CONTEXTUALISATION IN UNIVERSITY LEVEL PROBLEM-BASED LEARNING

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At many universities, student-centred approaches such as project-based, problem-based and challenge-based learning are being implemented into curricula. The goal of this article is to conceptualise how university education in mathematics may take place when applying the principles of problem-based learning (PBL). PBL is a rather well-known and effective education method for K12 education in mathematics (*e.g.* Merritt, Lee, Rillero & Kinach, 2017) and is also applied in postsecondary medical and engineering education where students learn to apply their knowledge to real-life problems and situations. Mathematics is a powerful, unavoidable and useful tool for a myriad of disciplines in engineering, economics, science and medicine—but also a discipline in its own right. Mathematics is an abstract body of knowledge but PBL, as well as project-based learning, *etc.,* builds on authentic real-life problems with relevance outside of academia, where students show that they understand the context of a problem and are able to carry out an analysis of this problem. How does mathematics fit with PBL? What might PBL problems in mathematics look like in a higher education curriculum? To what extent and how can the abstract part of mathematics benefit from a PBL environment?

Our starting point is how PBL is interpreted and implemented at Aalborg University (AAU) in Denmark. However, we are dealing with a deeper problem, that of how to interpret PBL in abstract disciplines such as mathematics, philosophy, or disciplines with major theoretical areas such as theoretical physics, or disciplines such as language and literature with substantial aspects that are not just concerned with solving real problems or societal challenges. So, although we are at AAU, our problem is of a more general nature.

Aalborg University—an example of PBL

At AAU, PBL is carried out through projects in which a team of students collaborate. PBL is implemented at all semesters and in all study programmes with some variation. PBL projects in each semester are carried out by groups of up to eight students and take up half the semester, while the rest of the semester contains courses that may or may not support the projects. Each group has its own supervisor(s) that act(s) as facilitator and usually produce a report to document their work. All curricula are situated on six shared general AAU PBL principles: (1) the *problem* is the point of departure of the project, (2) *projects* are organised as group work, (3) projects are supported by *courses*, (4) there is *collaboration* with a supervisor, external partners and other groups, (5) the learning outcomes of the project can be transferred to similar problems relevant to the student's future profession, it is *exemplary* and (6) students are responsible for their own learning, which includes project planning and organising meetings with the supervisor. Students, thus, work in a problem-based fashion with projects that apply theory to solve or explain authentic real-world, exemplary problems from society. As such, problem- and project-based learning are applied simultaneously; but in this article, we focus on the nature of the problems.

Generally, students are supposed to find a so-called initiating ill-defined problem in society within a given thematic semester framework. This expectation of student autonomy is in line with Sierpinska's conceptualisation of an authentic problem as "not one that is given by the teacher" (1995, p. 3). Sierpinska further describes the need for students to develop into autonomous learners, hence they should choose own solution strategies. In PBL, students analyse the problem in its context. This problem analysis is an in-depth study of the initiating problem, seeking to answer the six W's (why, what, where, when, whom, how). They ask, for example, "For whom is this a problem?", "When is this a problem?", *etc.* leading to a more precise and narrow problem statement, which then guides the rest of the work (see, *e.g.,* Askehave, Prehn, Pedersen & Pedersen, 2015). For example, if the initiating problem is transport logistics, the resulting problem statement could be how to optimise the route of a truck picking up eggs at farms. The problem analysis would here be a study of various aspects of transport logistics, such as economic and environmental issues or transportation time, and would eventually lead to the much more concrete and precise problem statement.

This interpretation of PBL is one of many found in higher education institutions throughout the world (see *e.g.,* Andersen & Heilesen, 2015), but it is beyond the scope of this article to describe the full range of various PBL interpretations. Moreover, there are other curriculum models that are not PBL but strongly resemble PBL in that they are also organised around relevant authentic problems, such as inquiry-based learning (IL). According to Hmelo-Silver, Duncan and Chinn (2007), apart from their origins, "in practice PBL and IL environments are often indistinguishable" (p. 100). Thus, PBL at university level is implemented and interpreted in different ways and has similarities to other pedagogies. Note that we are not talking about the practices of problem-solving (Schoenfeld, 1985) that relate to the small tasks that students solve in mathematics courses. PBL in mathematics concerns teaching *through* problems.

Nevertheless, regardless of interpretation, the question remains of how basic sciences such as mathematics in which students should be able to both carry out theoretical abstract work and be able to apply the discipline, can benefit from curriculum models that emphasize authentic real-life problems. Dahl (2018) argues that mathematics would suffer in a PBL curriculum if reduced to being a tool for other sciences, or just applied to solve problems in society. This would lead to a lack of fundamental and deep understanding of mathematics and impede further development of mathematical theories. This article aims to provide a taxonomy for how PBL can be applied in degree programmes in mathematics—both pure and applied. PBL can be beneficial to the learning of both aspects; when students work within PBL, they develop skills such as collaboration, communication, critical thinking, creativity *etc*. that are also beneficial within an abstract discipline.

What is the problem in mathematics?

Mathematics is a central tool for solving many problems in engineering and the social sciences, but it is also a theoretical and abstract science. If students study mathematics, they therefore must meet *both* mathematical worlds, that is, applied as well as pure mathematics. But how do you find authentic problems with relevance outside academia within an abstract science? At AAU, this is often handled by focusing on specific applications of mathematics, such as population growth or optimal route planning, where abstract mathematics can be applied to a real-life problem. However, the assumption of student autonomy in PBL is at stake here: one cannot expect that first-year students are able to find a relevant problem that they can formulate precisely and solve with the mathematical competences obtained at the current semester without significant help from the supervisor. Thus, students at AAU are often handed more or less fully formulated project descriptions with suggestions for problems, often in the form of a so-called project catalogue. The effect of this is often that the process of formulating an initiating problem and performing a problem analysis are seen as irrelevant concepts imposed by the supervisor. It is therefore not always clear to the students what exactly a problem analysis is, and why it is necessary. Since they have been given a problem by their supervisor, it is by default considered to be relevant. So, they immediately start studying the relevant mathematical theory, apply it to a relevant practical case and make conclusions. In higher semesters, the challenge grows since the contents of the studies are becoming increasingly research-based. Our experience stems from AAU, however the issue of students being able, or not able, to find relevant problems to be solved, within a given semester and learning objective requirement, is something we would argue is general to all higher education programmes in mathematics.

How should one carry out research-based teaching at a university focused on PBL? Mathematical research papers are traditionally only implicitly problem-based; there is typically no specific problem statement in the form of a research question, and the context of the problem is often only briefly mentioned or not at all. The structure of a typical research publication in mathematics is quite different from that of a typical PBL report. What could be interpreted as a problem analysis is often a short description of what others have and have not done, and there is often no separate conclusion. How does one reconcile these norms of mathematical research with the expectations of PBL? In particular: What is a problem in mathematics, and how is its context analysed?

We wish to argue that research in mathematics has always been ‘problem-based’, even if the theoretical problem which a study sets out to answer is not explained in great detail. Usually, the readers are other researchers who are perfectly aware of the existence and relevance of the problem. This is also seen historically. One example is the development of the Dedekind cut which defined and proved the existence of the real numbers. This happened towards the end of the nineteenth century, but what might be considered strange here is that we saw the first proof of the existence of irrational numbers from the Pythagoreans during the sixth century BCE—which more than 2500 years earlier. Why was Dedekind’s cut necessary then? This means that a PBL-problem for students in mathematical analysis could be why Dedekind’s work was necessary, which theoretical problems did his definition and proof solve? Digging into this would turn the learning process ‘upside down’ and not just teach students about the Dedekind cut but make them understand its importance.

What is mathematics?

Though mathematics is often considered a natural science it is fundamentally different. Mathematics is not empirical in nature and does not apply the scientific method fundamental to (other) natural sciences. Instead, it is abstract and axiomatic-deductive in nature. Mathematical results are products of deductive reasoning using accepted rules of inference on basic postulates. This is the ‘pure’ aspect of mathematics: *theory-building*, including definitions and stating and proving theorems, as opposed to the ‘applied’ aspect of *model-building* and calculations using models. Applications in other disciplines have been essential for the development of mathematical theories but these resulting theories are ‘pure’. Dealing with authentic problems only is not an obvious way of learning both aspects of mathematics. We may learn to apply mathematics and to model something mathematically, but the abstract and deductive nature of mathematics is not learned through such endeavours. However, there is more to the theory-building aspect of mathematics than applying inference rules to basic postulates or already established results. Any working mathematician relies heavily on mathematical intuition when understanding and developing new theories, and in conjecturing and proving new theorems. Mathematical intuition may be even harder to define succinctly but it has to do with understanding the ideas and concepts of mathematics, and with having a feeling of how ‘things behave’ in mathematics. To attain this intuition, or “mathematical foresight” (Maciejewski & Barton, 2016), about a particular mathematical topic, one must gain experience with the topic. It is not difficult to imagine that mathematical models inspired by authentic problems could be a playground for such experience, but this still leaves out the question about the relationship between the problem and the mathematics: Should the problem guide what kind of mathematics one learns, or should the mathematics in the curriculum dictate the nature of the 'authentic' problem? Also, can problem-based learning be applied (interpreted?) in such a way that the student also gains competences in the theory-building aspect of mathematics? In order for PBL to be truly relevant to higher education curricula, the last question must be answered by a yes. Otherwise, students will miss out on central parts of mathematics. Dahl (2018) argues that although mathematics is a body of abstract knowledge, PBL is still a fitting curriculum for developing higher-level mathematics if students can experience the processes of (re)inventing mathematical knowledge, and if PBL also includes problems relevant to a theory-building research community.

External and internal contextualisation: three types of PBL problems

Some problem settings encountered in mathematics degree programmes involve the use of mathematical theories in an application domain outside mathematics itself—a concrete problem setting that needs to be analysed using mathematics or one within another academic discipline, say, physics, computer science or sociology. We shall refer to this is as *external contextualisation,* as it involves putting the mathematical theory into a context external to mathematics itself.

However, a particular characteristic of mathematical problems is that some of them are internal to the subject area, since an important aspect of any degree programme in mathematics is that of being able to understand how mathematical theories are built—their internal structure, including definitions, theorems and valid principles of reasoning as well as connections to other mathematical theories. This is what we call *internal contextualisation,* as it involves a context within mathematics.

The analysis of a problem in the realm of pure mathematics should contain this kind of internal contextualisation where the focus becomes that of the surrounding theory, its structure and development. This focus always involves students making choices regarding what to focus on. Understanding how and why these specific choices are made is part of the general understanding of the subject area that a mathematician must master.

We now describe a taxonomy, three archetypes of PBL problems, and give concrete examples based on existing degree programmes in mathematics at AAU as to where these problems appear or can appear. However, the examples are also applicable to other institutions and countries.

Type 1: the purely external context

The primary learning goal of a Type 1 problem is to develop competences in the area of model-building: formulating a mathematical model for analysing a problem setting external to mathematics and using this mathematical model to solve this problem. In this setting, the problem becomes a mathematical version of an external problem. The main driver of the work is the desire to solve the external problem and to find a good mathematical model of it. In relation to mathematical modelling, this is applied in much mathematics education including in non-PBL curricula.

An example of an initiating Type 1 problem is the construction of a statistical model for predicting house prices in future sales using a dataset containing the prices of houses sold during a given period together with other data on the houses (size in square metres, number of bedrooms, *etc.*). While the aim of creating a good model for predicting house prices will guide the work, the curriculum may dictate that the class of models to be used is that of linear regression models. This would typically be the case in the early stages of an undergraduate degree programme, where it is often important to ensure that students learn about a specific mathematical theory through the project work. Although the class of models is restricted to that of linear regression models, the students must still choose which part of the theory to apply to the data, for example by choosing methods for estimation, model checking and model selection, as well as choosing one or more particular linear regression models for the data.

The same data concerning house prices may also be used at a later semester, for example for a Master's thesis that usually allows substantial freedom of choice with regard to mathematical theory. In this version of a Type 1 problem, the focus may be to pick an appropriate statistical model for predicting house prices without being restricted to linear regression models. The student may choose to use other generalised linear models or to use mixed models—depending on which class of models best solves the problem.

Type 1 problems allow for many ways in which the external problem can direct the choice of mathematical theory. In particular, the first version illustrates an inherent dilemma that every degree programme including Type 1 problems must handle. On one hand, students must be able to choose appropriate tools for their mathematical model, as this is an important competence for an applied mathematician. On the other hand, the choice of model needs to be one involving a field of mathematics which is part of the curriculum, so the choice cannot be entirely open. Either the learning goals must be less restrictive, or one should ensure (by the choice of external problem) that a suitable field of mathematics will be chosen. As we noted earlier, a central characteristic of PBL is that students need some degree of autonomy in finding, selecting, and refining the problem. To some extent this contradicts the PBL principle of exemplarity: for professional mathematicians, the problem completely directs the choice of theory—we might term these Type 0 problems—but such problems would in our opinion not be seen in a degree programme in mathematics, which is always restrained by either local curricula or national requirements.

Type 2: an internal problem setting motivated by an external context

The primary learning goal of a Type 2 problem is one of being able to understand a mathematical theory, but an important part of the work is a concrete, external case that uses model-building as a means to understand theory-building. The construction and analysis of a model is used to obtain a more precise understanding of and (re)construction of aspects of the theory used in the model.

An example of a Type 2 problem is a project about the Travelling Salesman Problem. The outset would be that of optimising the route for a salesman given a set of customers with specified addresses. Students would then in a problem analysis discuss the relevant mathematical model for such a problem, leading to the study of graph theory. Methods for optimising graphs could be developed, and various toy problems solved. The original problem would also be solved but only be of secondary importance. It could have been solved simply by guessing, by asking an experienced salesman or by inspection (such a problem might in a concrete setup have an 'obvious' solution that is clearly optimal), but this would not be satisfactory in terms of curricular guidelines. The actual goal is to learn graph theory and optimisation methods related to graphs, but the Travelling Salesman Problem is used as a motivation and to illustrate the abstract concepts from graph theory.

The bulk of the work would follow a typical mathematical structure with definitions, theorems and proofs within the topics of graph theory. The actual problem is motivational and illustrative but its details are less important. It could have concerned a paperboy and readers, a nurse visiting patients, or a biologist with field observation stations. Hence, although students have a real-world problem at the outset and use real data, the project is a mathematics project. Type 2 often involves some aspects of an analysis of why and how the mathematics works, the choice of mathematical theory and the simplifications needed. The actual data is used as a means to learn the subject of graph theory and how to apply it in concrete settings. Naturally, the students would be motivated by the actual problem, but the direction of the project, from formulation of an initiating problem to the actual problem formulation, would be heavily *guided* in a direction that would make it possible for the students to both give *an* answer to the initiating problem and make it possible for them to learn mathematical theory. It is this guiding of the problem analysis that distinguishes Type 2 problems from the others. However, it still leaves plenty of room for students to target their work. The competences in understanding a mathematical theory are therefore at the forefront, while the modelling aspect is of secondary importance.

We would also like to note that a similar initiating problem asked of engineering students might lead them in a completely different direction. Even if engineering students ended up with a similar problem formulation, their actual report and work would look completely different and most likely not contain mathematical proofs.

Type 3: a purely internal context

Type 3 has as its primary learning goal to develop competences in the areas of understanding connections between and limitations of mathematical theories. In this setting, the problem must be formulated as a question concerning how mathematical theories or aspects of a single mathematical theory are related or how and why limitations arise within a theory. This could for instance be the example mentioned above about the Dedekind cut. The main driver of the work is the desire to obtain a precise and rational (re)construction of (parts of) a mathematical theory. The student report must do more than simply present the *context of justification*, that is, to present the theory in a structured fashion (the usual definition-theorem-proof(-example(-counterexample)) structure); it must also present the aspect of the internal context that is the *context of discovery.* This includes an account of motivating counterexamples and reflections on the definitions and conjectures that were rejected or revised in the development of the theory under consideration, and while the account of the context of justification focuses on the product (a mathematical theory), the account of the context of discovery must include reflections on the process that led to the product and the structure of the internal context that constitutes the problem setting. The context of discovery resembles the proof-and-refutation process presented in Lakatos (1976), in which pupils in a fictional classroom debate the proof of Euler's formula $V - E + F = 2$ for regular polyhedra and the process shows how definitions and proofs developed through different types of counterexamples. The real history of Euler's formula is seen in footnotes, and the discussion among the pupils is to some extent a replica of the history.

An example of Type 3 is one of students working with the theoretical problem of the existence and uniqueness of solutions to ordinary differential equations (ODEs) and algorithms for finding solutions. The resulting report presents the theory of ODEs including theorems about existence and uniqueness of solutions and algorithms for finding solutions. The outcome is a precise, structured presentation of part of the theory of ODEs and its limitations. This includes describing and discussing the main obstacles that have arisen in the development of the topic and the attempts to deal with them that contributed to the development of the mathematical theory. This is the context of discovery. The benefit of approaching the topic of ODEs via Type 3 is that ODEs appearing in real-world problems are typically very restricted in type by the given problem, and the specific type of ODEs can be solved concretely in a given setup (either numerically or analytically). This is completely unproblematic from the point of view of the real-world problem but will not give much insight into the general theory.

In practice, a Type 3 project on ODEs could work as follows: The students are presented with an introduction document explaining some of the basics on ODEs including concrete examples of different types of ODEs and known results on ODEs, finishing with listing some different possible directions to explore. The task of the students is then to choose a direction and make a coherent, mathematical presentation of the chosen topic(s) including all the necessary mathematics needed to fill the gap between what the students already know and what is needed for a presentation of the topic, including its context of discovery. In other words, students at the same level with no additional knowledge of ODEs should be able to learn the selected topic(s) within the theory of ODEs by reading the report as if it were a tailor-made textbook. The list of possible directions in the introduction document is of course not exhaustive, but it is important to realise that the scope of any direction chosen for the project must be limited with the aid of a supervisor, since the students, being non-experts, will have no chance of knowing whether a topic can realistically be covered within the time frame of a single semester. Some almost identically looking mathematical questions may vary greatly in how extensive their answers are, even ranging from one being easily solved and another still being an open problem to the mathematical community.

In a sense, a Type 3 project is one big problem analysis that ends with stating a problem (and often a result) internal to mathematics. The challenge within the PBL perspective is that problems in PBL are required to be authentic and relevant outside of academia. Here it can be argued that the fundamental sciences are of course relevant outside of academia, even if some of the problems that they consider may not appear relevant to the rest of society in the short term. Fundamental science is part of a food chain that will often have consequences appreciated by those outside the subject area. A Type 3 project may contain reflections on how the theory could be applied outside the internal context, but such reflections are not essential in these projects. The importance of Type 3 projects is that they contribute to a deeper understanding of the internal structure of a fundamental scientific theory and how such a theory arises.

Discussion

In this article we have discussed the role of PBL in mathematics degree programmes based on our experiences from AAU and on the dual nature of mathematics: its 'applied' aspect of model-building and analysis and its 'pure' aspect of theory-building. We identify three archetypes of problems that can be used to understand the role of PBL in degree programmes in mathematics.

Central to this taxonomy is the notion of contextualisation. Making internal contextualisation explicit is important for projects in pure mathematics. We believe that for PBL to be truly fitting to mathematics and to bring forward the fact that mathematics is indeed a problem-based science, this exploration and understanding of the context of discovery ought to be a goal of such projects. Otherwise, we cannot truly answer yes to the question stated above: if problem-based learning can be carried out in such a way that the student also gains competences in the theory-building side of mathematics. Furthermore, we also find that these three archetypes are not only relevant to curricula defined around the principles of PBL but can provide useful inspiration for any other degree programmes in higher education mathematics, as these will most often comprise both pure and applied elements.

The overall question here, of course, is if it is better for students if they learn to work in a problem-based manner with all three types of problems—particularly those of Type 3. One might argue that since research mathematicians do not make explicit problem statements, then why should the students? As stated at the beginning, PBL is considered to be an effective method in K12 teaching and also develop competences and skills that are highly appreciated by future employers. But even more important, the extent to which the students are consciously aware of what they are actually doing in Type 3 projects makes it possible for them to understand and reflect more deeply on the mathematics than they can when their work is only implicitly problem-based.

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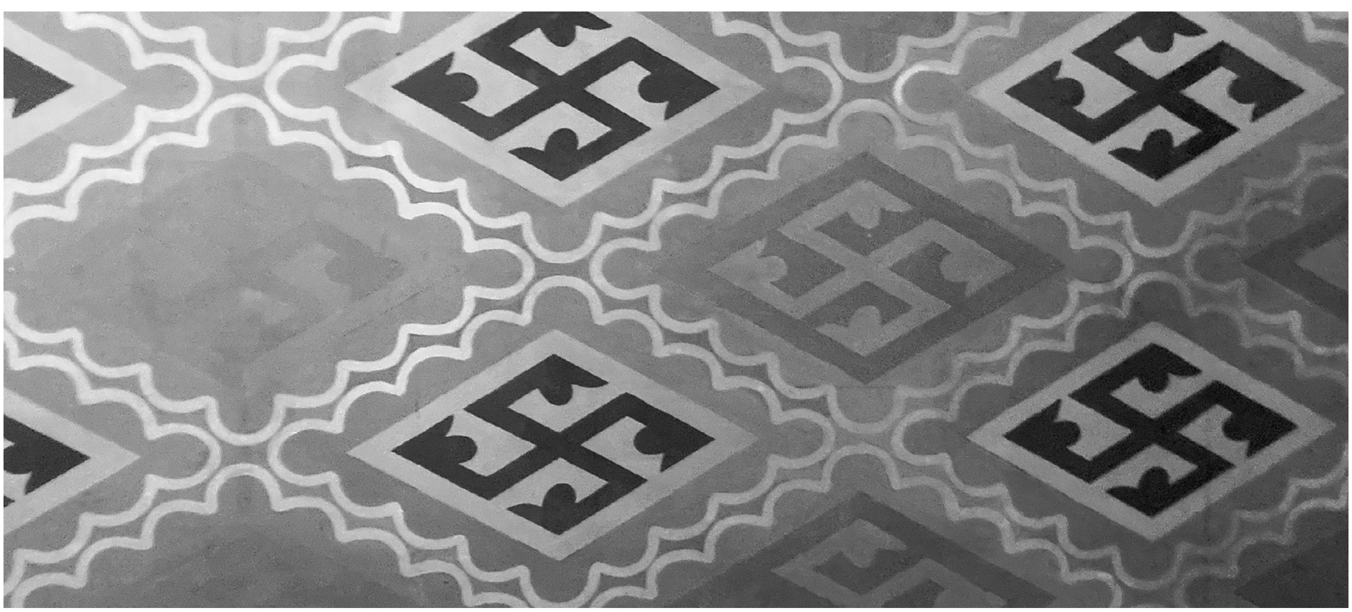

Wall painting from the Matthias Church, Budapest.