a car that travels at a constant speed, a common characteristic of cars in mathematical word problems, is completely unreal. Similarly, can two children mow a lawn at constant rates, no matter how long they mow? Is not the rate related to the weather? Will they work at the same rate whether they are alone or together? Whether they like or hate each other? If someone wants to fill a pool, why would they open a hose and a pump simultaneously? And how can we assume that the probability of being born in a specific month is 1/12? Is giving birth really random? From the student point of view, if real-life considerations can be ignored in mathematics classes, then why cannot a negative number of cars be considered a legitimate answer?

Knowing is a slippery notion

Statements such as, "He knows nothing about functions" or "She knows systems of equations very well" are common to teacher talk. What do such statements mean? What does it mean for someone to know something? What does it mean that someone does not know? If students get 80% of the answers correct on a test does that mean that they know 80% of the test material? If they get 100% of the answers correct does that mean that they know everything perfectly? If they fail the test does that mean that they do not know the material? Here, again, research can contribute to teaching. Studies conducted in the last two decades indicate that 'knowing' is not easy to pinpoint. Individuals may seem to 'know' in some situations and 'not know' in others. The following example illustrates this idea (discussed more fully in Even, 1993). It focuses on one university student's 'knowing' the simple 'fact' that functions are single-valued, i.e. that for each element in the domain there is only one corresponding element (image) in the range.

Today, when students first encounter the concept of function formally at school, univalence is often presented to them as an important characteristic, both by textbooks and teachers. Students then spend some time learning to distinguish between functions and non-functions. It is not surprising, therefore, that when asked to define a function, a student who was in the last year of his mathematical studies at the university, emphasized that every element in the domain is mapped to a *unique* element in the range. He was also able to distinguish between several functions and non-functions that were presented to him, using various representations, by observing whether there was an assignment of a *single* value to each element in the domain. He also correctly used the 'vertical line test' to support his decision to accept even 'strange-looking' graphs as functions. Such checking of students' knowledge, by asking for definitions and for their applications in various situations, would satisfy most teachers. So, ending the story at this point could have left the reader feeling that our student knew that functions have to be single-valued. However, in the midst of a mathematical discussion, the very same student suddenly claimed that a circle was a graph of a function. He did notice that circles do not conform to his own definition of functions, as they are not single-valued. He also explicitly said that he noticed that they do not pass the vertical line test. Still, he *felt* strongly that a circle must be a function, probably because he was so familiar with its graph and equation. What is it possible to say now? Did the student know that functions have to be single-valued? How imprecise and simplistic is the claim that someone does or does not know something. And how situated 'knowing' is

Conclusion

I have briefly presented four ideas derived from current research in mathematics education that seem to me to be of great relevance to the teacher. The choice of these ideas is, of course, related to my own knowledge, understanding, and beliefs about, experiences in and practice of, research in mathematics education and teacher education. Other researchers in mathematics education may make different choices. I aimed to convey what I feel the two dominant approaches in mathematics education – the cognitive perspective and the sociocultural perspective – can offer teachers to help them become more knowledgeable about knowing, learning, and teaching mathematics and be professionally courageous and modest in the sense described earlier.

Embracing the idea that knowing is a 'slippery' notion entails professional modesty in situations where the teacher conducts student assessment. The idea that knowledge and practices of learning and knowing are inseparable implies an expansion of focus from the products of teaching only, to include examination of traditional processes of teaching, as well as supporting taking the risk of changing them. That mathematical knowledge is constructed in ways that do not necessarily mirror instruction and that mathematical meaning is both subjective and sociocultural puts the student and the community in an important place with regard to learning. And implies continuous teacher considerations of extremely complex aspects, supporting professional modesty in claiming to have 'the right way of teaching' and professional courage to be attentive to students when making and changing instructional decisions.

At the current stage of research in mathematics education, its main contribution to practice may be to raise teacher awareness and deepen teacher understanding of the complicated nature of mathematical knowledge, knowing, and learning. It is not realistic to expect research to provide either simple answers to complex questions or recipes. However, helping teachers become knowledgeable about what research says about issues, such as the four ideas discussed in this article, has the potential to make a contribution to practice both by problematizing things taken as simple, and by clarifying puzzling ones

The ideas discussed above may serve as a guide for pre-service and in-service teacher educators when designing courses and activities for mathematics teachers. An example of such learning experiences for teachers is provided in Even (1999). One component of a three-year professional development program focused on the first idea presented in this article. The participating teachers read and discussed research articles that exemplify common inquiry into the learning of specific mathematical topics (algebra, calculus, geometry, and probability) in the last three decades. Gradually, the teachers began to understand what research suggests about students' learning and understanding of various mathematical topics. They began to develop an appreciation of the idea that students construct their knowledge in ways that are not necessarily identical to the instruction. They also started to conceptualize and make explicit naive and implicit knowledge.

The next step was to examine theoretical knowledge acquired from reading and discussing research in the light of the teachers' practical knowledge, and *vice versa,* to build upon and interpret their experience-based knowledge using theoretical knowledge. The teachers chose one of the studies presented in the course, shaped it according to their interests and conducted a mini-study with students. The teachers were required to write a report that would describe the students' mathematical thinking in light of the relevant academic background and 'compare' their 'study' with the original study, thus integrating their personal experiences, theoretical knowledge, and practice.

The first step, reading and discussing research articles, contributed to the teachers learning, in general, that students construct their knowledge. The second step, the mini-study, made this theoretical idea more specific, concrete and relevant for the teachers. They learned what the constructivist view might mean in a practical context. Some teachers examined specific cases that exemplified that learning processes are complicated no matter how 'clear' the instruction may be. Others examined cases that exemplified that, contrary to expectation, students are able to deal with sophisticated mathematical ideas. Conducting a mini-study with real students provided opportunities for examining theoretical matters by particularizing them in a specific context. It also provided opportunities for integration of theoretical knowledge with knowledge learned in practice. It involved learning about *real* students. The design of such learning experiences has the potential to encourage integration of research-based knowledge with knowledge learned in practice, as a means to challenge teachers' existing conceptions and beliefs, and promote intellectual restructuring. This, in turn, supports professional modesty to admit that the teacher's assumptions about what students know may have been inappropriate, and the courage to take a risk and try new ways of teaching.

Conducting such learning experiences for teachers indicates that professional modesty and courage are essential not only for the teaching of mathematics. They are as important for the teacher educator and the provider of professional development for teachers of mathematics. Similar to teaching mathematics to students, teacher education involves, among other things, interpreting what teachers are saying and doing, examining taken-for-granted assumptions about what it means to know and learn how to teach mathematics, and to teach mathematics teaching. It also involves making teacher-education conjectures (e.g. making an on-the-spot change in a session plan in response to teachers' talk and action or trying new methods to conduct professional development for teachers). Making such conjectures, similar to the case of teachers, requires the teacher educator to have the modesty to admit that their conclusion may have been inappropriate and the courage to take a risk and change

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Transforming: views of the same statue of Evangeline in the grounds of the Grand Pré National Historic Site, near Wolfville, NS, Canada The sequence of photographs shows her aging as I walk around her, but I could not see this with my own eyes. (Sculptors, Louis-Philippe Herbert, posthumously completed by Henri Herbert; photographs, Laurinda Brown.)