AN ANALYTIC FRAMEWORK OF REASONING-AND-PROVING

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In the discipline of mathematics, the development of new knowledge often passes through several stages, of which the development of proofs is typically the last. Earlier stages frequently involve the identification and arrangement of significant facts into meaningful patterns, the use of the patterns to formulate conjectures and the testing of these conjectures against new evidence, the revision of the conjectures to address possible counterexamples, and the effort to understand and provide arguments about whether and why things work the way they do (Lakatos, 1976; Polya, 1954). These activities are important because they aid mathematicians (as well any other doer of mathematics) in understanding the terrain associated with the phenomenon under examination, building a foundation for the development of proofs (Boero *et al.*, 1996; Mason *et al.*, 1982).

In school mathematics, the development of proofs has often been treated as a formal process (primarily in high school geometry) isolated from other mathematical activities. However, this treatment of proof is problematic, because it does not afford students the same level of scaffolding that professional users of mathematics are afforded to make sense of and establish mathematical knowledge. It is thus important that students be assisted to develop proficiency in, and understand the relations among, all major activities that are frequently part of the process of making sense of and establishing mathematical knowledge: 'identifying patterns', 'making conjectures', 'providing non-proof arguments', and 'providing proofs'. In this paper, I use the term *reasoning-and-proving* (RP) to describe the overarching activity that encompasses these four activities; the choice of a hyphenated term reflects an emphasis on viewing these activities in an integrated way. I capture the last two activities under the notion of 'providing support to mathematical claims' and the first two activities under the notion of 'making mathematical generalizations'. Following Polya (1954), I associate the making of mathematical generalizations with the transportation of mathematical relations from given sets to new sets for which the original sets are subsets. Algebraic generalizations have perhaps attracted the most attention in the literature (*e.g.*, Kieran, 1989; Radford, 2006), but the notion of generalization in this paper transcends content areas (*e.g.*, algebra, geometry) and representational forms (*e.g.*, algebraically, pictorially).

Although RP is central to doing mathematics, many students and teachers of mathematics face serious difficulties with the different activities that comprise RP, especially the development of proofs (*e.g.*, Balacheff, 1988; Healy & Hoyles, 2000; Knuth, 2002; Stylianides *et al.*, 2007). In order for the field to build the knowledge base that will support the efforts to effectively promote mathematical ideas such as RP among students, it is important to develop analytic tools that can support different kinds of investigations focusing on these ideas, such as textbook analyses and examinations of teaching and learning (Stein *et al.*, 2007). These analytic tools can also provide the means to connect research findings from different investigations, thereby supporting the development of integrated research programs and the accumulation of knowledge across different domains. This paper contributes to this research and development endeavor in the particular domain of RP by:

- proposing an analytic framework that can serve as a useful platform for conducting different kinds of investigations with a focus on RP; and
- illustrating the utility of the analytic framework in the context of two research studies: a textbook analysis and an examination of a teacher professional development session.

Analytic framework

The conceptualization of RP that I have outlined sets the stage for an analytic framework that includes three components: mathematical, psychological, and pedagogical (Fig. 1). Before I elaborate on the three components of the framework and discuss what kind of inquiry on RP they can support, I make four comments about the framework. First, I do not claim that the framework captures every possible activity related to RP. For example, an activity that is not adequately captured in the framework is 'reasoning by analogy', which is central to the work of professional mathematicians (Polya, 1968) and to students' engagement in problem solving (English, 1997; Reid, 2002). Second, I do not suggest that the development of proofs in the discipline or in school mathematics passes, or should always pass, through the identification of patterns and the making of conjectures. Students may try to prove, for example, a statement that is given to them by the teacher or that appears in the textbook. Also, professional mathematicians often try to prove already established conjectures or generate alternative proofs to already established theorems. Third, I do not suggest that conjectures always develop inductively through pattern identification. Conjectures can also be generated, for example, by analogy or deductively from already established theorems (Koedinger, 1998). Fourth, my focus on a mathematical, psychological, and pedagogical perspective (each deriving from the corresponding framework component)

	Reasoning-and-proving			
Mathematical Component	Making Mathematical Generalizations		Providing Support to Mathematical Claims	
	Identifying a Pattern	Making a Conjecture	Providing a Proof	Providing a Non-proof Argument
	• Plausible Pattern • Definite Pattern	• Conjecture	• Generic Example • Demonstration	• Empirical Argument • Rationale
Psychological Component	What is the solver's perception of the mathematical nature of a pattern / conjecture / proof / non-proof argument?			
Pedagogical Component	How does the mathematical nature of a pattern / conjecture / proof / non-proof argument compare with the solver's perception of this nature? How can the mathematical nature of a pattern / conjecture / proof / non-proof argument become transparent to the solver?			

Figure 1: The analytic framework

does not suggest that these perspectives are unrelated to, or are more important than, other perspectives that have been used in the study of RP, such as cultural-historical, social-interactionist, and semiotic (for a semiotic analysis of students' strategies in algebraic generalization of patterns, see Radford, 2006).

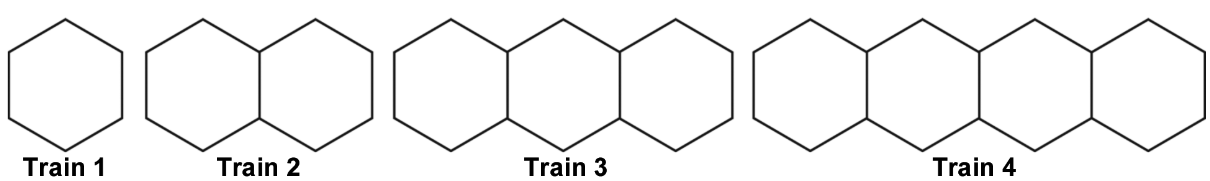
Mathematical component

The mathematical component of the framework includes the four activities that comprise RP together with a further breakdown of some of these activities to capture important distinctions. Its primary feature is that it integrates many well-known activities related to engagement with proof, thereby allowing one to examine this engagement in an integrated way. As suggested by its name, this component of the framework can support inquiry on RP activity from a mathematical perspective. In this perspective, the observer (or examiner) of the activity (*e.g.*, classroom activity or activity described in textbooks) is considered to be a mathematically proficient person (*e.g.*, the researcher or the teacher) who analyzes the given activity using mathematical considerations. For example, the observer may identify a student argument as an empirical argument irrespectively of the student's perception of the argument (the student may believe that his argument qualifies as a proof).

Identifying a pattern. A major challenge in mathematics education is to develop students' abilities to make generalizations on the basis of mathematical structures (structural generalizations) rather than on the basis of perception or the evidence offered by the regularities found in a few examples (empirical generalizations) (Bills & Rowland, 1999; Küchemann & Hoyles, in press). Students' ability for structural generalizations is particularly important when they engage with patterns, which denote general mathematical relations that fit given sets of data. A considerable body of research (Becker & Rivera, 2006; Bills & Rowland, 1999; Küchemann & Hoyles, in press; Zazkis & Liljedahl, 2002) indicates that students tend to make empirical rather than structural generalizations when they engage with tasks like 'The Hexagon Trains Task' (Fig. 2).

The hexagon task gives rise to what I call a *definite pattern*, because it is possible mathematically for a solver to provide conclusive evidence for the selection of the pattern, Perimeter of Train $n = (4n + 2)$. Of course, there are many equivalent ways (both algebraic and non-algebraic) to represent the pattern. The specific pattern is determined by the mathematical structure of the task, which specifies the process by which each train in the pattern is created: 'The first train in this pattern consists of one regular hexagon. For each subsequent train, one additional hexagon is added'.

For the pattern shown below, compute the perimeter for the first four trains, determine the perimeter for the tenth train without constructing it, and write a description that could be used to compute the perimeter of any train in the pattern. The first train in this pattern consists of one regular hexagon. For each subsequent train, one additional hexagon is added. The first four trains in the pattern are shown below.



Find as many different ways as you can to compute (and justify) the perimeter.

Figure 2: The Hexagon Trains Task

Without this structure it would be impossible mathematically for the solver to decide, for example, whether Train 5 would consist of 5 hexagons. [1] If this structure were missing, the task would give rise to what I call a *plausible pattern.*

In plausible patterns, it is not possible mathematically for a solver (given the information in a task) to provide conclusive evidence for the selection of a specific pattern over other patterns that also fit the data. However, the solver may select a specific pattern based on other criteria. For example, she may select the simplest or most evident ('natural' in Zazkis & Liljedahl's, 2002, terms) pattern that fits the data. Example 1 illustrates the notion of plausible patterns by presenting a pattern that is not uniquely determined.

Example 1

The table shows how two variables relate. Find a pattern in the table and use the pattern to complete the missing entries.

a	0	1	2	3	4
b	1	2	4		

One pattern that fits the data in the table is $b = 2^a$. Based on this pattern, the missing entries are 8 and 16. Another pattern that fits the data is $b = \frac{1}{2} \cdot a \cdot (a + 1) + 1$. According to this pattern, the missing entries are 7 and 11. From a mathematical standpoint, however, any answer to the task could be correct. One could say, for example, that the missing entries are 1 and 2, considering the first three values of *b* in the table as the 'unit of repeat' of the pattern. More boldly, one could say that both missing entries are 0, considering that all the values in the pattern after the first three are 0.

While an observer can determine the nature of a pattern (definite versus plausible) using strictly mathematical criteria, solvers can take different paths (correct or incorrect, from the observer's standpoint) when they engage with patterns. Specifically, although in a definite pattern a structural generalization is required for the unique identification of the pattern, one might (incorrectly) offer an empirical generalization for it. In a plausible pattern, however, one is not required to provide a structural generalization to be correct (*cf.* the 'zero' solution in Example 1). Yet, it is possible for a solver to provide a structural generalization in identifying a plausible pattern. Consider again the hexagon task but without the two sentences that specify how each train in the pattern is created. A solver could *assume* a particular structure (*e.g.*, a linear pattern) and provide, based on this assumption, a structural generalization for the pattern.

Making a conjecture. In this framework, a *conjecture* is defined as a reasoned hypothesis about a general mathematical relation based on incomplete evidence. The term 'reasoned' is intended to emphasize the non-arbitrary character of the hypothesis. The term 'hypothesis' indicates a level of uncertainty about the truth of a conjecture and denotes that further action is needed for its acceptance or rejection (Cañadas & Castro, 2005; Reid, 2002). As Harel and Sowder (1998) put it, "[a] conjecture is an observation made by a person who has doubts about its truth" (p. 241).

Although the activities of making conjectures and identifying patterns are clearly related (as parts of the more general activity of making mathematical generalizations), there are two important differences between them. First, in conjecturing, a hypothesis is formulated that has a domain of reference which extends beyond the domain of cases that gave rise to it, whereas in pattern identification, the statement of a pattern does not necessarily extend beyond the domain that gave rise to it (Reid, 2002). Second, in conjecturing, a hypothesis is set forth that, although accompanied by an expression of conviction about its truth, is not considered to be true or false and is subject to testing, whereas in pattern identification, a relation that fits a given set of data is presented in a way that does not communicate necessarily possible doubt about its truth.

Providing a proof. Following Stylianides (2007), a *proof* is defined here as a valid argument based on accepted truths for or against a mathematical claim. The term 'argument' denotes a connected sequence of assertions. The term 'valid' indicates that these assertions are connected by means of accepted canons of correct inference such as *modus ponens* and *modus tollens*. The term 'accepted truths' is used broadly to include the axioms, theorems, definitions, modes of reasoning, and representational tools that a particular community may take as shared at a given time. An argument that qualifies as a proof makes explicit reference to 'key' accepted truths that it uses.

The term 'valid' in the definition of proof should be understood in the context of what is typically agreed upon in the field of mathematics nowadays. Of course, this is not to say that this term has universal meaning in mathematics nowadays, but it is beyond the scope of this paper to elaborate on this issue. Also, deciding what belongs to the set of accepted truths of a particular community at a given time or which accepted truths should be explicitly referenced in a proof are delicate issues. Addressing in detail these issues is again beyond the scope of this paper. It is clear, however, that a possible response to these issues would have to consider the audience for the proof. For example, *modus ponens* is a major accepted truth that is, however, never referenced in mainstream mathematical proofs because it is considered to be basic knowledge. In this sense, we may say that *modus ponens* is not considered a key accepted truth worthy of reference in the context of mainstream mathematical proofs. Another issue concerns the knowledge that can be considered as shared within a community, such as a classroom community. This knowledge does not necessarily reflect the understanding of each individual student. As Lampert (1992) noted, the individual learners of a classroom community 'go their separate ways with whatever knowledge they have acquired' (p. 310). Accordingly, when I talk about the set of truths that are accepted by a community, I do not imply that each member of the community understands in the same way the elements of this set. Rather, I mean to refer to the elements of this set that can comfortably be assumed and used publicly without justification.

The framework distinguishes between two different kinds of proof: generic examples and demonstrations. A *generic example* is a proof that uses a particular case seen as representative of the general case (Balacheff, 1988; Mason & Pimm, 1984; Rowland, 1998; also similar to Harel & Sowder's, 1998, 'transformational proof'), whereas a *demonstration* is a proof that does not rely on the 'representativeness' of a particular case (similar to Harel & Sowder's, 1998, 'axiomatic proof' and Balacheff's, 1988, 'thought experiment'). Valid arguments by counterexample, contradiction, mathematical induction, contraposition, and exhaustion are examples of demonstrations. I do not associate demonstrations with any particular representational form such as the use of algebraic notation. For example, the following argument by a third-grader for the claim 'odd + odd = even' would qualify as a demonstration (assuming that the argument addressed appropriately contextual features of the classroom community where it was developed):

> All odd numbers if you circle them by twos there's one left over. So, if you add two odd numbers, the two ones left over from the two odd numbers will group together and will make an even number. This is because all even numbers if you circle them by twos there's none left over. (adapted from Ball & Bass, 2003, p. 39)

Providing a non-proof argument. In this framework, a *non-proof argument* is defined as an argument for or against a mathematical claim that does not qualify as a proof. The framework distinguishes between two kinds of non-proof arguments: empirical arguments and rationales.

An *empirical argument* is an argument that provides inconclusive evidence for the truth of a mathematical claim (similar to Harel & Sowder's, 1998, 'empirical justification' and Balacheff's, 1988, 'naïve empiricism'). In particular, the solver may conclude that a claim is true after checking a proper subset of all the possible cases covered by the claim, or after considering the full range of possible cases without however showing that she did so.

Contrary to the well-known notion of empirical argument, the notion of *rationale* is new and has been introduced in the framework to capture arguments for or against mathematical claims that are neither proofs nor empirical. For example, an argument counts as a rationale (*vs.* a proof) if it does not make explicit reference to some key accepted truths that it uses (in the context of a particular community where these truths can be considered as key), or if it uses statements that do not belong to the set of accepted truths of a particular community. Consider the third-grader's demonstration presented earlier without the first and last sentences: 'If you add two odd numbers, the two ones left over from the two odd numbers (after circling them by twos) will group together and will make an even number'. This argument does not make explicit reference to the definitions of even and odd numbers being used; these definitions can be considered as key for the development of the argument in the context of a third-grade classroom community where these definitions are just emerging. There are also other accepted truths in this argument that are not referenced explicitly, such as the transitivity of equality. However, requiring explicit reference to such intuitively obvious properties for arguments to qualify as proofs could shift the emphasis from proof as a vehicle to sense-making to proof as a ritual procedure (Schoenfeld, 1991). For another example of a rationale, consider a situation where a student is trying to prove that 'odd + odd = even' using the statement 'even + even = even', but this statement does not belong yet to the set of accepted truths of the particular classroom community. The following student argument would then count as a rationale: 'Since even + even = even, I can get the sum of any two odd numbers by adding 1 to each of two even numbers. Therefore, odd + odd = (even + 1) + (even + 1) = (even + even) + 2 = even + 2 = even, since 2 is an even number'.

Psychological component

The psychological component of the framework focuses on the learner. An inquiry on RP from a psychological perspective would examine the solver's perception of the mathematical nature of a *mathematical object related to RP* (pattern, conjecture, proof, non-proof argument) as this perception is reflected, *e.g.*, in the solver's solution of a task or in the solver's comments about another solver's solution of a task. Thus, contrary to what happens in the mathematical component of the framework, the psychological component requires that terms such as 'proof' and 'conjecture' be interpreted in a subjective sense (Harel & Sowder, 1998, 2007).

To illustrate the psychological component of the framework, consider a solver producing an argument in response to a task that asks him to prove a claim. The important question from a psychological perspective would then be whether the solver considers the argument he produced to be a proof. The solver may believe that his argument is a proof, even though it may be an empirical argument according to the judgment of an observer (*e.g.*, a researcher or a teacher) who evaluates the argument using a mathematical perspective. The distinction between the solver's perception of the argument and the nature of the argument (as derived by application of a mathematical perspective) bears some similarities to Balacheff's (1988) distinction between the notions of 'explanation' and '(mathematical) proof': the validity of the former relates (initially, at least) to the person who articulates it, whereas the validity of the latter depends on rules of discourse shared by the wider community of mathematicians.

For another example, refer back to the hexagon task and assume that a solver approaches the task empirically by generalizing the pattern from the numerical values in a table, but believes that her solution uniquely determines the pattern. Although this approach to deriving the pattern is limited from a mathematical standpoint, an observer who uses a psychological perspective to examine the solver's work can conclude that the solver appears to perceive the pattern as definite. Of course, such a conclusion would not imply or presuppose that the solver was aware of the distinctions between, or the language of, plausible and definite patterns.

Pedagogical component

The pedagogical component of the framework uses both the mathematical and the psychological components. An inquiry into RP from a pedagogical perspective would focus on two primary and interrelated issues. The first issue concerns how the mathematical nature of a mathematical object related to RP (as derived by application of a mathematical perspective) compares with the solver's perception of this nature (as derived by application of a psychological perspective). Possible discrepancies emerging from this comparison can help identify potential foci for teachers' pedagogical actions aiming to support the refinement of their students' understandings of the nature of a particular RP object (see Stylianides, 2007, for a teaching approach that is consistent with this pedagogical perspective in the particular area of proof).

The second issue is an extension of the first: once instruction compares and identifies possible discrepancies between students' perceptions of particular mathematical objects (the activity of RP being a case in point) and conventional understandings of these objects in the mathematical community, instruction needs to actively seek ways to help students gradually refine their perceptions toward the conventional understandings (*e.g.*, Ball, 1993; Harel & Sowder, 2007; Stylianides, 2007). In cases where this goal is achieved with respect to a particular mathematical object, we may say that the mathematical nature of the object has become 'transparent' (versus 'non-transparent') to the solver. Specifically, the mathematical nature of a RP object is said to be *transparent* to the solver if the solver's perception of the object coincides (after an instructional intervention, etc.) for valid reasons with the mathematical nature of the object. If, however, the solver's perception of the object does not coincide with the mathematical nature of the object, or if it coincides with it for invalid reasons, then we may say that the mathematical nature of the object is *non-transparent* to the solver.

For example, an empirical argument is said to be transparent to a solver if the solver recognizes what makes the argument empirical such as the fact that the argument provides inconclusive evidence for a claim, having established its truth only on a proper subset of all the possible cases. If, however, the solver believes that the empirical argument is a proof, then the mathematical nature of the argument is said to be non-transparent to the solver. For another example, a definite pattern is said to be transparent to a solver if the solver recognizes by application of a valid method why the pattern is uniquely determined. If, however, the solver is convinced that the pattern is definite by an empirical argument, then the mathematical nature of the pattern is said to be non-transparent to the solver.

Two studies where the analytic framework was used

In this section, I explain how the analytic framework was used in two studies with a focus on RP: a textbook analysis [2] and an examination of an episode from a teacher professional development session (Stylianides & Silver, in press). My goal here is not to present in detail the two studies, but rather to illustrate the utility of the framework in each of them. The first study adopts a mathematical perspective, whereas the second study adopts a combined mathematical/psychological/pedagogical perspective.

Study 1: A textbook analysis

In many classrooms, mathematics textbooks are an important resource for providing tasks for students and teachers' work (*e.g.*, Tarr, 2006). To date, however, we lack detailed knowledge of how RP is treated in contemporary textbooks. Also, little research has focused on developing and applying analytic tools to investigate this issue and address questions like the following: To what extent are tasks that design opportunities for students to engage in RP represented in mathematics textbooks? How are these tasks distributed across the different constituent activities of RP and their subcategories? Is this distribution different across grade levels and content areas?

Because of the nature of a textbook analysis, it was meaningful to adopt a mathematical perspective in this study. Who would be the solver whose subjective experience would be examined had I also adopted a psychological or a pedagogical perspective? I used the mathematical component of the framework to investigate the opportunities for RP that were designed for students in the algebra, number theory, and geometry units of the *Connected Mathematics Project* (CMP; *e.g.*, Lappan *et al.*, 1998/2004). CMP is a popular middle school (ages 11–13 years) mathematics textbook series in the United States that aims to embody the curriculum reform proposals of the National Council of Teachers of Mathematics (NCTM, 1989, 2000). My analysis of CMP involved: (1) identification of the textbook tasks that designed opportunities for students to engage in at least one of the activities that comprise RP, and (2) coding these tasks using as codes the different RP (sub)categories in the framework. Multiple coding of a task was possible.

An important issue faced in the analysis was how to make decisions about what opportunities each task was designing for students, especially because the *actual formulation* of the tasks – that is, how the tasks played out in classroom practice – is not available when one analyzes textbooks. To address this issue, I developed a way to make reasonable inferences about the *expected formulation* of the tasks – that is, the path students were anticipated to follow in solving the tasks as this was reflected in what the textbook authors wrote in the students' textbook and the teacher's edition of their textbook series. I determined the expected formulation of tasks by solving them in the order they appeared in the textbook series and by considering together the following three factors: (1) the approach suggested by the students' textbook, (2) the approach suggested in the teacher's edition, and (3) the students' expected knowledge and understanding when encountering a certain task. I determined the latter by looking at what preceded the given task in the student's textbook and considering it to be known to the students. Below is an example of how I coded a CMP task.

> **Example 2** (adapted from Lappan *et al.*, 1998/2004, p. 29)
> Make a conjecture about whether the sum of two even numbers will be even or odd. Then justify your answer.

This task was triple-coded as *identifying a pattern (definite pattern), making a conjecture,* and *providing a proof* (*demonstration*). This is because in the expected formulation of the task the students were anticipated to: (1) examine a few cases and notice the definite pattern that the sum they got in all cases was an even number; (2) use the pattern to formulate the conjecture that the sum of any two evens is even; and (3) use their representation of even numbers as rectangles with a height of two square tiles to provide the following demonstration: 'The sum of two even numbers is even because we can combine two rectangles with height of two square tiles (*i.e.*, two even numbers) to get another rectangle with height of two square tiles'.

I tested the inter-rater agreement of the coding scheme by comparing my codes with the codes of a second rater in a sub-sample of tasks that offered the opportunity for different kinds of codes. One reliability value was based on our decisions on whether each task in the sub-sample was designed to engage students in RP (92.7%; kappa statistic = .9). A second reliability value was based on our decisions on how to sort the RP tasks to the seven RP subcategories (88.5%; no kappa statistic because tasks could be classified in multiple subcategories).

About 40% of the 4578 tasks in the sample were designed to engage students in RP. Of the RP tasks in the sample,

- 62% were designed to offer students opportunities to give rationales,
- 18% to identify definite patterns,
- 6% to identify plausible patterns,
- 3% to make conjectures,
- 2% to provide empirical arguments, and
- 12% to provide demonstrations.

Generic examples were virtually non-existent in the sample (0%). The latter result suggests a possible limitation in CMP students' opportunities to engage in proofs, because generic examples can provide middle school students with a powerful and easily reached means of conviction and explanation (Bills & Rowland, 1999; Rowland, 1998), especially when they lack the mathematical language and notation to develop arguments detached from any particular case (Balacheff, 1988). Another notable finding is that the RP tasks were distributed unevenly across grade levels and content areas, with the sixth-grade number theory unit concentrating the highest proportion of proofs.

To conclude, the major strength of the analytic framework in this study is its unifying power: it offered a reliable tool to examine in detail the opportunities designed in the algebra, geometry and number theory CMP units for students to engage in all constituent activities of RP. The findings from the analysis addressed the questions that the study aimed to address and raised some other important questions related to how RP can best be promoted in mathematics textbooks more generally. For example, the findings related to the sixth-grade number theory unit raised the issue of whether it matters how the RP opportunities are *allocated* in a textbooks series (*e.g.*, across grade levels and content areas), or whether what really matters is to have these opportunities *in* the textbook series.

Study 2: An examination of an episode from a teacher professional development session

In this study, we utilized a combined mathematical/psychological/pedagogical perspective and used the *identifying a pattern* category of the framework to analyze an episode with a group of 12 middle school teachers who were identified by their schools as leaders and were experienced users of CMP. The teachers were engaged in the hexagon task (*cf.* Fig. 2). One of the goals of the teacher educator for the session was to engage the teachers in pattern identification and to initiate a discussion on issues related to teaching patterns to their students.

Using the language of the analytic framework and adopting a *mathematical perspective,* we can say that the hexagon task provided an opportunity for teachers to look for the underlying mathematical structure of a *definite pattern* and to use that structure to derive a rule that could be used to compute (conclusively) the perimeter of any train.

Our analysis of the episode using a *psychological perspective* showed that many teachers approached the problem the same way the literature suggests that many students would approach it: they calculated the perimeter of the first few trains and identified the pattern in ways predicated on the basis of the regularities found in these terms (empirical generalization), without connecting the generalization with the process by which each subsequent train was constructed from the previous one (structural generalization). In essence, this approach ignored the information in the task that made the pattern *definite*, thus treating the pattern as if it were *plausible*. Two teachers, for example, used the numbers 6, 10, and 14 to find the pattern $4n + 2$. Here is a description of the empirical approach that these teachers followed to solve the task based on data gathered by measuring the sides of the first three trains:

> The constant difference of 4 between one train and the next suggested to us that there must be a factor of $4n$ somewhere in the formula. Once we figured this out, we started looking for the appropriate correction factor by trial and error.

Despite the fact that many of the teachers approached the pattern empirically, most of them seemed to have no doubt about its definite nature. The apparent discrepancy between their approach to the pattern and the mathematical nature of the pattern suggests, according to the *pedagogical component* of the framework, that the definite nature of the pattern was *non-transparent* to them. Yet some of the teachers who identified the pattern based solely on numerical values expressed doubts about whether the algebraic expression they found was an accurate representation of the pattern. Also, one teacher, Nicole, noted the following struggle she faces in her teaching: 'I teach this unit and my students would struggle with this, and I also struggle with how to help them'.

After some prompting from the teacher educator, another teacher, Tanya, came up with a structural generalization: she presented a solution that was connecting the pattern to the figures and was referring to the process by which each train in the pattern was created. Tanya's solution, which suggested that the definite nature of pattern was *transparent* to her, helped make the pattern transparent to some of her colleagues as well. For example, Nicole remarked that Tanya's solution was an eye-opener to her. She also noted that, in her teaching, she would only explain the pattern from the numerical values in a table (empirical generalization), implying that now she would seek something better than that (structural generalization). This remark suggests that she, and possibly other teachers in the episode, would normally enact in their classrooms a pattern task such as the hexagon task the same way they initially approached the hexagon task themselves during the session. Specifically, they would guide their students to derive the pattern based on numerical values from a table in isolation from the mathematical structure in the task.

In this study the analytic framework helped us in two important ways. First, it offered us useful language to describe the mathematical terrain relevant to the hexagon task. Second, it helped us compare teachers' initial understanding of the pattern in the task with the mathematical nature of the pattern and examine how this understanding developed during the session. Although the teachers in the session were dealing with a definite pattern, they initially tended to treat the pattern numerically and generalize it without connecting it to the mathematical structure of the task that uniquely determined the pattern. Later in the session, however, the discussion helped make the pattern transparent to some of the teachers. Making the pattern transparent to the teachers helped some of them gain insights into their teaching.

Other potential contributions of the analytic framework to research

The analytic framework has the potential to support studies in the domain of RP that would focus on each of the three major components of instruction – students, teachers, and textbooks (Cohen & Ball, 2001) – thereby facilitating examination of the relationships among these components. For example, the interplay between what is included in the textbooks (written curriculum) and how the textbooks are enacted by teachers and their students in classrooms (implemented curriculum) is still not well understood in the particular domain of RP but also more broadly (Remillard, 2005). Understanding these relationships is important because it can help connect research findings from different investigations that focused on the different components of instruction.

The analytic framework can be used to study the relationships between written and implemented curriculum in the domain of RP: To what extent do teachers implement in their classrooms textbook tasks related to RP in the ways intended by the textbook authors? A textbook analysis like the one I described earlier would offer a good picture of the place of RP in the written curriculum. To study the place of RP in the implemented curriculum, one can use the mathematical component of the framework to code the tasks based on their actual formulation (as opposed to their expected formulation that I used in the textbook analysis) in the classroom. One can expect that application of the analytic framework in a study of the implemented curriculum will be less challenging than its application in a study of the written curriculum: it is easier to code tasks based on observation of how they played out in classroom practice, rather than based on inferences about the path students are expected to follow to solve them. However, in a study of the implemented curriculum some of the framework categories would need to be further refined to capture important complexities that are hard to capture in analysis of textbooks. One possible refinement concerns the providing a proof category. Textbooks do not typically offer information about how students are likely to represent their arguments, so a textbook analysis cannot easily examine the representational tools used in arguments and proofs. However, the availability of actual student work permits this kind of examination in an analysis of classroom practice (see Stylianides, 2007).

With regard to issues of attained curriculum, each of the activities that comprise RP in the analytic framework corresponds to an activity for which researchers can design tasks to construct a comprehensive portrait of students' understanding of RP. For example, researchers can design tasks to examine students' ability to produce proofs and also their ability to evaluate whether their arguments qualify as proofs. Specifically, the analytic framework can offer a useful tool to analyze the student arguments both from a mathematical perspective (*e.g.*, to characterize a particular student argument as an empirical argument) and from a psychological perspective (*e.g.*, to characterize the student's perception of whether his empirical argument qualifies as a proof). Comparison of the findings to be obtained from application of the two perspectives can provide insights into students' understanding of proof and inform pedagogical interventions that will aim to promote students' understanding in this domain. In a study of university students' understanding of proof, I gave 39 preservice elementary teachers two test items that asked them to prove two mathematical claims and then evaluate whether their arguments qualified as proofs. Preliminary analysis of the preservice teachers' written responses to the two items showed that about a quarter of these responses qualified as proofs. Also, in all of these responses the preservice teachers clearly recognized the arguments they produced as proofs. From the rest of the responses, which ranged from non-proof arguments to non-arguments, about a third were clearly recognized by the preservice teachers as non-proofs. These findings provide a new lens to viewing students' understanding of proof: an unsuccessful attempt to prove a claim does not necessarily imply limited understanding of the notion of proof.

Finally, in cases where studies on the connections between the written and implemented curriculum reveal high fidelity of curriculum implementation, researchers can map the findings of textbook analyses onto the findings of studies on students' understanding of RP. This mapping can inform textbook revisions in order to better support instruction that would help students develop their understanding in this domain.

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Notes

[1] When I characterize a pattern as definite, I do so with the caution "that it is sometimes extraordinarily difficult to achieve understanding, certainty, or clarity in mathematics" (Hersh, 1979, p. 40). In the case of the hexagon task, one could argue that there is ambiguity in how additional hexagons are added to form subsequent trains (*e.g.*, one could claim that the new hexagon does not necessarily need to be added serially with only one of its sides overlapping with one side of a previous hexagon in the train).

[2] Stylianides, G.J. (2005) *Investigating students' opportunities to develop proficiency in reasoning and proving: a curricular perspective,* Unpublished doctoral dissertation, University of Michigan, Ann Arbor.

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