Communications

Action Proof vs Illuminating Examples?

G. WALTHER

In this note I'll comment briefly on Z. Semadeni's recent article on action proofs in this journal [1].

About five years ago in my inaugural lecture at Dortmund University I suggested "illuminating examples" as a tool for simplification in mathematics teaching and teacher education. This work was mainly stimulated by private discussions with Z. Semadeni in Warsaw, a lecture by A. Kirsch on "premathematical" proofs [2] and a lecture by Kirsch [3].

There were at least two reasons why I found and still find these ideas so interesting.

— One is due to the role the concept of activity plays in Semadeni's conception, which fitted so well in a theoretical framework called "Active Learning, Operative Programs, Operative Principle" [4] to be developed in those years in Dortmund

— Second, I saw a chance to generalize Semadeni's conception in the following ways

In my opinion it is not necessary to restrict the representation of action proofs to the enactive/iconic mode. Representation is only one aspect; it seems to be more important which kind of structured activity the enactive/iconic/symbolic represented (proof-) situation allows and demands (c.f. my example in [5])

Another question concerns the regulation of activity in an action proof. The examples given in [5] and in Semadeni's article are more or less general statements of the type

$$\text{For all } n \in \mathbf{N} : A(n) \qquad (1)$$

Of course (1) cannot be proved by presenting some examples—as sometimes pupils do.

P. Lorenzen suggested in [6] an interpretation of the "all"-quantifier which makes use of the mathematical activity of proving and allows the processing of "concrete" examples (but is not restricted to a special mode of representation). His dialogical interpretation of the all-quantifier requires a so-called "Proponent" to give for any natural number "suggested" by the "Opponent" a proof for $A(n)$. Obviously this process can also be viewed as a segment of an inner dialogue in the mind of only one person.

If e. g. the Opponent suggests $n = 5$, $n = 12$, then $A(5)$, $A(12)$ are *examples* supporting the general statement (1). But success in establishing $A(5)$ and $A(12)$ does not necessarily imply success e. g. in proving $A(8)$. So the Proponent (student or teacher) is confronted with the problem: given an arbitrary natural number n', how can he defend ("immunize") himself against the obligation to prove $A(n')$ anew?

From the psychological point of view it is obvious to try this defence with some special examples and to look for aspects of proof which are *invariant* regarding a transfer to other arbitrary examples. These invariants then constitute a *proof-strategy* for the general statement (1). So in my opinion the question concerning the regulation of activity in an action proof is answered by the concept of proof strategy. It is important to notice that a proof-strategy for an action proof has its roots in actions on mathematical objects and not only in the objects themselves. What are "illuminating examples"? These are generic examples of the type $A(5)$ and $A(12)$ which allow a proof strategy to emerge

From the pedagogical point of view it is important that these illuminating examples, as "intellectually honest" tools for simplification, facilitate mathematical communication and make it possible to conceive the formalization of a proof strategy as a next step in the learning process.

Hence there is certainly no conflict between action proofs and illuminating examples. The latter, however, seem to be a bit more general.

As I learnt from [7], quasi-general (orig. *quasi allgemeine*) examples—which as far as I understand it, really *are* illuminating—have also been used in the history of mathematics for mathematical communication. In Euclid's famous theorem "Prime numbers are more numerous than any assigned multitude of prime numbers" a proof strategy is developed with a generic set of three prime numbers. Various other examples from history can be found in [7].

References

[1] Semadeni, Z. (1984) Action Proofs in Primary Mathematics Teaching and in Teacher Training, *FLM* Vol 4(1), 32-34

[2] Kirsch, A. (1979) Beispiele für prämathematische Beweise, in: Dörfler and R. Fischer (Edit.) *Beweisen im Mathematikunterricht*, Wien, pp. 261-274

[3] Kirsch, A. (1977) Aspects of Simplification in Mathematics Teaching, *Proc. 3rd. Int. Congr. on Math Educ.* 98-120

[4] Wittmann, E. (1981) Beziehungen zwischen operativen "Programmen" in Mathematik, Psychologie und Mathematikdidaktik, *Journ. für Math. Didaktik* 2(1), 83-95

For the Learning of Mathematics **4**, 3 (November 1984)
FLM Publishing Association. Montreal, Quebec, Canada

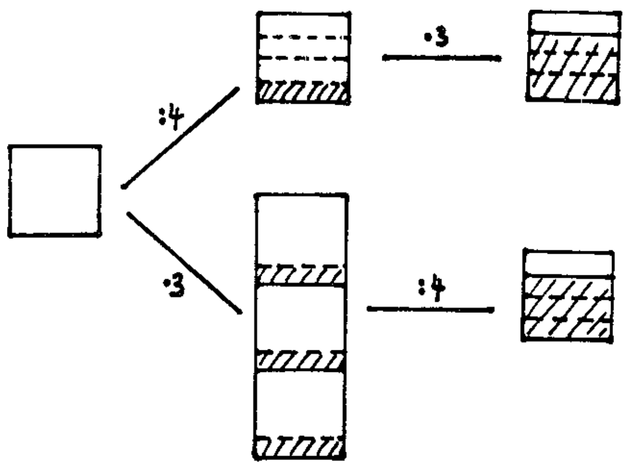
[5] Wittmann, E. (1981) The Complementary Roles of Intuitive and Reflective Thinking in Mathematics Teaching, *Educ. Stud. in Math.* 12, 389-397
[6] Lorenzo, P. (1969) *Normative logic and ethics*, Mannheim
[7] Freudenthal, H. (1953) Zur Geschichte der vollständigen Induktion, *Arch. Int. d'Hist. Sci* 22, 17-37

Appendix: Three illuminating examples

1. *Example from the teaching of fractions*
"Concrete" fractions (m/n) U, where U is a unit such as cm, dm^2, and m, n are natural numbers, can be defined as $m((1/n)U)$ the meaning of which is: m times the n-th part of U. The latter formulation obviously refers to actions by which starting with U, (m/n) U can be constructed. It is not trivial that

$$m((1/n)\ U) = (1/n)(mU)$$

Pupils can get insight into this equation by means of illuminating examples.
Let U be a unit, measuring e. g. area.
We want to show $3((1/4)U) = (1/4)(3U)$.



It is, of course, not satisfactory to compare merely the results of the two ways.
We get a strategy if we really perform with sheets of paper—or imagine—the following actions:

(a) Fold the paper into 4 equal parts.
(b) Take 3 of these parts together.
(c) Repeat step (a) with each of 3 sheets of paper
(d) "Choose" in each of the sheets exactly one part
(e) "Glue" these parts from (d) together.

From a heuristical point of view step (c) is obvious: try to use already established knowledge which is in relation to the task to solve. The next step (d) is crucial. Instead of dividing the row of sheets anew into 4 parts the more tricky, and in our situation convenient, procedure is used. At first glance this choosing procedure seems to be too far from childrens' thinking. It becomes, however, accessible if the sheets are not put in a row but piled up.

The strategy described above obviously works independently of the special numbers 3 and 4.

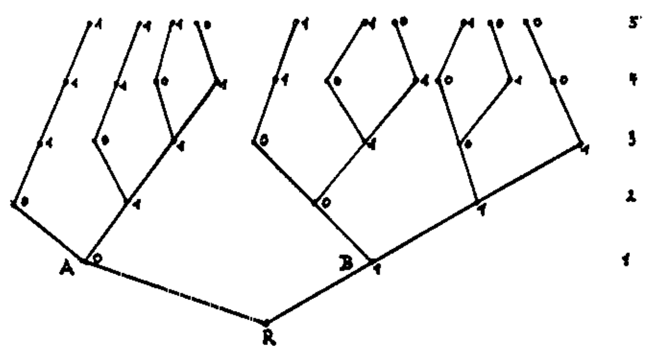
But there is yet another direction for generalization and abstraction. The same strategy can also be used for units other than area, e. g. length, volume, weight.

2. *Relation between binomial coefficients*
Let the binomial coefficient $\binom{m}{n}$ $(m, n \in N, m \leq n)$ be introduced as the number of different ways to put n pebbles into m boxes so that at most one pebble is in each box. The fundamental relations

$$\binom{m}{n} = \binom{m}{m-n}$$
$$\binom{m}{n} = \binom{m-1}{n} + \binom{m-1}{n-1}$$

can be made accessible by means of an illuminating example: take $m = 5$, $n = 3$.

To describe the distributions of 3 pebbles in 5 boxes as a structured whole a tree is used. The layers of the tree correspond to the boxes. The signs 1 or 0 at a node of the tree indicate if a pebble is out or is not put in the box in question



Interchanging 0 and 1 shows that

$$\binom{5}{3} = \binom{5}{5-3}$$

At the point R the tree splits up into two subtrees; if A,B are taken as new starting points then clearly

$$\binom{5}{3} = \binom{4}{3} + \binom{4}{3-1}$$

The right hand side of this equation is related to the following counting strategy: the first box is kept empty, count the number of ways in which 3 pebbles can be distributed over 4 boxes; then put 1 pebble in the first box and count the number of ways in which the remaining 2 pebbles can be put into 4 boxes.

3. *Euler's φ-function is multiplicative*
$\varphi(n)$ can be defined as the number of elements in a reduced residue system mod n. Experiments e. g. with $n = 24$ show that $\varphi(24) \neq \varphi(6) \cdot \varphi(4)$, $\varphi(24) \neq \varphi(2) \cdot \varphi(12)$, but $\varphi(24) = \varphi(3) \cdot \varphi(8)$. How can this good behavior of φ be explained?

From gcd (3,8) = 1 follows: a natural number is relatively prime to 24 iff it is relatively prime to both 3 and 8. So let us look at the rectangular array:

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24

As an immediate consequence of this construction we have

1. Each column belongs to the same residue class mod 8.
   Each row forms a complete residue system mod 8.
   Each column forms a complete residue system mod 3
   (These properties follow from gcd (3,8) = 1.)
2. The φ (8) rectangular boxes contain all numbers ≤ 24 which are relatively prime to 8.
3. From 1., in each of these boxes there are exactly φ (3) "circled" numbers ≤ 24 which are relatively prime to 3.
4. Hence φ (24) = φ (8) · φ (3).
   Obviously this proof strategy can be generalized to arbitrary natural numbers *m,n* with gcd (*m,n*) = 1

Institut für Mathematik und ihre Didaktik
Pädagogische Hochschule Kiel
Olshausenstrasse 75
2300 Kiel 1, West Germany

Unobtrusive Influences on the Learning of Number Sense

MARTIN COOPER

With the rapidly increasing use of hand-held calculators in both the classroom and everyday life, many educators have called for the teaching of methods of estimation and approximation to be introduced into mathematics curricula.

Edwards [1984] believes that the justification for teaching of computational estimation lies in the need to develop "number sense", an attribute which includes mainly mental arithmetic and some sort of capacity to compare numbers. When discussing the use of calculators in their "foundation list of mathematical topics", Cockcroft et al [1982] state that "emphasis should be placed on using appropriate procedures for checking the reasonableness of the answer which has been obtained". They believe, moreover, that "simple calculations should in general be carried out mentally or on paper and not with a calculator".

In his recent article on computational estimation, Edwards 1984] suggests a number of "agreed procedures" for the teaching of estimation. In particular, he gives details of procedures for estimating sums, differences, means, products, quotients and percentages. Levin 1981], also, provides a number of estimation techniques, these being based on the concept of measurement rather than on whole numbers.

While strongly supporting the suggestion that estimation and approximation should be taught, especially as a means of inculcating number sense, I cannot help wondering whether the formal teaching of estimation is the best way to achieve the objective. The presentation of a topic in a "block" of syllabus material often results in the learning, testing and forgetting of the subject unless there is frequent subsequent reinforcement. When doing calculations in front of a class, my own approach is always to carry out an estimation of the reasonableness of the result as a matter of course, and to encourage my students to carry out rough checks on their own work. This has applied to the teaching of physics as well as mathematics to a wide age-range of students. As an example of carrying out a rough check using a "rounding to whole numbers" approach, Levin [1981] illustrates the estimation of the product of 589 and 49.3 by multiplying 600 by 50 and then adjusting downward a bit. Although his claim that techniques of this sort are seldom taught formally in the classroom, one wonders how many teachers try to instill the habit of estimation in an informal, unobtrusive and persistent manner.

It is perhaps worth observing that in order to carry out the sort of estimation illustrated above, it is necessary actually to carry out a multiplication. Although this may be done by means of a calculator, it is surely more efficient to be able to perform such a simple operation in one's head or on paper, as suggested in the Cockcroft report. This action, of course, requires knowledge of the sort formerly learned as "multiplication tables". My point here is that although calculators have removed the tedium of performing more complex operations, including long division and the summing of lists of many-digit numbers, they have not eliminated the advisability of knowing the sums and products of many pairs of simple numbers, and the reciprocals and percentage equivalents of certain simple fractions. Indeed, I believe that knowledge of this sort is a part of the "number sense" which should be abroad in our society.

Although certain estimation techniques are probably best learned in a formal setting, I believe that most should be taught unobtrusively. The advantage of an informal, pervasive approach to number sense is that it may then be absorbed by osmosis, as it were, the reinforcement being natural and un-forced.

To conclude, I must admit to some scepticism about the introduction of estimation and approximation into mathematics curricula (even for unobtrusive treatment) without a major effort being made at the level of teacher education. Although many teachers can guide the learning of formal estimation techniques, I suspect that a vast number have not been visited by the notion of inculcating number sense. This is an area where mathematics-teacher educators may perhaps make a contribution.

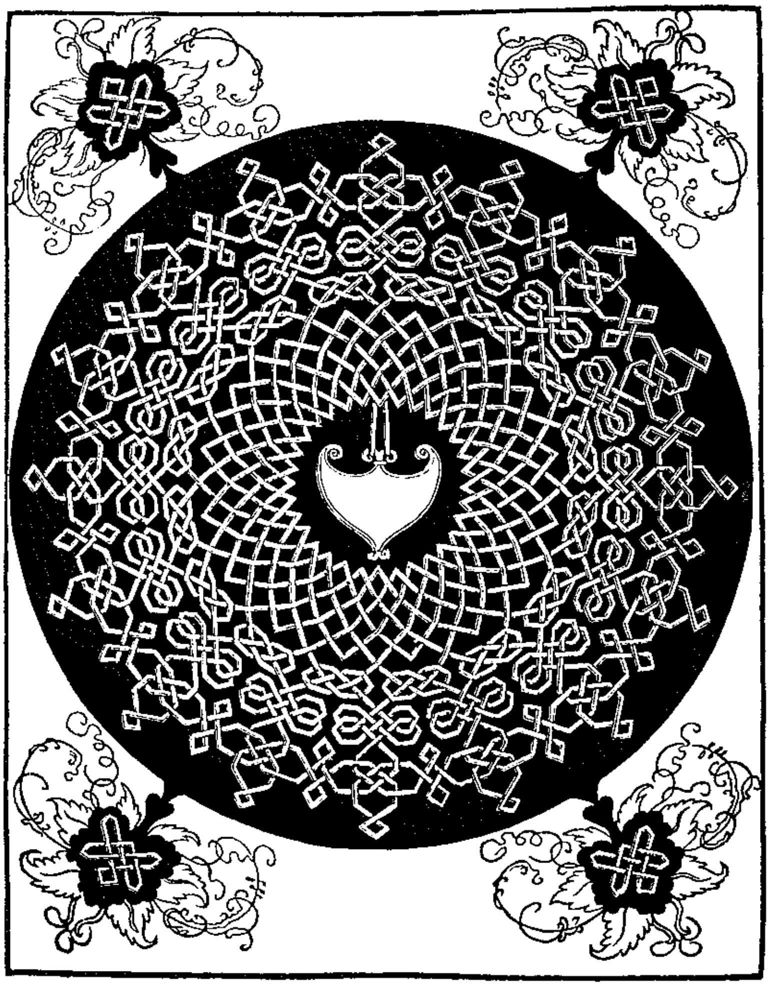
References

Cockcroft, W.H , *Mathematics Counts*: Report of the Committee of Inquiry into the Teaching of Mathematics in Schools. London, HMSO, 1982

Edwards, A , "Computational estimation for numeracy". *Educational Studies in Mathematics*, 1984, 15

Levin, J.A , "Estimation techniques for arithmetic: everyday math and methematics instruction", *Educational Studies in Mathematics* 1981, 12, 421-434

Faculty of Education
University of New South Wales
Sydney, Australia



Woodcut by
Albrecht Dürer