On Equivalent and Non-Equivalent Definitions: Part 1

GREISY WINICKI-LANDMAN, ROZA LEIKIN

> "When I use a word," Humpty Dumpty said, in rather a scornful tone, "it means just what I choose it to mean - neither more nor less." (Lewis Carroll, *Through the Looking-Glass*)

This article deals with the concept of mathematical definition as well as with definitions of different mathematical concepts. It focuses on the possibility of defining a mathematical concept in several ways based on logical relationships between different mathematical statements related to the concept. The choice of one of these statements as definitional plays a crucial role in the learning process.

In this first part of the article, we discuss mathematical and didactic characteristics of a definition and consider implications of specific changes in the defining conditions of a concept in general. We then exemplify this analysis by means of the concept of a *straight line tangent to a curve*.

On definitions

One of the more important questions in mathematics education is: "What is the best way to introduce a new mathematical concept to a learner?" However, the question which has to be asked beforehand is: "How should the concept be defined?"

According to Poincaré (1909/1952):

> For the philosopher or the scientist, it [a good definition] is a definition which applies to all the objects to be defined, and applies only to them; it is that which satisfies the rules of logic. But in education it is not that; it is one that can be understood by the pupils (p. 117)

Mathematical characteristics of a definition

Many mathematicians (among them: Khinchin, 1968; Solow, 1984; Vinner, 1991) refer to some logical principles that must be met when defining any mathematical concept. Among others, they point out the following.

1. Defining is giving a name. The name of the new concept is presented in the statement used as a definition and appears only once in this statement.
2. For defining the new concept, only previously defined concepts may be used.
3. A definition establishes necessary and sufficient conditions for the concept.
4. The set of conditions should be minimal.
5. A definition is arbitrary.

For every mathematical concept there are a variety of statements which constitute necessary conditions - the concept *properties* - or sufficient conditions, that is *indications* of the concept. Some of the statements establish both necessary and sufficient conditions and thus define the concept.

By examining logical connections between the statements related to the concept, an equivalence class of defining statements may be established. Every statement that belongs to this class may *be chosen arbitrarily as a definition*, while the others become theorems that constitute necessary and sufficient conditions of the concept.

As mentioned above, each of the statements from the established equivalence class may be chosen as the definition: *mathematically, there is no difference among them*. As Poincaré pointed out, the requirement that the statement used as a definition fulfills these logical principles is not a sufficient condition for using this definition while teaching mathematics. There are *didactic considerations* that may influence the introduction of a new mathematical concept to the learner.

Didactic considerations for a definition

The 'optimum' definition for a particular mathematical concept is one that has to be both mathematically correct and didactically suitable. While considering didactic aspects of a definition, the following questions may guide the analysis.

1. What is the most appropriate statement for the definition of the concept among those that belong to the equivalence class mentioned above?
2. What are factors that may influence this selection?
3. How does this selection influence the teaching sequence and how is that sequence influenced by it?

Each didactic decision made while teaching is based on the discussants' beliefs and attitudes toward the nature of mathematics and the nature of the learning process. In the discussion of the didactic suitability of a definition, this article relies on the following conceptions.

- According to a constructivist approach to learning, the learner builds knowledge on the basis of known concepts (familiar for the learner, and ones with which the learner can operate) and by building connections between them. Therefore, in a didactically suitable definition of a new concept, only concepts already known to the learner should be included

- The level of development of the learner is defined by his/her current knowledge and by the knowledge which is in their 'zone of proximal development' (ZPD), which determines the dynamics of the development of the learner. To enable the learner's intellectual development, the teacher may relate only to concepts that belong to the ZPD of the learner (Vygotsky, 1982)
- The statement selected as definition must be based, as much as possible, on the intuitions of the learner (Fischbein, 1987).
- It is desirable that the definition will be elegant (Vinner, 1991).

Every definition determines a set of objects that fulfill the conditions of the definition. These objects are said to *exemplify* the concept. If two different statements define two concepts and their corresponding sets of exemplifying objects are non-disjoint sets, then the following relationships between the definitions are possible: the definitions can be equivalent; one of the definitions can follow from the other one; the definitions can 'compete'.

This can be illustrated as follows:

Let statement I be a definition of a concept a.

Let statement II be a definition of a concept b.

Let A be the set of the objects exemplifying the concept a

Let B be the set of the objects exemplifying the concept b

Let α be the set of defining conditions of concept a.

Let β be the set of defining conditions of concept b

Equivalent definitions

If the sets of objects are equal, then the statements are equivalent definitions of the same concept

Statement I $\Leftrightarrow$ Statement II iff $A = B$

$A = B$ iff $\alpha = \beta$.

For example, a parallelogram may be defined as a quadrilateral having two pairs of parallel sides or as a quadrilateral with bisecting diagonals. These two statements are equivalent definitions of 'parallelogram'. The learning sequence for these statements, i.e. acceptance of one of them as a definition and the other as a theorem, depends on didactic considerations only.

Consequent definitions

If one of the sets of objects is a proper subset of the other, then the corresponding sets of defining conditions established by these statements are also connected by a relation of inclusion and one of the definitions follows from the other (see Figure 1).

Statement I $\Rightarrow$ Statement II iff $A \subset B$

$A \subset B$ iff $\alpha \supset \beta$.

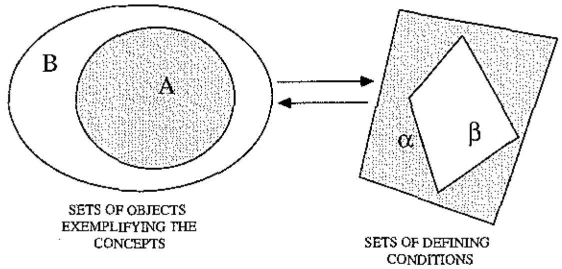

Figure 1 The sets of objects that represent two different concepts and their defining conditions

If A is a subset of B, then there exist certain conditions that may be added to the conditions established by Definition II in order to produce a statement equivalent to Definition I. In this case, Definition I is called 'stronger' than Definition II. The relationship between the sets of objects representing certain concepts is *inverse* to the relationship between the sets of defining conditions of these concepts. Figure 1 presents this inversion. The inverse inclusion of the sets of objects exemplifying the concepts and concepts' defining conditions may cause difficulties in understanding the connections among the concepts. The presented consi-derations are followed by two different learning sequences

- Learning according the first sequence may begin from the concept b exemplified by the objects from set B, and advance to the concept a exemplified by objects from the set A by adding defining conditions (see Figure 1). For example, while teaching special quadrilaterals, one may start from the concept of parallelogram (b) and move toward the concept of square (a). In this case, part of the conditions that were sufficient for b become insufficient for a
- The other possibility is to start from the concept a (exemplified by objects from the set A) and to move forward to the concept b (exemplified by objects from the set B), by removing some of the defining conditions (see Figure 1). For example, while teaching special quadrilaterals, one may start from the concept of square (a) and advance to the concept of parallelogram (b). On the one hand, in this case part of the conditions that were not sufficient for a become sufficient for b. On the other hand, some of the necessary conditions for a become unnecessary for b

These two learning sequences may be applied in practice for teaching many mathematical concepts included in the secondary mathematical curriculum. Usually, only one of the sequences is chosen and this choice may influence the construction of the learners' knowledge and consequently affect their difficulties involved in the learning process of this particular concept

Competing definitions

If the sets of objects intersect, and the intersection set is a proper subset of each of them, then we will say that the statements are *competing* definitions for two different mathematical concepts. For example, the following definitions compete.

(I) An isosceles trapezoid is a quadrilateral having an axis of symmetry which is not a diagonal.

(II) A parallelogram is quadrilateral having central symmetry

On the one hand, according to these definitions, a rectangle is both an isosceles trapezoid and a parallelogram On the other, an isosceles trapezoid which is not a rectangle does not have central symmetry, and a parallelogram which is not a rectangle does not have an axis of symmetry which is not a diagonal. Thus, these two definitions can be considered as competing.

On the definition of the straight-line tangent

According to the Israeli secondary school mathematical curriculum, students encounter the concept of straight-line tangent (tangent) during the last four years of mathematics in secondary school It appears in different contexts with different connotations. In Euclidean geometry, a tangent to a circle is defined and problems based on its necessary and sufficient conditions are solved. In algebra, a tangent to a parabola is used when investigating quadratic functions. In analytic geometry, problems are solved connected with tangents to conic sections. In calculus, tangents to the graphs of various functions are considered.

Thus, across the secondary school curriculum, definitions of tangents are connected with different curves: however, the definition of tangent always involves a pair of concepts - a straight line and a curve

General definition of a straight line tangent to a curve

Courant and Robbins (1941) provide a commonly accepted definition of a tangent as follows:

> [Let $P(x, y)$ and $P_1(x_1, y_1)$ be two points on a curve.] The straight line joining P to P_1 we call t_1; it is a secant of the curve, which approximates to the tangent at P when P_1 is near P. [...] Now if we let x_1 approach x, then P_1 will move along the curve toward P and the secant t_1 will approach as a limiting position, the *tangent t* to the curve at P. (p 416; *italics in original*)

In light of the previous discussion of different didactic approaches (see Figure 1), the concept of tangent may be examined from two different perspectives Specifically, there are two approaches to studying the concept of a straight line tangent to a curve

According to the first approach, studying the concept begins from studying the tangent to a general curve, followed by studying the tangent to the graph of a function, and concluding by studying the tangent to the graph of a specific function (to a specific curve) As mentioned above, the objects that exemplify the concept of tangent consist of two geometric figures: one constant - a straight line - and the other variable - a curve. Reduction of particular types of curves (set *B* in Figure 1) leads to the enlargement of the properties of the straight line tangent to a curve

According to the second approach, one begins by studying the tangents to a specific curve (for example, the circle) and then moves toward the general definition of the tangent to a curve (from set *A* to set *B*, in Figure 1)

As it was pointed out before, both of these two approaches may be applied in learning the concept However, it seems that in the case of the concept *straight line tangent to a curve* only one sequence is appropriate: from set *A* to set *B*. This restriction may follow from the fact that at the stage when students are cognitively prepared to learn the concept of *tangent to a circle*, they are not ready to learn the general concept of *tangent to a curve*.

The relationship between changing a curve and properties of a tangent to the curve

The traditional definition of a tangent to a circle is frequently perceived as a general definition of tangent:

> A straight line tangent to a curve (circle) is a straight line that has only one common point with the curve (circle) (D1).

Shifting from a circle to another curve - a parabola, for instance - many students may 'have no difficulty' defining a tangent to parabola by making a 'small change': the word 'parabola' substitutes for the word 'circle'; so, their definition would be:

> A tangent to a curve (parabola) is a straight line that has only one common point with the curve (parabola) (D2)

The problem is that the set of objects exemplifying this definition of tangent includes lines parallel to the axis of symmetry of a parabola that intersect it in one point (see Figure 2)

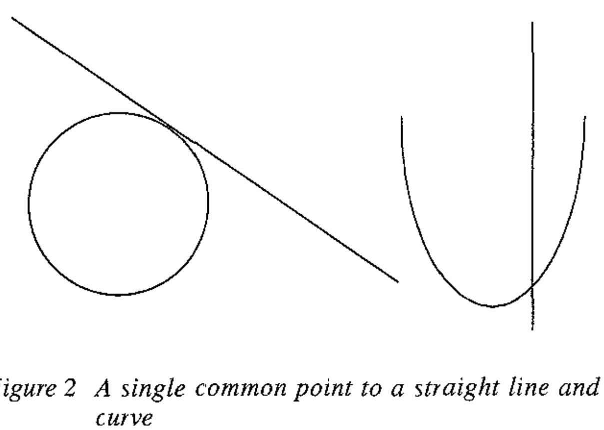

Figure 2 A single common point to a straight line and a curve

One of the definitions of tangent to a parabola presented in the ninth-grade curriculum is the following:

A tangent to a curve (parabola) is a straight line that leaves the curve (parabola) in the same half-plane defined by the straight line, and has only one common point with it (D3).

It can be seen that this statement includes the defining condition of the definition D1 but also includes an additional condition: *to be in the same half-plane defined by the straight line* Statement D3 defines also a tangent to a circle, but its collection of defining conditions is not minimal: therefore, this statement is not chosen as a definition of a tangent to a circle.

Additionally, note that both circle and parabola are convex figures; thus, the condition *to have only one common point*, which is common to all straight lines tangent to convex curves, was a necessary condition both for a circle and for a parabola The fact that a circle is a closed figure made this condition also sufficient (i.e. defining). A parabola is an open curve, thus the condition *to have only one common point* was not sufficient and the condition to *be in the same half-plane defined by the straight line* had to be added to define a straight line tangent to a parabola (and more generally to any convex, open curve) Figure 3 exemplifies this general relationship between straight lines tangent to convex figures

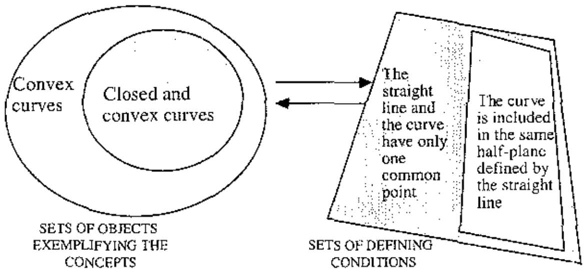

Figure 3 The sets of the objects that represent two different concepts and their defining conditions

Application to the learning of the concept of a straight line tangent to a curve

The examples presented in the previous sub-section demonstrate the difficulty associated with the inversion between the inclusion of the set of objects exemplifying two concepts and the inclusion of the sets of their defining conditions of the same concepts

Different curves provide their tangents with different properties (necessary conditions) Included among the properties of the tangent considered in secondary school mathematics are the following.

1 A tangent has only one common point with the curve.

2. There is a point at a common distance from all the tangents

3. All of the curve is on one side of a tangent line

4. The slope of the tangent line - if it exists - is equal to the value of the derivative of the curve's equation at the point of tangency.

5 A tangent is the graph of the linear approximation of the curve's equation (if it has one) at the point of tangency.

6. A tangent is a limiting position of secant lines passing through the point of tangency.

Table 1 shows a list of curves and some properties of the tangent lines to them As presented in the table, only one property is common to all these curves. Some of the properties are common only to some curves

<table>
<tr><th>The curve
properties</th><th>a circle</th><th>parabola</th><th>ellipse</th><th>hyperbola</th><th>polynomial functions</th><th>square root function</th><th>rational function</th><th>trigonometric functions</th><th>logarithmic function</th><th>general</th></tr>
<tr><td>1 A tangent has only one common point with the curve</td><td>|</td><td>|</td><td>|</td><td>|</td><td></td><td>|</td><td></td><td></td><td>|</td><td></td></tr>
<tr><td>2 There is a point at an equal distance from all the tangents</td><td>|</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>3. All the curve is on one side of a tangent.</td><td>|</td><td>|</td><td>|</td><td></td><td></td><td>|</td><td></td><td></td><td>|</td><td></td></tr>
<tr><td>4 The slope of the tangent is equal to the value of the derivative of the curve's equation at the tangency point.</td><td></td><td></td><td></td><td></td><td>|</td><td></td><td>|</td><td>|</td><td>|</td><td></td></tr>
<tr><td>5 A tangent is the graph of the linear approximation of the curve's equation at the tangency point</td><td></td><td></td><td></td><td></td><td>|</td><td></td><td>|</td><td>|</td><td>|</td><td></td></tr>
<tr><td>6. A tangent is a limit position of secant lines passing through the tangency point.</td><td>|</td><td>|</td><td>|</td><td>|</td><td>|</td><td>|</td><td>|</td><td>|</td><td>|</td><td>|</td></tr>
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Table 1 Straight-line tangents to different curves and their properties

In many cases, teachers and students use one of the properties 1, 3, 4, 5 (see above) as a general definition of a tangent to a curve The fact that the majority of the curves considered in the secondary school mathematics have tangents with these properties has several implications:

- these properties may be perceived as defining conditions for tangents to all curves;
- properties (that is, necessary conditions of a concept) may be perceived as sufficient conditions;
- the learning sequence may induce incorrect transfer of the properties from one curve to another This may result from lack of attention to the fact that changes in the set of the concept's properties imply changes in the set of objects exemplifying the concept (see Figures 1 and 3)

Although, property 1 (*A tangent has only one common point with the curve*) does not hold for every curve, for the majority of the curves considered in high-school mathematics this property is fulfilled in a small neighborhood of the point of tangency Thus, this condition may be incorrectly considered as a defining condition for a tangent to a curve.

As most of the functions learned in the framework of school calculus are concave or convex in the whole domain, then property 3 (*All of the curve is on one side of a tangent line*) may be erroneously conceived as a general property for all cases. (Not enough emphasis is placed on cases like $f(x) = x^3$ at $x = 0$ - see Vinner, 1991.)

A particular issue in school mathematics calculus is the connection between the following three concepts: straight-line tangent, linear approximation and derivative Property 4 (*The slope of the tangent line – if it exists – is equal to the value of the derivative of the curve's equation at the point of tangency*) and property 5 (*A tangent is the graph of the linear approximation of the curve's equation at the point of tangency*) are related to these connections. The existence of the derivative at a point is equivalent to the existence of a linear approximation at this point. On the other hand, a tangent may exist at a certain point without the existence of a derivative or a linear approximation (e.g. $f(x) = \sqrt{|x|}$ at $x = 0$).

Property 6 (*A tangent is a limiting position of secant lines passing through the point of tangency*) is the only property among those presented above that provides a necessary and sufficient condition for a tangent to every curve. If the limit does not exist at a certain point of a curve, then there is no tangent to this curve at that point. Therefore, this condition may be used as a definition of the concept of *a straight line tangent to a curve*.

This section focused on the spiral development of the concept of tangent in the learning process. In our opinion, it is very important to involve students in discussions focusing on connections between properties of tangents to different curved lines and on differences between general and specific properties of tangents to different curved lines As in all the other topics of the curriculum, it is important that learners encounter a variety of examples and counter-examples which emphasize specificity and generality of the cases and that are related to necessity and sufficiency of different conditions of the concept of a tangent.

In every one of the stages of learning the concept of tangent, the teacher must choose the definition which is most didactically suitable It is possible that 'equivalent definitions' may all be suitable at a certain stage. In this case, it is important to provide the learners with various statements equivalent to the chosen definition, in order to develop their mathematical thinking and to enable them to build logical structures that support their mathematical knowledge

Concluding remarks

We suggest that it is important to develop mathematics teachers' awareness of the ideas discussed in the article These include:

1. Every definition provides both necessary and sufficient conditions for the concept. Awareness of this idea enables using the definition for mathematical proofs in two directions:

   (a) in order to prove the properties of a concept, if the conditions of the definition denote additional characteristics;

   (b) in order to prove the existence of sufficient conditions for the concept, if the conditions of the definition follow from these statements.

2. Each of the statements from the equivalence class of one of the definitions of a concept may be chosen as a definition. This choice is arbitrary and relies on both mathematical and didactic considerations.

3. The relationship between the sets of the objects is inverse to the relation between the sets of their defining properties Awareness of this issue may prevent incorrect transfer of conditions of one concept to a more general one.

Teachers' professional development should include activities focusing on the issue of equivalent and non-equivalent definitions. In the second part of this article, we discuss ways in which teachers' sensitivity to both the role and attributes of mathematical definitions in the learning process may be developed.

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