SENSE-MAKING IN MATHEMATICS: AN ACT OF COMPREHENSION OR CREATION?

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Sense-making is a widely-used theoretical construct in mathematics education research. Although it has been used with different meanings in mathematics education research, the notion of sense-making remains largely underspecified in the literature. It is generally understood as a means of overcoming the separation between mathematical concepts and mathematical conceptions, as described by Sfard (1991):

> The word 'concept' (sometimes replaced by 'notion') will be mentioned whenever a mathematical idea is concerned in its 'official' form—as a theoretical construct within 'the formal universe of ideal knowledge'; the whole cluster of internal representations and associations evoked by the concept—the concept's counterpart in the internal, subjective 'universe of human knowing'—will be referred to as a 'conception'. (p. 3)

In this view, students strive to make sense of a given mathematical concept by developing conceptions that best fit the mathematical concept in question.

However, research suggests that students develop conceptions not only to attribute meaning to an already existing mathematical concept, but also to give meaning to a mathematical concept that has yet to become as a result of ongoing experiences. In this latter form of attributing meaning, that we call 'meaning-making', learning mathematics is not about comprehending an existing concept, but about creating a new concept. In this article, we argue for an explicit distinction between sense-making and meaning-making, and how both contribute to the learning of mathematics. To this end we resignify the semiotic-cognitive standpoint (see Duval, 2006) to clarify how an act of creation differs from the general notion of sense-making as an act of comprehension.

Theoretical foundation

The theoretical foundation presented here is based on central insights offered by the German mathematician and philosopher Gottlob Frege (1848–1925), which have informed a variety of theoretical viewpoints on learning mathematics [1]. Here we draw on these insights to further our understanding of at least two critical aspects of learning mathematics. First, we share Frege's (1892a) claim that a mathematical concept is not directly accessible through the concept itself, but only through objects that act as proxies for the concept. Here we understand mathematical concepts as, in a sense, ideal entities and mathematical conceptions as personal and subjective entities.

Secondly, mathematical objects (unlike objects in the natural sciences) cannot be apprehended by the human senses (*e.g.* we cannot 'see' the object). They do not exist independently of a representation. Mathematical objects can only be apprehended through a certain 'mode of presentation' (Frege, 1892b). In other words, mathematical objects are given to us in a certain way, as signs or other semiotic means, such as gestures, images or linguistic expressions. That is, access to mathematical objects necessarily involves the use of representations (Duval, 2006).

However, since we only have access to mathematical objects through representations, a representation can be mistaken for the object to which the representation refers. Already more than a century ago, Frege (1892a) pointed out the possibility of confusing objects with their representations, a potential confusion that led Duval (1993) to state what has become known as the cognitive paradox in learning mathematics:

> Comment des sujets en phase d'apprentissage pourraient-ils ne pas confondre les objets mathématiques avec leurs représentations sémiotiques s'ils ne peuvent avoir affaire qu'aux seules représentations sémiotiques? L'impossibilité d'un accès direct aux objets mathématiques, en dehors de toute représentation sémiotique rend la confusion presqu'inévitable. [2] (p. 38)

Therefore, in learning mathematics, it is an 'epistemological requirement' (Duval, 2006) to distinguish the mode of presentation of an object from the object itself. Frege (1892b) called the mode of presentation of objects the *sense*$_F$ ('Sinn') of a representation and distinguished it from the *reference*$_F$ ('Bedeutung') of a representation, that is, the object to which it refers (the subscript F indicates that these terms refer to Frege, 1892b). In this respect, any expression or representation refers to something in a certain way, and its sense$_F$ is that way of presentation. Frege (1892b) argued that two representations can have different senses$_F$ even though they have the same reference$_F$, that is, they refer to the same object. Take, for example, the expressions '$2 + 2$' and '$2 \cdot 2$'. Although they have the same reference$_F$ (*i.e.* the number four), they have different senses$_F$—different ways of arriving at the same number. Different expressions such as '$2 + 2$' and '$2 \cdot 2$' can then be associated with different *thoughts*$_F$ ('Gedanken'), that are, different ways of knowing the same number, such as the sum or product of two numbers.

On extracting meaning and giving meaning

There are several strategies or approaches in which students can make sense of a mathematical concept. While there has been something of a proliferation of approaches to sense-making in learning mathematics over the past few decades, two approaches have made a significant contribution to our understanding of learning mathematics: extracting meaning and giving meaning (for a discussion, see Pinto & Tall, 2002). Previously, extracting meaning and giving meaning have, however, been studied primarily as sense-making strategies in which the concept in question was already given.

Extracting meaning

Extracting meaning can be seen epistemologically as realised through the manipulation of objects and the reflection of variations of senses$_F$ in the manipulation of objects. For example, Duval (2006) argued that by systematically varying one representation of an object and reflecting on the resulting variations in another representation of the same object, a student can recognise what is mathematically relevant and "dissociate the represented object from the content of these representations" (p. 125). In this view, by coordinating between different representational systems a student can distinguish between a mathematical object and its representation and dissociate the sense$_F$ of a representation from the object that is represented.

Consider, for example, the concept of fractions. A common approach in schools is to systematically vary the numerator and denominator of a fraction, with the expectation that the thought$_F$ of 'part of a whole' can be captured. By systematically varying the (a) denominator and (b) numerator of a fraction and reflecting on the resulting variation in the iconic representations (as shown in Figure 1), a student can extract, for example, the meaning of the object ¾ and develop the corresponding conception of a whole divided into four equal parts, three parts of which are coloured.

Although one might expect that the concept in question is developed by an individual or group of individuals, and that the mathematical object would be actualised after it has been developed by the others, the experience provided refers to a mathematical object that has a being to begin with.

Such a view holds that students internalise extracted mathematical structures and relations associated with their actions and reflections of their actions on objects. These mathematical structures and relations, although associated with the student's actions, are seen as a somewhat ideal realm that exists prior to the student's attempts to know it. According to such a view, a student develops conceptions that best fit the mathematical concept to which the student intends to refer.

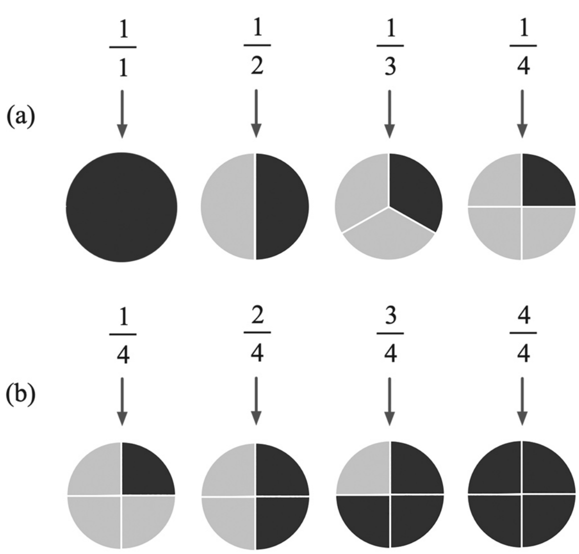

Figure 1. Systematic variation of (a) the denominator and (b) the numerator of a fraction and reflection on the resulting variation in the iconic representations.

Giving meaning

Sense-making in mathematics can be understood not only in terms of extracting meaning, but also as giving meaning to the objects of students' thinking. The understanding of sense-making as giving meaning to the objects of students' thinking has been particularly advanced by examining how students use different conceptions, and assign them to, the same object actualised in different contexts and expressed in different representations. For example, consider the object ¾. There are many different ways of bringing to mind ¾, even within the same representation system (see Figure 2a and Figure 2b).

Figure 2a, for example, expresses the thought$_F$ 'part of a whole', which can be apprehended by the conception of dividing a whole into four equal parts and taking three of those four parts. Figure 2b, on the other hand, expresses the thought$_F$ 'part of several wholes', which can be apprehended by the conception of taking three wholes, each divided into four equal parts, and then taking one part of each whole. Here it is implied that, unlike Frege (1892b), who construed sense$_F$ in a disembodied fashion as a way in which an object is given to an individual, an individual can assign different senses$_F$ to the same object when giving meaning.

However, which senses$_F$ are assigned to an object depends on which conceptions are activated in the immediate context. Conceptions can then direct the formation of the modes of presentation under which a student refers to an object. As such, it is a student's conceptions that guides the formation of a sense$_F$, not just the object to which a representation refers.

This suggests that when students give meaning, they are not quite engaged in developing conceptions that best fit a given mathematical concept, as is the case in extracting meaning, but that they are in some way engaged in creating a concept that best fits their conceptions.

In particular, our research suggests that students can give meaning to objects that are yet to become resulting from processes during ongoing experience (see Pinto & Scheiner, 2022). This means that although an object has no being before students attempt to know it, students can create new conceptions that direct their thinking towards potential

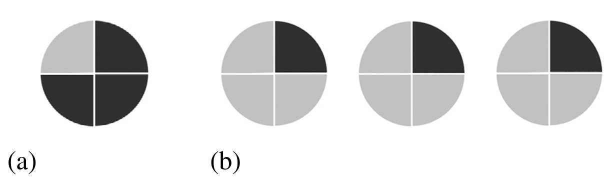

Figure 2. (a) Part of a whole. (b) Part of several wholes.

objects, or more precisely, students can create conceptions that allow objects that are yet to become to be assigned a new $sense_F$. In such cases, learning mathematics is not about reflecting an existing concept, but rather about creating a concept that best fits students' conceptions.

For example, with respect to the object ¾, the conception of 'dividing a whole into four equal parts and taking three of those four parts' to apprehend the $thought_F$ of 'part of a whole' (see Figure 2a), and the conception of 'taking three wholes, each divided into four equal parts, and considering one part of each whole' to apprehend the $thought_F$ of 'part of several wholes' (see Figure 2b), can be coordinated to promote the emergence of new conceptions—such as that of 'taking three out of four'—for apprehending $thoughts_F$ that are yet to come (see Figure 3).

Such $thoughts_F$ relate to an alternative conception to the direct identification in Figure 3 as identical to the initial $thought_F$ of dividing the whole into four equal parts and taking three of them. On a ground that is not dominated by a single epistemological possibility of concept development, one can choose to look for new potential relations between $senses_F$ and $thoughts_F$ and create an object that has yet to become. In this way, it is possible to take into account the diversity of meanings underlying fractions that become through acts of creation.

The unified way of presenting the object ¾ as shown in Figure 3 may relate the $thoughts_F$ of the object ¾ as a part–whole relation (*e.g.* as three parts of a whole divided into four equal parts) to the $thought_F$ of reconstructing one quantity (*e.g.* the whole) by repeated addition of the other (*e.g.* one part of the whole). The comparison between such quantities may give meaning and anticipate the object ¾ as a ratio of two quantities (*e.g.* the ratio of coloured and non-coloured parts of a whole). As such, the representation in Figure 3 can serve as a resource to extend to yet unknown mathematical relations. From an epistemological point of view, the concept of fraction has been developed in mathematics from the ratio of two quantities. The latter was conceived as an answer to a comparison problem of how many times one quantity fits into another.

For this transition resulting from an intentional shift in attentional focus, the representation in Figure 3 could trigger how many times the whole fits into the coloured parts by reversing the sequence of actions involved in dividing a whole into equal parts (or in the part–whole relation). This may be supported by an understanding that a whole is filled through repeated addition of n equal parts of size $1/n$. The latter conception, in turn, could trigger another transition that allows for improper fractions such as m/n, where *m* is greater than *n*.

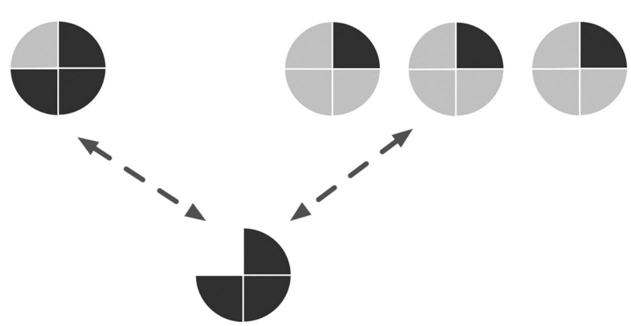

Figure 3. On the creation of new conceptions to give new meaning to an object.

Sense-making and meaning-making

In the previous section, we discussed sense-making in terms of extracting meaning and giving meaning. For example, in seeking to make sense of a given mathematical concept, a student may extract meaning by manipulating mathematical objects and reflecting on the actual $senses_F$ of those objects, while also giving meaning to $senses_F$ by activating and assigning conceptions. In doing so, students develop mathematical conceptions that fit the given mathematical concept. Seen in this way, sense-making is an act of comprehending a given mathematical concept by establishing a concept-to-conception direction of fit.

However, giving meaning can also involve processes in which students create new conceptions to give meaning to potential $senses_F$ of a mathematical concept. This insight assumes that students journey towards a new meaning of a mathematical concept that fits their conceptions, instead of merely activating conceptions that fit the mathematical concept in question. We suggest here that the journey towards new meaning—in such cases—is better referred to as meaning-making rather than sense-making, and in this light offer the following descriptions of sense-making and meaning-making:

> *Sense-making as an act of comprehension:* Sense-making refers to the act of comprehending a given mathematical concept by developing and activating mathematical conceptions on the part of the students that fit the mathematical concept in question.
>
> *Meaning-making as an act of creation:* Meaning-making refers to the act of creating a mathematical concept that best fits mathematical conceptions developed by the students. In this act of creation, students may attribute new $senses_F$ to mathematical objects, which thereby come into being and bring about new forms of meaning of a mathematical concept.

This distinction between sense-making and meaning-making brings to the fore central differences that have not yet been made explicit in the field (see Table 1).

Direction of fit In sense-making, students seek to comprehend a given mathematical concept by establishing a concept-to-conception direction of fit. That means, students develop mathematical conceptions that best reflect the mathematical concept in question. In meaning-making, on the other hand, students are not so much concerned with reflecting a given mathematical concept, but with creating a mathematical concept that best fits their mathematical conceptions, thus establishing a conception-to-concept direction of fit.

Ontological assumptions When students seek to make sense of a mathematical concept, they or others (*e.g.* teachers or researchers) treat the mathematical objects that fall under the concept as if they had a being. In such cases, students or

Table 1. The distinction between sense-making and meaning-making.

	Sense-making	**Meaning-making**
Underlying act	Act of comprehension	Act of creation
Direction of fit	Concept-to-conception	Conception-to-concept
Ontological assumptions	Mathematical concepts and objects have a being (meaning as a means to comprehension)	Mathematical concepts and objects come into being (meaning as a means of creation)
Epistemological assumptions	Students come to know a mathematical concept by grasping thoughts$_F$ (senses$_F$ are bearers of actual meaning of mathematical objects)	Students shape a mathematical concept by anticipating thoughts$_F$ (senses$_F$ are triggers for potential meaning of mathematical objects)
Orientation	Normativity	Intentionality

others regard the mathematical concept as something given. The meaning of a concept emerges (from Latin *emergere*, to become visible) in the process of making sense and is thus a means to comprehension. In contrast, in meaning-making, students create new conceptions by recontextualizing or transforming existing conceptions which are then directed towards objects that are yet to become. Put differently, students create conceptions to create future possibilities. Here, the meaning of a mathematical concept evolves (from Latin *evolvere,* to make more complex) by transforming conceptions and is thus a means of creation.

Epistemological assumptions In sense-making, students come to know a mathematical concept in apprehending the thoughts$_F$ that are expressed in the different contexts in which mathematical objects actualise. Their senses$_F$ are, in a way, bearers of actual meanings of the mathematical objects. That is, the apparent 'objectivity' of objects appears in such senses$_F$. In contrast, in meaning-making, students form a mathematical concept in anticipating thoughts$_F$ of mathematical objects that are yet to come into being. Senses$_F$ are then not so much bearers of actual meanings to be extracted from an object, but triggers for the creation of new, potential meanings to be given to an object.

Orientation Sense-making as an act of comprehension is oriented towards normativity—that is, it is about developing conceptions that fit the mathematical concept in question. In this way, students extract the meaning of the mathematical concept and give it a meaning that is as normative as the mathematical concept is assumed to be. In contrast, meaning-making as an act of creation is oriented towards intentionality—that is, students intend to create a mathematical concept that best fits their conceptions. Students express a yet-to-be-realised state of the mathematical concept, that is, they express a way in which the concept can be or should be. In this way, students direct their minds to future possibilities in which the mathematical object could be realised. That is, instead of creating conceptions that fit an apparently given mathematical concept, students create a concept that fits their conceptions. Students thus create new forms of meaning, suggesting that the meaning of a mathematical concept depends on its actual use and intentions, rather than being inherent to a concept.

Conclusion

In this article, we explored—from a semiotic-cognitive standpoint—the nature of sense-making in learning mathematics, especially in relation to two sense-making approaches: extracting meaning and giving meaning. Central to this exploration was the observation of an aspect of learning mathematics that is not well documented and explained; that individuals give meaning not only to mathematical objects that have a being, but also to objects that have yet to become. This observation led us to make a distinction between sense-making and meaning-making in learning mathematics, two terms that are often used interchangeably in the literature. We positioned sense-making as an act of comprehension and meaning-making as an act of creation, a distinction that is not made explicit in the field.

Furthermore, the distinction between sense-making as an act of comprehension and meaning-making as an act of creation highlighted important differences in terms of the ontology of a mathematical concept, the functions of senses$_F$ and the orientation towards normativity and intentionality.

These differences between sense-making and meaning-making in the way they are presented here, however, should not be understood as suggesting learning mathematics is either an act of comprehension or an act of creation. Learning mathematics, from our perspective, involves both acts of comprehension and acts of creation. Indeed, acts of comprehension and acts of creation can be in an interplay that keeps their directions of fit (concept-to-conception or conception-to-concept) and their orientations (normativity or intentionality) in a productive tension [3]. Instead of being concerned with either the concept-to-conception direction of fit or the conception-to-concept direction of fit, learning mathematics as both sense-making and meaning-making balances these opposing directions of fit. Our analysis of students' activation of conceptions in different contexts expressing multiple thoughts$_F$ (see Scheiner & Pinto, 2019) offers an interpretive account for the synergy that may emerge from the tensions between striving for normativity and being directed by intentionality.

These insights promote the recognition that is the interplay of acts of comprehension and acts of creation that constitutes the reciprocal relationship between mathematical concept and mathematical conception, with oppositions and tensions in the respective directions of fit being balanced or embraced rather than necessarily resolved. They also promote the recognition that mathematical concepts are not absolute or pure. Instead, as de Freitas and Palmer (2016) suggested, “we must study concepts for their indeterminacy or difference, for how they are alive and flexible” (p. 1204).

Notes

[1] In the philosophy of mathematics, there has been a long debate about whether mathematical concepts and objects exist in the mind and are created by the mind, or are outside the mind and independent of human thought (for an overview, see de Freitas, Sinclair & Coles, 2017). Frege, for instance, argued that they are independent of our sensations, intuition and imagination, and of all mental constructions—to counter psychologism with its central thesis that all human understanding can be analysed in terms of subjective mental processes. For Frege, concepts and objects are objective insofar as they can be grasped by more than one human mind. The focus here, however, is rather about the *being* of concepts and objects (as also with Frege), and we add here, about their *becoming* (which was not the case with Frege).

[2] How can students in the learning process not confuse mathematical objects with their semiotic representations when these can only be related to semiotic representations? The impossibility of direct access to mathematical objects, beyond any semiotic representation, makes the confusion almost inevitable. (Our translation)

[3] This view is based on the theoretical principle that deeper processes in learning mathematics can be identified when tensions, conflicts and paradoxes between fundamental, even opposing, theoretical perspectives are acknowledged and utilised (see Scheiner, 2020).

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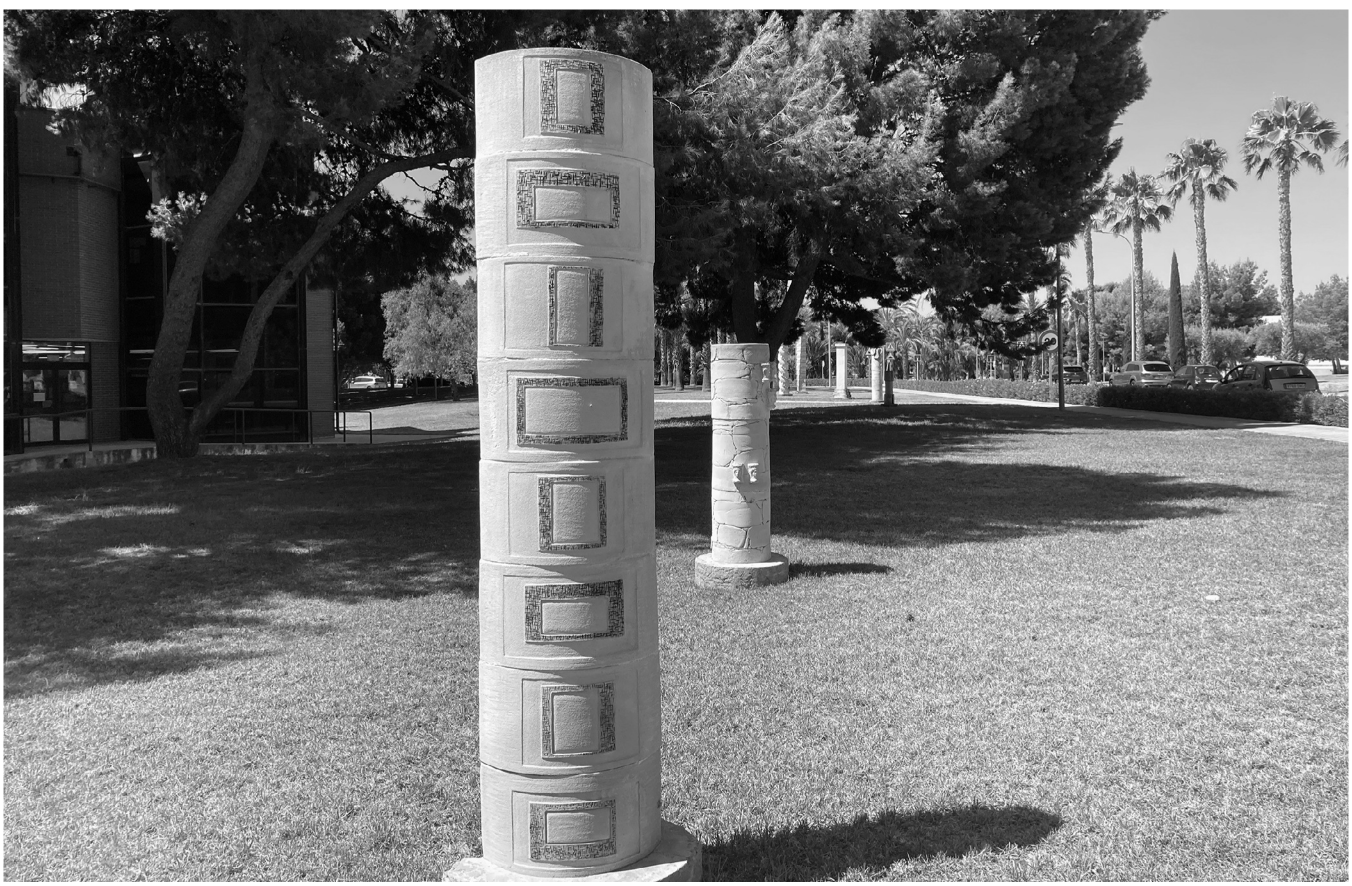

Sculptures on the San Vicente del Raspeig campus of Universitat d’Alacant