Why is Intuition so Important to Mathematicians but Missing from Mathematics Education?

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In my recent *Communication* (Burton, 1999), I outlined some features of a interview-based study of mathematician's views and research practices. I have been exploring how research mathematicians come to know the mathematics they develop, with a view to substantiating different learning strategies which consequently might inform practices when working with less sophisticated learners. This article follows up in more detail on the brief comments I made there about what my interviewees had to say about the topic of 'intuition'.

Intuition in coming to know mathematics

According to several writers, intuitions and insights, for many mathematicians, are of great importance For instance, Reuben Hersh (1998) considers that "intuition is an essential part of mathematics" (p. 61) and provides a list of meanings and uses given to this word:

> 1 Intuitive is the opposite of rigorous ...
>
> 2 Intuitive means visual ...
>
> 3 Intuitive means plausible, or convincing in the absence of proof ...
>
> 4. Intuitive means incomplete ...
>
> 5 Intuitive means based on a physical model or on some special examples This is close to 'heuristic'...
>
> 6. Intuitive means holistic or integrative as opposed to detailed or analytic ...
>
> In all these usages intuition is vague It changes from one usage to another One author takes pride in avoiding the 'merely' intuitive [...] Another takes pride in emphasizing the intuitive. (pp 61-62)

While his conviction about the importance of intuition to the process of doing mathematics is clear, these meanings describe the results of applying intuition. But, as Hersh points out, these uses are much less specific about why intuition is so important, what it is, where it comes from, how reliable it is and how it feels. All of these points were addressed by the participants in the interviews I conducted and provide an organising frame for this article.

Like some of the mathematicians interviewed, Ian Stewart (1995) refers not to intuition but to 'instinct'. However, again, he is more concerned with the mathematical outcome than the human process:

> The entrepreneur's instinct is to exploit the natural world. The engineer's instinct is to change it. The scientist's instinct is to try to understand it - to work out what's really going on. The mathematician's instinct is to structure that process of understanding by seeking generalities that cut across the obvious subdivisions. (p. 13)

Raymond Wilder (1984, pp. 41-45) distinguishes three roles for intuition: in the evolution of concepts, in research and in teaching, separating the 'knowledge component' from the 'intuitive component' of mathematics learning. He points out that:

> without intuition, there is no creativity in mathematics [but] the intuitive component is dependent for its growth on the knowledge component. (p. 43)

From the ways in which these terms are used, both in the literature and by the participants in my study, it seems to me that what they represent could be placed on the boundaries between thinking and feeling and that there is little doubt that intuition, or insight, should be understood in a subjective way. Indeed, the use of 'intuition' or 'insight' to describe, albeit loosely, part of the ways in which mathematicians come to know allows me to challenge the positivist control of the epistemological agenda, opening it towards a view of coming to know which is, I believe, both more coherent, from an intellectual and personal perspective, and more holistic. This then permits a further challenge to the historical view of mathematics which has positioned it as pre-eminently cognitively driven and, 'consequently', male. By juxtaposing the two sexist stereotypes, cognitive, '*ergo* male', and intuitive, '*ergo* female', in my view, can help to explain some of the pervasive labelling of mathematics as a male domain.

My interests lie in exploring how my participants regarded intuition and insight, whether these terms featured in their discourse about coming to know mathematics and, if so, in what ways. Were there any differences between the men and the women? Their heterogeneity of views around intuition, insight or instinct was more notable than any similarities Jack Smith and Kedmon Hungwe (1998) have also drawn attention to the same phenomenon.

Is intuition important?

> If we look at mathematical practice, the intuitive is everywhere. (Hersh, 1998, p. 61)

Although many of the participants in my research agreed with this, for example:

> one must use intuition all the time;
>
> the ability to pick up that kind of connection in mathematics is mathematical intuition and is a central feature;
>
> I don't think you would ever start anything without intuition;

it was certainly not the position taken by all. A female mathematician said:

> I don't think intuition plays a part;

and a male observed:

> I think intuition is a sort of word which is loaded, fashionable for people to disown or adopt

Similar contradictions were reported by Efraim Fischbein (1987):

> According to Poincaré, no genuine creative activity is possible in science and in mathematics without intuition, while for Hahn (1956) intuition is mainly a source of misconceptions and should be eliminated from a serious scientific endeavor (p 4)

Some of the mathematicians were concerned about the meaning of the word, although not necessarily about the phenomenon:

> I don't think I would call it intuition;

and others, with possible reflections on its sex-relatedness, linked intuition to "something mystical" with one saying that:

> intuition is not a word I would use.

The participants were responding to a request to talk about their work on a particular mathematics problem of their choosing and to tell me about how they thought about the problem, where and when other than at their desks, how they knew when they had arrived at an important point, whether or not they were (often) mistaken. If, throughout this section of the interview, no reference was made explicitly to intuition or insight or implicitly to anything that seemed to fit that description, I would note that they had not referred to intuition and ask if there was a reason why not. If their reference had been implicit and/or descriptive, I asked: "Would you call that intuition?"

Fewer than five out of the seventy mathematicians took a completely negative stance - there is no such thing as intuition in mathematics - but it is hard to be exact about the number since the conversations were exploratory in nature. However, it might be the case that a participant who asserted that intuition was irrelevant to mathematics still recognised something that others might call intuitive but labelled it quite differently and that I did not discover this during our conversation Nonetheless, the overwhelming majority (83%) recognised something important which might be called intuition, insight or, less frequently, instinct at play when they were coming to know mathematics.

What is it?

Vera John-Steiner (1997) offered:

> Metaphors, analogies, and the delineation of new patterns is the stuff of insights. (p. 203)

How did the mathematicians describe what one explained as "seeing the best way forward - that's intuition"?

> Mainly you get the Ahas! when you wake up in the morning Think of the problem before I go to sleep, wake up with a solution which hasn't yet been incorrect. More often than not I'm right
>
> In my field, there are lots of places where you have intuitions about connections between different areas of mathematics and relativity and so on. But you cannot base a theory on intuition; you must show the connections rigorously.

Ten out of the seventy favoured the term 'insight', but could not always say what it was:

> insight is seeing a connection;
>
> if the light switches on when I look at a problem, I have had an insight;
>
> having an insight is having a feel for how things connect together I think it is probably possible to manage without it in that, by sheer bulk of discovery and trial and error, the same picture might emerge but that flogging to death involves a lot more work, a lot more blind alleys So the insight is a short cut.

For others, 'insight' and 'intuition' were distinct:

> You depend upon developing insights - that is what was involved in having a summary view which is a linking of insights I would use the word 'intuition'. I think the two are connected. You are always thinking in a not-straightforward, deductive way You are always looking for some hint from within that this might be an interesting thing to do, or you should check up on this, that sort of thing
>
> Insight is a sort of flash; intuition is more murky and definitely develops because if you have a false one, that intuition is modified.

A few ruminated on how they might describe this process which they recognised, what one called "I wonder if's":

> ponderings, maybe ...;
>
> an enhanced understanding that perhaps comes to you suddenly;
>
> maybe it is understanding;
>
> pattern matching is the best way of describing it.

Although most of the participants had a strong sense of making use of something that they called 'intuition', 'insight' or 'instinct', there was no agreement about whether these terms represented different, distinctive states and, if so, how these states might be recognised. Any agreement to be found rested in words like 'connections', 'ways forward', 'understanding', 'a sense of the possible or even likely'. There was agreement that intuition was not always enough, of course, but that it was reinforcing to feel that you saw a direction. One respondent offered a caution:

> intuitions can come close to prejudice, which can be a very blinding thing.

If you have it, from where does it come?

> Intuition isn't direct perception of something external It's the effect in the mind/brain of manipulating concrete objects - at a later stage, of making marks on paper, and still later, manipulating mental images. This experience leaves a trace, an effect, in your mind/brain (Hersh, 1998, p 65)

For one participant, relying upon knowledge and experience explained why it was *not* intuitive:

> On the basis of what I know, and what I have done before, I feel pretty sure that this kind of method will work on this kind of problem. It is not intuition, because it comes from experience.

Those mathematicians I interviewed who recognised intuition, insight or instinct as a factor in their working lives explained its genesis in the following terms:

> My intuitions are based on my knowledge and my experience. The more I have, the more robust my intuitions are likely to be
>
> The insight into what is going on, whether it is intuition that gives you that I don't know, but that is what you are getting to It is not an Aha! It is a case of gradually fitting the bits together until you have got it right It is small steps in different directions It is incremental.
>
> Looked at from the point of view of anybody else, it looks like intuition. But, for me, it was more concrete, a translation from the abstract to something much more concrete.

Knowledge and experience were, then, the main factors which the participants used to account for their success in achieving intuitions or insights, although there were a few mathematicians who believed that you either had it or you did not - which was presumably a genetic explanation. As you will see below, some of the female mathematicians recognised intuition in others but not in themselves. None of the mathematicians talked about working on their intuitions to improve their frequency or reliability, and those who did make reference to their students were, in the main, disparaging in this regard.

Given that the only explanations for intuitions were both products of learning (i.e. knowledge and experience), I was struck by this lack of sensitivity to their own, and their students', development If growing familiarity with mathematics and developing knowledge, together with experience of solving problems, are the basis for mathematical intuition on which research success depends, there is a serious pedagogical challenge here. Why it is that in conventional classrooms these two, knowledge and experience, are seen as separate and the role of intuition ignored?

Intuitions and feelings

The mathematician quoted above who described an insight as a "short cut" went on to point out that:

> Often you can't trust it!

Indeed, trusting your intuitions or insights, or not, featured quite prominently in our conversations, but more so for the females than the males. The following quotations are all from female mathematicians:

> You might have an intuitive feeling, but it would be based on your knowledge and your experience. But I think you need a firmer basis for doing something than intuition but perhaps I am too cautious
>
> Sometimes intuition is wrong You think something is going to work one way and you battle on and it doesn't work When I do that, trying to see how it really is and trying to explore, I think I am trying to get an insight into what is going on. One must use intuition all the time. My intuition would be something that says I think the argument might work this way.
>
> I don't think I am intuitive enough. If you know something is a good problem, or not, you are intuitive - have a good sense of whether something can be done. I don't trust my own intuition [:] There are intuitive mathematicians.

Mathematical activity, for many of these mathematicians, was driven by curiosity and the resultant pleasure when something was resolved or strong feelings of frustration when an intuition proved untrustworthy. Satisfaction was associated with finding a pattern, making a connection, eradicating a difficulty - coming to know Such activity was described in very emotive terms:

> When I think I know, I feel quite euphoric So I go out and enjoy the happiness without going back and thinking about whether it was right or not but enjoy the happiness.
>
> There are lots of different ways in which one understands something The most gratifying is that sudden wave of insight in which suddenly something all becomes clear That is sadly rather rare.

For me, the indivisibility of the cognitive and the affective is made clear in these quotations. Far from understanding being something which is *only* driven by knowledge, there is both a *need* to know and an associated *pleasure* in knowing which is its own reward.

> Most mathematicians do mathematics for the very good reason that they like and enjoy doing it. (Mordell, 1984, p 157)

The intuitive or insightful leap is integral to this, despite the experience that it is not always reliable and despite a possible disconnection from proof or the disciplinary requirement that the leap will have to be justified by rigorous argument.

Under these circumstances, it would seem to me a natural development that a mathematician would want better to comprehend her or his own processes of achieving an intuition or insight and would see it as a necessary part of their pedagogical responsibilities to encourage students to recognise the process and become more sensitive to it and to using it. Indeed, Wilder (1984) advocated that:

> the new curricula should try to turn teaching of the knowledge component into a process whereby the student's intuition is actually used and developed further in acquiring new knowledge (p. 43)

But far from reflection being integral to their processes, many of the mathematicians would agree with their colleague who said:

> I don't think about it, I am just curious and I do it.

And, intuition is not uncontested. For these mathematicians, intuition embodies the subjective, which explains why reliability is so important, since having *good* intuitions presumably means that you are successful at your work, write lots of papers and achieve recognition. The subjective is thereby transformed into the production of 'objective' mathematics. Hersh (1998) speaks of intuition succumbing to the drive for mathematical 'infallibility'. He puts it succinctly, although leaving the reader with another query - what is a 'faculty'?:

> *the study of mental objects with reproducible properties is called mathematics.* Intuition is the faculty by which we consider or examine these internal, mental objects. (p. 66, original emphasis)

The apparent need for mathematics to be infallible and objective requires the removal of the evidence of the very human beings who created and worked with the mathematical objects, missing:

> entirely the stumbling human process that created those results in the first place. (Smith and Hungwe, 1998, p. 46)

The results, I believe, can be seen in the writing style in which mathematics is offered to the world (see Morgan, 1998 and Burton and Morgan, forthcoming), the competitive and cut-throat academic atmosphere of which some of the mathematicians spoke, indeed the observed masculinisation of the discipline which is, in part, explicable by this process of objectivisation. And yet, mostly, they agreed on the centrality of something intuitive or insightful. Perhaps they felt it is best to leave this as uninterrogated as possible in case something which might undermine the power of the positivist paradigm is displayed. Alternatively, if intuition is central to creating mathematics, then teachers have an obligation to create the conditions for its nurturance.

Mathematics education: the nurturing of intuition and insight?

> One of the things I find about students, undergraduates in particular, is that they seem to have very little intuition. They are dependent upon being spoon-fed. The ability to look at a problem from different angles is crucial. They tend to look from one angle and you cannot see a way through. If you look at it from different angles, mysteriously perhaps something dawns on you and you find a way through.

Not all the mathematicians were as dismissive of the students nor of their own responsibility for assisting learning. One, a female, put it this way:

> Ph.D. students don't have the depth of experience to have such realisations. I think you have to help them by posing the questions and leading them. You model the process. I think maybe you explain it as you are going along, you try and lead them along the way so that they experience looking for the answers.

John-Steiner (1997) appealed for:

> instruction in diverse problem-solving strategies just as there is instruction in varied laboratory techniques during the apprenticeship of young scientists; however, this is seldom the case. Instead young men and women are exposed primarily to what Medawar has called "the art of the soluble" (pp. 182-183)

Responding to her question of "whether it was possible to help students to rely upon intuition as part of their preparation for becoming physicists", John Howarth said:

> Intuitive solution of problems is important. Essentially it is finding the answer to a problem before you have solved it. Students are tempted to believe that physical intuition is something that you either have or don't have. We certainly all have different talents, but the process can certainly be encouraged - that's one of the things that teaching is about. Teachers can encourage the talent by example and by describing their own approach to problem solving. They can also take the time to explore the student's process with him or her. (pp. 183-184)

I recall many years ago running an M.Ed. class on mathematical problem solving with David Singmaster in which we engaged overtly in the *ponderings*, *what ifs*, *it seems to me thats*, *it feels as thoughs* with respect to tackling problems for which we did not have a known solution. In the words of one of my research participants, we were modelling a process of exploring our intuitions out loud not to become better solvers of problems, but to become more reflective about the process and, perhaps consequently, better learners and teachers. We were drawing on the features above, highlighted by the mathematicians, to emphasise that producing mathematics, at any level, is dependent upon *learning through enquiry*, that is, making connections, building understanding through knowledge *and* experience, developing a sense of the possible or even likely.

As Smith and Hungwe (1998) pointed out:

> If guessing and the resulting cycle of inquiry does not become visible to students, they are left with only public mathematics – the carefully crafted propositions and polished arguments they see in their texts. (p. 46)

However, what is particularly noticeable is that, except for the important book by Fischbein (1987), the literature of mathematics education does not address intuition as a focus of interest in classrooms. And yet, Jon MacKernan (1996) has asked:

> What's so awful about using intuition or using inductive arguments? [...] without them we would have virtually no mathematics *at all*; for, until the last few centuries, mathematics was advanced almost solely by intuition, inductive observation, and arguments designed to compel belief, not by laboured proofs, and certainly not through proofs of the ghastliness required by today's academic journals. (p. 16, original emphasis)

The research mathematicians provided clear directions towards a mathematical pedagogy that could be as engaging, exciting and rewarding for learners, as it was for them in their own practices. These directions were firstly towards valuing and nurturing intuitions and also recognising the importance of making connections or links in the building of mathematical meaning. Far from believing that you simply do, or do not, 'have' the intuitive process, it seems to me that some of the participants were reporting that they had learned, over time, to recognise and trust their intuitions and that, with such acknowledgement, the quality of this process was improved. Otherwise, how can their reliance on experience and knowledge as its source be explained?

Both experience and knowledge are educable (even though they are embedded in process and affect, as well as cognition). We can learn to be more generous, more sensitive, more tolerant if the learning process has those objectives alongside others. In the case of mathematics, there is a persistent attempt to erase the subjective and affective in favour of:

> mathematics that is as dry as dust, as exciting as a telephone book. (Davis and Hersh, 1983, p. 169)

A similar argument is often offered for teaching the basics first: without the basics, learners are ill-equipped to delve into the non-basics. Those who exploit this argument fail to define what is 'basic' and for whom and, in any case, are often the first to complain about students who have failed in their acquisition of them.

My argument, on the contrary, is that acquiring facility in making mathematical arguments requires some knowledge but that the objects are there to serve the function of making a convincing argument and that learning to do that is a more important part of learning mathematics and, in any case, cannot be done in the absence of content. (For research evidence to support this, see Boaler, 1997.)

Two mathematicians, quoted above, said:

> Having an insight is having a feel for how things connect together.

> You depend upon developing insights – that is what was involved in having a summary view which is a linking of insights

However, those who have studied students learning mathematics, whether at school or at university, find that they tend to believe that remembering fragments is more important than thinking. In a recent article, Jo Boaler, Dylan Wiliam and Margaret Brown (1998) report that 68% of the students in the 'most-able' class, that is presumably and possibly the future mathematicians, prioritised memory over thought, confirming Boaler's (1997) finding from a previous study that 64% of the top achieving pupils believed that remembering was more important than thinking. Likewise, Kathryn Crawford and her colleagues (1994) found that 82% of the university mathematics undergraduates whom they studied used a reproductive strategy (learning by rote memorisation or by doing examples).

To use Wilder's classification (see above), the students were the products of a system which is knowledge- rather than intuition-based. Wilder went on to quote mathematician Edward Moise:

> mathematics is capable of being learned as an activity, and that knowledge which is acquired in this way has a power which is out of all proportion to its quantity. (p. 44)

But activity makes particular demands upon the pedagogical situation to which student control, enquiry, argument and justification are all necessary contributors. In turn, these learning approaches rely upon a sense of community with expectations of exploration and creativity towards coherent and connected learning in which intuition plays a central role. It is clear that there are mathematicians who recognise, along with many mathematics educators, the power of this learning paradigm to achieve where traditional teaching has failed. Unfortunately, their influence does not seem to be having an impact upon the teaching of either school or university students.

Conclusion

Intuition, insight or instinct was seen by most of the seventy mathematicians whom I interviewed as a necessary component for developing knowing. Yet none of them offered any comments on whether, and how, they themselves had had their intuitions nurtured as part of their learning process. While many referred to the centrality of their intuitions to how they came to know within the research process, some were very dismissive of their students', and others of their own, power to bring intuition into play. More importantly, some considered intuition as something you either had or did not have.

For me, the importance of applying the epistemological model (Burton, 1995) that this research set out to compare with mathematicians' descriptions of their processes of coming to know is that it draws attention to the complexity of these processes and the variability of positions adopted by those working in the field. A mathematician, such as the one who said:

> I don't think I ever have heard mathematicians talk about intuition;

can see, from the above, that that position is *not* adopted by many, indeed most, of their colleagues and might think in more detail about what these differences represent and imply. Further, those who do believe that intuition is an important feature can be challenged by this work to answer the following question: if it is so important, how could you set out to nurture it in your students?

These practising research mathematicians speak with such enthusiasm and joy of their practices. However, with the notable exception of the work of Fischbein, accounts of the deliberate nurturing of intuition and insight is absent from the mathematics education literature, even from process-based research, and, despite the claim for the centrality of it to mathematical work, it is equally absent from practices with students. I would like to encourage mathematicians, indeed anyone who has responsibility for the learning of mathematics, to open mathematical activity to include the subjectivity of intuitions, to model their own intuitive processes, to create the conditions in which learners are encouraged to value and explore their own and their colleagues' intuitions and the means that they use to gather them. This seems to me to be a necessary step which provides a justification for, but is prior to, the search for convincing argument and, ultimately, proof.

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