Teacher Intervention in Small-Group Work

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Introduction

Peer interaction in small groups has become an important area of research in mathematics education (see, for example, Kieran and Dreyfus, 1998; Healy *et al.*, 1995; Cobb, 1995; Hoyles *et al.*, 1991; Hoyles, 1985; Webb, 1989). Such interaction is seen to provide support for the construction of mathematical meaning by pupils, since it allows more time and space for pupil talk and activity. However, as with any teaching strategy, working with small groups is not unproblematic. Pupils might struggle to communicate with and hence learn from each other, and might reinforce rather than challenge mathematical misconceptions.

The teacher's role in dealing with these difficulties is central to the success of small-group work. But how do teachers intervene and mediate appropriately in pupils' interactions? Curriculum reform in a number of countries encourages teachers to intervene less rather than more, and when they do intervene, to do so in ways which do not impose their own understanding on pupils. It is hoped that this will enable pupils to trust and develop their own meanings. Attempts to change the amount and the nature of teacher intervention raise new questions and challenges for teachers. How long should they wait before challenging and trying to correct misconceptions? How long should they avoid attempts to reorient discussions in more mathematically fruitful directions? How do teachers manage to hear the contributions of all pupils in the group and enable pupils to communicate more effectively with each other? And how do teachers intervene appropriately in pupils' discussions in order both to affirm pupils' current thinking and to enable the development of new mathematical knowledge?

This article provides an analysis of the interactions between one teacher and a small group of three pupils. This teacher was successful, I will argue, in developing different intervention strategies in her interactions with the group. In so doing, she struggled to find ways to help her pupils move beyond their partial conceptions. This teacher's successes and difficulties may be illuminating for other teachers, teacher-educators and researchers, and may help us to think more clearly and carefully about the kinds of questions and choices that teachers face when mediating pupils' mathematical meaning in small groups.

The study took place in a Grade 9 classroom at a well-resourced school in Johannesburg, South Africa. The class consisted of five groups, with between three and five pupils in a group. The pupils worked on a number of tasks relating to the area and perimeter of shapes on a geoboard. Each group had one geoboard, a set of elastic bands and dotted grid paper to work with. I developed the tasks in consultation with the Grade 9 teachers at the school, all of whom used the tasks with their classes. The tasks were designed to fit into their teaching programme on area and to enable me to observe pupil interaction and activity in small groups. The teacher who is the focus of this article added to my task requirements that the pupils should present their findings to the whole class at the end of the week.

This article is concerned with one of these tasks, where the pupils were required to calculate the areas of a number of shapes on the geoboard (see Figure 1). I have chosen to discuss this task because the pupils generated a particularly interesting response to it, an innovative method for calculating area which contains both correct and incorrect mathematics.

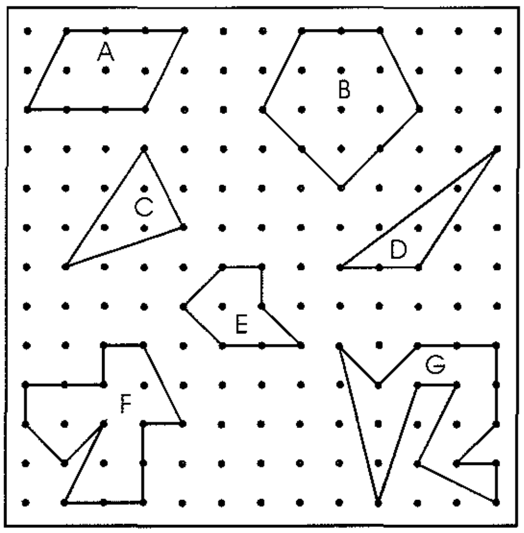
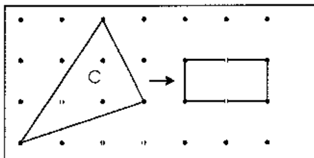


Figure 1

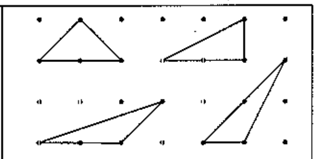
A method to calculate area

The pupils recognised that they could calculate the areas of rectangles and squares on the geoboard relatively easily by adding unit squares. However, it was difficult for them to do this with triangles and other, more complex shapes. For these shapes, the pupils generated the following procedure: if you transform a shape on the geoboard into a different shape, as long as the number of hooks in the two shapes remains equal (counting the hooks inside the shape as well as on the perimeter), then the area of the new shape will be equal to that of the original. This procedure enabled the pupils to transform most shapes to rectangles and so they were able to calculate an area (see Figure 2a) This conjecture of the pupils is clearly problematic in that it claims that when transforming a shape on the geoboard, conservation of the area of the shape depends only on conserving the number of hooks. On the other hand, the pupils' method does work in many cases, particularly with triangles which have four hooks (see Figure 2b), which can be transformed into unit squares and do have area 1



(a)

Triangle C with six hooks is transformed into a rectangle with six hooks and so is considered to have an area of 2



(b)

Triangles with four hooks with areas of 1

Figure 2

In generating and developing their method, the pupils engaged in mathematical processes such as developing and justifying hypotheses, generalising, specialising, checking theories against evidence and trying to resolve contradictions. I argue that the pupils' mathematical activity and the teacher's interventions reflect aspects of a Lakatosian dialogue (Lakatos, 1976) The pupils encountered counter-examples to their method, arising both in their own work and from the teacher. They dealt with these in ways which resonate with Lakatos' 'monster-adjusting', 'exception-barring' and 'surrendered' positions The teacher tried to encourage aspects of 'lemma incorporation': she encouraged the pupils to try to find out which aspects of their method were sound and could be maintained and which were in need of refinement or change.

However, the ways in which the teacher intervened - primarily with indirect challenges and probing and trying to find counter-examples and refutations - although resonating with a Lakatosian approach, did not enable the pupils to move beyond their incorrect notion of the conservation of area under this particular invariant-preserving set of transformations. So, I argue, one reason for the teacher's difficulties lay precisely in the area of her success, namely her different kinds of engagements and interventions with the pupils' work.

As we shall see, the teacher was not always able to intervene appropriately because she did not have access to what happened in the group when she was not there. The pupils' varied approaches to counter-examples reflected their different approaches to the task and different understandings of their method. These understandings were never expressed to the teacher, so although the teacher interacted substantially with this group, she did not get to hear the range of understandings that became evident through my analysis

Thus, she was unable to work with some of the meanings which influenced interactions in the group and could not mediate between the different positions in the group in order to help the pupils to develop a more accurate concept of area. I am not suggesting that this results from any deficit on the part of the teacher. On the contrary, I have argued elsewhere (Brodie, 1994) that there were substantial dynamics and power relations at play among the pupils which contributed to the silencing of one pupil, and the dominance of another, in interactions with the teacher

Consequently, the process of conjecture and refutation that took place within this group and with the teacher was not sufficient to refute a mathematically incorrect notion of area, nor to enable the development of more accurate and appropriate mathematical knowledge among the pupils In successfully maintaining aspects of a Lakatosian dialogue with her pupils, this teacher was faced with new challenges in trying to help them learn mathematics

Conjectures and refutations

Lakatos' (1976) imaginary dialogue between a mathematics teacher and a group of (university) students trying to prove the Euler formula for polyhedra elaborates the processes of discovery and development of mathematical ideas Lakatos rejected the Euclidean programme which asserts that mathematics progresses only through rigorous, logical deduction from fixed definitions and self-evident axioms. Rather, Lakatos argued, mathematics grows through a series of conjectures, attempts at proof and refutations Partial proofs suggest counter-examples, which purportedly refute either the conjecture or parts of the proof or both, and therefore can stimulate improvements to the proof and/or the conjecture.

Lakatos' project was to establish a case for mathematics as a quasi-empirical activity, to elaborate the logic of quasi-empirical mathematical discovery and to argue that mathematical discovery is intimately connected with justification through proof For him, a proof did not establish a theorem with absolute certainty: rather, it was an attempt at justification, a partial proof, which could be improved upon through critique via attempts at its refutation

According to Ernest (1991), a quasi-empirical perspective argues that:

> Mathematics is a dialogue between people tackling mathematical problems Mathematicians are fallible and their products, including concepts and proofs, can never be considered final or perfect, but may require renegotiation as standards of rigour change, or as new challenges or meanings emerge (p. 35)

Although links between philosophies of mathematics and educational practice are indirect, a Lakatosian view of mathematical discovery would support current attempts to bring mathematical investigation, problem solving, discussion and dialogue into the curriculum These pedagogies aim to give school pupils opportunities to engage in processes of discovery, to make conjectures and to attempt to justify and refute them quasi-empirically. Lakatos' dialogue is offered as a rational reconstruction of the history of the development of certain methods of doing mathematics and his student participants take on various positions and views expressed by mathematicians in the past.

Similar dialogues are unlikely to occur in school classrooms, not least because of the mathematical sophistication of his 'students'. However, aspects of some of these positions may be evident in the ways in which pupils and teachers deal with counter-examples in their attempts to develop and justify mathematical ideas. In my study, the pupils came across counter-examples as they attempted to develop and articulate their method, and in dealing with these they exhibited what Lakatos terms 'monster-adjusting', 'exception-barring' and 'surrender'.

'Surrender' occurs when the conjecture is rejected because it has been refuted by a counter-example. This reflects an approach which holds that theorems and proofs are either true or false, and the aim of mathematics is to verify true theorems. For Lakatos, attempts to deal with various counter-examples were preferable to surrender.

A 'monster' is a counter-example which is a "pathological" (p. 14) example of the mathematical object, and does not comply with intuitive notions of the object. Because of some of the monster's bizarre features, the conjecture does not hold for it. Two approaches can be taken with monsters: 'monster-barring', which redefines the object to eliminate monsters and thus maintains the truth of the conjecture; and 'monster-adjusting', in which the monster is interpreted differently in such a way that the conjecture does apply to it.

An 'exception' is a less "aggressive" (p. 24) term than a 'monster', and is a counter-example which suggests a domain for which the conjecture and the proof do not hold. The approach of "exception-barring" redefines the domain of validity of the original conjecture in such a way that it specifically excludes the exception. In this way, it improves the conjecture, rather than redefining the focal object in more restrictive ways, as monster-barring does. Lakatos' teacher points out that there are two kinds of exception-barring. *Ad hoc* exception-barring redefines the domain of the conjecture in an *ad hoc* way each time a new exception is found. *Strategic* exception-barring systematically analyses the proof and the exceptions, establishes which part of the proof does not hold for the exception, and then incorporates that aspect into the original conjecture. This is close to his method of 'lemma incorporation' and leads to proof-generated statements of theorems as improvements on naive conjectures.

Lakatos' teacher rejects the methods of monster-barring, monster-adjustment and exception-barring because these allow the mathematician to retreat to a safe domain, rather than systematically examine the objects and their definitions, the conjecture, the proof and the relations among them. He argues for the method of 'lemma incorporation' where the counter-examples force an analysis of the proof to identify which lemmas are refuted by the counter-examples. These lemmas are then incorporated into the conjecture, thus improving it rather than restricting it. For Lakatos, this is the *logic* of mathematical discovery: systematic analyses of purported proofs and putative refutations which improve conjectures and proofs. Thus, the role of proof in mathematical discovery comprises a systematic means for improving conjectures, and dialogue between mathematicians, or between teachers and their students, provides a means to enable this process.

Lakatos' teacher might represent the kind of teacher envisaged by curriculum reforms. He hears and recognises all his students' positions and he encourages debate among them. He intervenes occasionally in his students' discussions, and does so strategically. (It is a small class, though, and there is no indication of small groups.) He challenges his students' positions in order to encourage and develop their thinking. At the same time, he is not shy to put his own perspectives into the discussion and to argue his points in the same way that his students do. He tries to convince them of his perspective, but they are not easy to convince. As Lakatos wryly points out, these imaginary students are very advanced.

What happens in a real school classroom when students and their teacher engage in conjecturing and refuting? How does a real teacher work with conjectures and counter-examples in order to help develop her pupils' mathematical knowledge?

Monster-adjusting

The episode below shows the teacher interacting with the pupils about their method. They had explained the 'four-hook' part of their method to her, that triangles with four hooks have an area of 1 because they can be transformed into a square. She listened carefully to the pupils' ideas and her response to them was that it was an interesting conjecture, worth attempting to test and justify. She offered a counter-example to their conjecture, a triangle with four hooks on the perimeter, but whose area is not equal to 1 (see Figure 3). [1]

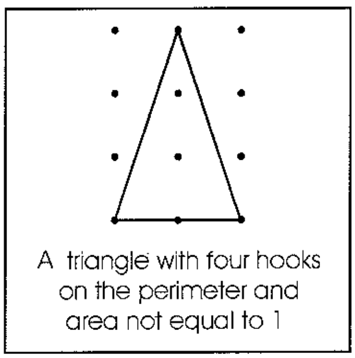


Figure 3

Teacher:	But this one here's got four hooks
Thabo:	No, one, two, three, four, five, six.
Teacher:	Oh I see, {you count
Thabo:	{You count there.
Teacher:	Aah, even the ones on the inside
Nandipha:	Yes.
Teacher:	O.K. You must write it down, cause that'll

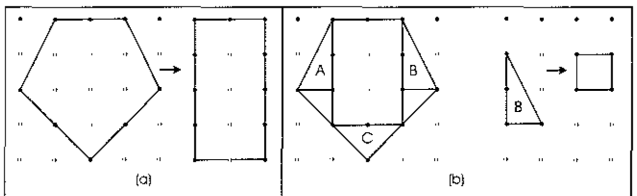
be nice for your presentation, but I'm gonna look for a, for a counter-example I dunno if there is one, oh I see because the minute you counting your inside ones, but I would've liked to say, that's got four hooks, but no it's got five, you see, and it doesn't have an area of, you saying four hooks has an area of one, you must write down your conjecture, O.K. [*walks away*]

The teacher's interventions were in the spirit of a Lakatosian process of mathematical discovery through conjectures and refutations. She encouraged the pupils a number of times to write down and test the conjecture. She also offered a counter-example which Thabo and Nandipha refuted by reinterpreting the 'monster'. They argued that the hooks inside the figure must be counted as well as the hooks on the perimeter. The teacher accepted this 'monster-adjustment' in this instance, but was still concerned about the validity of the conjecture. She left the group at that point, but promised to continue looking for counter-examples.

The teacher's interventions at this point were entirely non-impositional. She did not tell the pupils that she thought there might be a problem with their conjecture. Rather, she provided a challenge through a counter-example, and encouraged the pupils to do the same. In expressing doubt, she perhaps hoped that the pupils might take on that doubt and continue to test and improve their conjecture. Their response was to 'monster-adjust' the specific counter-example and to remain with the method as a useful way to calculate area.

Exception-barring

The pupils encountered a second counter-example when working on a pentagon (shape B in Figure 1). The pentagon has fifteen hooks and can be transformed into a rectangle with fifteen hooks, three hooks on one side and five on the other. This rectangle has an area of 8 and therefore, according to the pupils' conjecture, the pentagon also has an area of 8 (Figure 4a). The pupils also worked out the area of the pentagon by dividing it into a rectangle of area 6 and five triangles, three of area 1 and two of area $\frac{1}{2}$ (Figure 4b), and obtained the answer 10. The pupils used the 'four-hook' part of their method to work out the areas of the triangles within the pentagon (Figure 4b)



(a) The pentagon with fifteen hooks and area of 10 is transformed into a rectangle with fifteen hooks and area of 8

(b) The pentagon is divided into a rectangle and triangles and the areas of triangles A, B and C are calculated to be 1 because they have four hooks and can be transformed into a square

Figure 4

Thabo's first response to this exception was despondency that their method did not work. But he quickly decided to 'exception-bar' pentagons from the domain of the conjecture and to claim that the conjecture worked for triangles, but not other shapes. He was convinced that it worked for triangles because it worked for the triangles within the pentagon.

Thabo: *Ja*, this is it works with triangles, that's one of ours the presentation, it works, it works

Nandipha: But it didn't work with other shapes

Thabo: Yes, but it works with triangles.

[*silence*]

Thabo: Yes, yes, Nandipha, it works with triangles, it works, it works, look at this, look at this *ne*, all the shapes here we can divide them into, into, into, into, into, we can somehow cut them into uh, squares and triangles *ne*, let's, let's continue and see, let's see *wabona* [can you see]?

So Thabo continued to believe that their conjecture was a useful one because it worked for triangles and because all other shapes could be divided into rectangles and triangles (a very mathematical reductive approach). By means of monster-adjustment and *ad hoc* exception-barring, he managed to hold on to the original conjecture relatively intact. I have argued elsewhere (Brodie, 1994, 1996), that the two girls in the group, Lerato and Nandipha, did not seem to be as convinced about the method as he was, but kept silent at important points in the interaction so that his view dominated. In fact, they had different understandings of the task and the method which their silence kept hidden from the group and from the teacher. Their interactions around the pentagon exception reflects their different understandings.

Thabo's understanding of a method was that it should be a formal articulation of the ways in which they had found the area. He believed that the area of the pentagon was ten, but that they needed to articulate how they had worked it out.

Thabo: It's te[n], eh *ja*, we, we, know Nandipha *hore* [that] it's ten, but then we find, how we found it

For Thabo, the contradiction between the two answers, 10, which he believed was correct, and 8 which was given by the method, was a real one. Given his understanding of a method as a formal articulation of informal calculations, he needed to find a way to resolve this contradiction

Lerato believed that a method is developed in advance of using it. For her, a method should be a formula which they could apply to get the answer. She did not accept the answer 10, because she was unsure of the method which Thabo and Nandipha had used to work it out. Moreover, she did not believe that they could work out the answer because they did not have yet have a formula

Lerato: No, *wena* [you] you just assuming *hore* [that] the answer is ten

Thabo: Eh, it is ten but then we can find a method.

Lerato: It's not.

Thabo: O K

Nandipha: What is it, Lerato?

Lerato: I don't know, we all don't ...

Lerato did not accept either of the two answers, 8 or 10, and therefore did not see a contradiction. She therefore did not need to bar the exception. Her approach to the task at this point was to find a formula before she calculated the areas, rather than to test a conjectured method on the areas that they had already worked out. Thus, she was not engaged in the process of conjecture and refutation in the same way that Thabo was.

Nandipha did accept the answer 10, but did not see the need for articulating their method She was happy to merely calculate the answers

Nandipha: Ish, mmm, *maara waitse wena kobotsitse* [but you know I told you]? One, two, three, four, five, six, seven, eight, nine, it's ten

Thabo: It's te[n], eh *ja*, we, we, know Nandipha *hore* [that] it's ten, but then we find, how we found it, in eh, in eh.

Nandipha: Ag, just cut it up man it's supposed to be .

Nandipha accepted the answer 10 and was not focused on the method, so for her the contradictory results were not important, and she also did not participate in Thabo's exception-barring move. However, we have seen in the previous extract that she did engage in monster-adjusting together with Thabo when the teacher challenged them So at times she did engage in the process, but she was not as fully invested in it or in the method as Thabo was.

The pupils' discussion about the pentagon happened when the teacher was working with other groups and she did not find out about it at all during the week The pupils either forgot or chose not to raise it when she came to work with them Thus, the teacher could not mediate between the different understandings that were at work in the group. However, the pupils' different views of and participation in generating the method had important ramifications for their interaction with each other, and for perpetuating the area misconception which remained with them for the whole week. The pupils' different views are seen again in their responses to another counter-example.

Surrender

The extract that follows took place late in the week while the pupils were writing up their work for presentation to the class. Thabo again wanted to articulate their method and to test it. He suggested that they see if it works with a triangle with seven hooks (Figure 5a) He calculated the area of this triangle in two ways First, he extended their method to deal with shapes that have a number of hooks that cannot easily be made into a rectangle. In these cases, he used a rectangle with half a triangle on the end In this particular example, a six-hooked rectangle with an extra half a triangle gives seven hooks, the same number as the triangle, and has an area of $2\frac{1}{2}$ (Figure 5b).

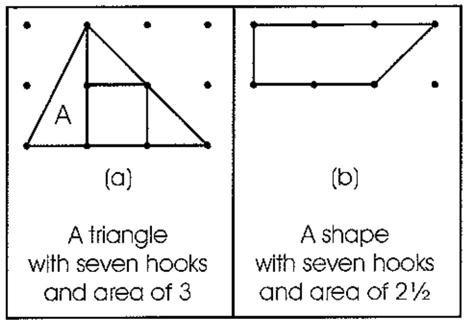


Figure 5

His second approach was to divide the triangle into a square and smaller triangles (Figure 5a), and to use the 'four-hook' part of their method to calculate the areas of the smaller triangles This gave an area of 3 Obtaining these two different values for the area in the case of a triangle, which he could not bar as an exception as he did the pentagon, led to despondency and near-surrender for Thabo:

Thabo: Oh, no this, ish, it's not gonna work

Nandipha: Ag.

Thabo: It's not gonna work.

Nandipha: So, it doesn't make a half

Thabo: You know why, you know why, our thing's not gonna work, our invention, cause *cava hier ne, cava hier* [look here], our, we supposed to get two and a half *ne*, *ne*, the, the, the area, the here's it's going to be one, two, it's two already, *ne*, so you can't tell me this is a half cause you've got four things, give us three. [2]

The two girls responded differently, reflecting their differential participation in the process and their different understandings of the task Nandipha's response to the different results was to say:

Nandipha: So come we'll make another one.

which can be interpreted as a willingness to ignore the contradiction and to try to find another example which would confirm their conjecture Lerato was willing to accept that the method did not work and was sometimes supported by Thabo.

Lerato: The method doesn't work.

Thabo: Our method doesn't work anymore, let's check, let's go back

As argued earlier, Lerato was never really convinced of the validity of the method, and this might explain her willingness to surrender easily While Thabo was also on the point of surrender, he was more invested in the method than she was, and suggested that they check it again.

Thabo: Let's forget the hooks, let's forget the hooks, let's forget the hooks, but then it works with the square, *ne*, so then from there, if you if the triangle is bigger than that, if it needs five hooks *ne*, then five hooks is one and a half.

He was torn between wanting to discard the conjecture and the fact that he had used the 'four-hook' part in most of the examples and it did work (his reference to the square above). He continued to try to find a way to make the method work. Soon after this the teacher approached the group

Teacher intervention

We saw earlier that the teacher challenged the group about their method with general counter-examples but did not express her own doubts about the method directly to them. [3] In the extract below, the pupils bemoaned to her the fact that their method did not work. She asked them to explain the method to her and tried to get them to be more precise about which part was not working. Again, we see a teaching approach which attempts to get the pupils to articulate their ideas and hence evaluate them themselves rather than rely on the teacher to evaluate them

Lerato: It's the method.

Thabo: The method doesn't work anymore.

Teacher: The method doesn't work?

Thabo: Yes, the method we showed you

Teacher: What method did you say?

Thabo: Of triangles and ..

Teacher: Explain the method to me

Thabo: You know, we shown you one, how to find, a uh

Nandipha: A shape

Thabo: A area of a thing, of a triangle, so but now it doesn't work anymore.

Teacher: What doesn't work, how did you say that they must do it? [4]

Nandipha: We said *ne*, eh, first you divide it miss into triangles, for instance *ja*, if we got six hooks *ne*, try to make a square with six hooks, so you've got six hooks on a triangle, and so you've got six hooks on a triangle

Teacher: Oh, O K you still using that

Nandipha: And then, then

Thabo: But it flopped.

Nandipha: But it doesn't work

Teacher: It flopped, well then write, where, on, on, on what, when did you find out that it flopped, and how did you find out that it flopped, write that, it's a va, it's a valuable finding, O.K, and then write to me an alternative method for the students.

Nandipha: We can't find an alternative.

Thabo: [*laughs*]

Teacher: Come on, you've answered all the questions beforehand, don't tell me you can't find an

Thabo: Yeah, we have, we ...

Teacher: . alternative method you've, you've answered all the questions and you did it correctly

The teacher did not engage the pupils about what was mathematically incorrect with their method. Rather, she tried to convince them that their work had not been in vain, that they had managed to calculate many of the areas and that they should try to describe how they had done it. She seemed to be trying to lift their spirits and convince them that they still did have something valuable to contribute. In the spirit of Lakatosian discovery, she tried to tell them that even developing something that does not work is useful, if they can understand why it does not work. In doing this she reinforced Thabo's reluctance to give up the four-hook part of the method. She continued:

Teacher: O K , now what I want you to do, and this is something that's valuable to learn, when you discover an invention, and it flops, it's still a valuable process, the fact that you thought you came up with something, and then you disproved it, mathematicians do it all the time, they come up with theories, they prove it, and then some guy comes a hundred years later and he disproves it, and both were valid, the one guy tried, and the other guy disproved it, that is what maths is about, O K., so don't feel that it was a flop, you failed. You didn't fail, O K.; the fact that you found your error in your theory means that you succeeded, it's fine, O.K , but now you have to give me another way that you actually found these, if your method flopped, how else did you find all of these areas?

The teacher talked to the pupils about the processes of doing mathematics and how if ideas do not work it need not mean failure She made two points, firstly that finding a mistake can lead to mathematical understanding, and secondly that some of what they had done must have worked, because they had managed to work out the areas of some of the shapes: thus, she took a Lakatosian view of conjecturing and refuting, and she worked, implicitly, with a version of 'lemma-incorporation', which is that refutations should enable you to refine your conjecture

In line with this, she continued to challenge the pupils to justify their method to her, while still acknowledging the worth of their thinking. They were able to justify the four-hooked triangles again. The fact that they could do so may have served to re-convince them that their method did work more generally. They did not raise the pentagon and the seven-hooked triangle for discussion with her. Thus, she was not able to engage with the pupils on the full extent of their method. She knew that there was a problem and that they had experienced difficulties. She tried to provide challenges in order to get them to articulate and resolve their difficulties. However, because she could not engage with the detailed contradictions that had emerged and which they could not resolve on their own, she could not deal with their fundamental misconception about area. She may even have unwittingly reinforced it.

Discussion

The teacher's interactions with the group show that she had developed a high level of skill with a particular kind of intervention, a refutational strategy in the spirit of Lakatos. She engaged with the pupils' ideas and tried sensitively to affirm and challenge them simultaneously. She required the pupils to convince her, and hence themselves, that some of their ideas were correct and did work. However, it was precisely this strategy, together with the fact that she could not always be present to hear the full extent of the pupils' thinking, that led to some of the limitations of the group's work in this case.

This raises the question: in what ways could this teacher have intervened differently, given what she could not know about the pupils' approaches to the counter-examples when she was not there? How could she have intervened to re-orient the discussion in more fruitful ways and to challenge seriously the pupils' misconception relating to area? Much of the discussion that follows suggests alternative possibilities for teacher action. These possibilities come out of the extensive research analysis and so they are grounded in what did happen in the classroom. However, they were not enacted and therefore we cannot know what their consequences might have been. Therefore, the discussion is necessarily speculative. Nevertheless, it helps my argument by suggesting that different, in some cases less-Lakatos-like interventions, might have enabled a better development of the area concept for the pupils.

A key question that arose for me about the pupils' work was: why the hooks? What was it that led to the pupils' fixation on the hooks rather than the lengths of lines when calculating the areas of triangles? [5] My analysis shows that the pupils' ideas about the relationship between the hooks and the area arose early in the week, when, in trying to work out the length of a line on the geoboard, they became confused about whether they should count the hooks or the spaces between the hooks. Their discussions at that point reinforced their thinking about the hooks. Although the teacher could not identify exactly what the problem with the method was, she did realise that the hooks were problematic, and she might, at some point, have tried to refocus the pupils' attention more directly onto the spaces, and hence the lengths of the lines.

A second possibility for a different kind of intervention might have been to remind pupils of their previous knowledge relating to area. During the week, the teacher could have reminded the pupils about the formulae for area which they had learned in previous years and asked how these correlated with their method. This may have served, again unbeknownst to the teacher, to reinforce Lerato's views and to allow her thinking to predominate, which in turn may have led to different developments of the method and the area concept. It is likely that the teacher did not talk about formulae because she wanted to allow for a more investigative task, in line with my research agenda.

As it happens, the teacher had been worried about why the pupils had focused on the hooks. However, she did not explicitly challenge them on this: rather, she chose to engage and challenge them on the issue of transforming the four-hooked triangles, which was the most mathematically appropriate part of their method. This suggests that she wanted to affirm their thinking and encourage them to do their own refuting of their conjecture. As discussed above, this may have reinforced rather than challenged their incorrect thinking. How else might the teacher have affirmed the pupils' thinking?

Pick's theorem, which relates the area of a shape on a lattice to the lattice points on the boundary and inside the shape (Coxeter, 1969), states:

> the area of any simple polygon whose vertices are lattice points is given by the formula $\frac{1}{2}b + c - 1$ where b is the number of lattice points on the boundary while c is the number of lattice points inside. (p. 209)

The proof of this theorem breaks down shapes into 'unit cells' of parallelograms with four vertices and no lattice points inside the parallelogram, or triangles with only three lattice points (half a parallelogram). The areas of these are then added to prove the theorem. This theorem resonates with the pupils' ideas and could have provided an exciting point of contact between the pupils' method and accepted mathematical knowledge. However, given the current mathematics curriculum in South Africa, very few teachers would have seen this theorem. Neither the teacher nor I had come across it. [6]

Another point where the teacher might have affirmed as well as challenged the pupils was to build on Thabo's simultaneous inclination and unwillingness to surrender the conjecture. They had complained to her that their method did not work, and yet Thabo still wanted to use the 'four-hook' part of the method. Among the group members, there were different views as to which part of the method did and did not work, but none of the pupils had thought about *why* it did not work. An explicit discussion about why the method did not work and yet why they did not want to give it up might have given the teacher the opportunity to give greater substance to her comments about the processes of mathematical discovery. It may also have reminded the pupils to bring up the counter-examples about which she had not yet heard. Had the teacher had access to these counter-examples, she almost certainly would have been able to work in more depth with the pupils about the role of counter-examples in refuting and refining conjectures.

This raises the issue of pupils' responsibilities in interacting with the teacher. My analysis has shown that the teacher missed much of what happened in the group, merely because she was working with other pupils. Much of what she missed could have been useful in helping to develop the pupils' thinking and concepts. Part of the responsibility for successful group work should lie with the pupils in becoming more aware of their difficulties when the teacher is not there, in order to raise them with her when they get the opportunity. This is an extremely difficult undertaking for pupils, but it is an important skill for group work and should form part of learning how to learn in groups.

Conclusions

In this article, I have argued that working with pupils in small groups brings new challenges for mathematics teachers. I have shown how one particular teacher developed ways of working with a group which was intended to encourage the pupils to participate in and come to understand what is entailed in the processes of discovering mathematical ideas. I have argued that the teacher had adopted strategies different from those usually seen in mathematics classrooms, which laid the basis for further development of ideas around conjectures and refutations.

However, I have also argued that her achievements were at the expense of the pupils' development of the particular concept of area. The extent to which she would be able to deal further with this problem is not known, but doubtful, given the constraints of the syllabus. This teacher's Lakatosian approach, together with her lack of access to much of the discussion in the group, came together to reinforce incorrect mathematics, and did not allow her to get to the core of the pupils' difficulties.

My analysis shows that the pupils could not separate out different parts of their method in order to try to reconcile them, nor either resolve the contradictions which had emerged, nor decide which parts were valid and which were not. In Lakatos' terms, they could not break down their thinking into lemmas which they could then analyse through the discovery of monsters and exceptions, and so they could not incorporate their lemmas and counter-examples to help them construct a better conjecture.

This is the area where they needed the teacher's guidance most. She tried to provide it as best she could, but was hampered in two ways. First, she did not and could not know the extent of their difficulties and their different views of the method, because she had not been part of many of their discussions. She was not even aware of the two counter-examples on which they had worked. Second, her emphasis on a Lakatosian *process* of mathematical discovery rather than on its products prevented her from intervening more directly to challenge their thinking.

This teacher made choices about intervening in pupils' mathematical thinking and around the tensions of simultaneously affirming and challenging pupils' ideas. There were also some choices that she could not make, because, due to the nature of teacher interactions with small groups, she did not have enough information. Her choices had consequences for the pupils' mathematics learning. In interaction with small groups, all teachers will be faced with similar choices and their consequences. I hope that the analysis of the nature of some of those choices in the case of this one teacher might help illuminate for other teachers, teacher-educators and researchers the choices that they do make together with their possible consequences.

Notes

[1] Two points about the transcripts: the pupils' names have been changed, and in their discussions the pupils speak English for most of the time and sometimes switch into their main languages, Sotho and Xhosa. They also use some Afrikaans expressions. In the transcripts, these expressions are italicised in the original spoken language and then translated into English immediately afterwards.

[2] He means that the right-angled triangle A has 4 hooks and so has an area of 1, which, when added to the 2 already obtained, gives a total area of 3

[3] The teacher said to me that she knew that there were problems with the method but she could not quite pinpoint them during the lesson, which is why she did not tell them directly what was wrong. She also might have been willing to be less directive because she knew that my research aim was to capture pupil interaction

[4] At this point, the pupils were working on part of the task which required them to explain their method to younger pupils

[5] This was not the pupils' first introduction to the area of plane figures: they had worked with the notion previously and had had no difficulty calculating the areas of rectangles

[6] I am grateful to Chris Breen for referring me to this theorem

References

Brodie, K. (1994) *Small Group Work and Ownership of Mathematical Knowledge*, unpublished M.Ed. research report, University of the Witwatersrand, Johannesburg

Brodie, K. (1996) 'Small group work: learning and teaching', *Pythogoras* **41**, 7-14.

Cobb, P. (1995) 'Mathematical learning and small-group interaction: four case studies', in Cobb, P. and Bauersfeld, H. (eds), *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures*, Hillsdale, NJ, Lawrence Erlbaum, pp. 25-129

Coxeter, H. S. M. (1969) *Introduction to Geometry*, New York, NY, John Wiley.

Ernest, P. (1991) *The Philosophy of Mathematics Education*, London, Falmer Press

Healy, L., Pozzi, S. and Hoyles, C. (1995) 'Making sense of groups, computers and mathematics', *Cognition and Instruction* **13**(4), 505-523

Hoyles, C. (1985) 'What is the point of group discussion in mathematics?', *Educational Studies in Mathematics* **16**(2), 205-214

Hoyles, C., Sutherland, R. and Healy, L. (1991) 'Children talking in computer environments: new insights into the role of discussion in mathematics learning', in Durkin, K. and Shire, B. (eds), *Language in Mathematical Education: Research and Practice*, Milton Keynes, Bucks, Open University Press, pp. 162-175

Kieran, C. and Dreyfus, T. (1998) 'Collaborative versus individual problem solving: entering another's universe of thought', in Olivier, A. and Newstead, K. (eds), *Proceedings of PME22*, Vol. 3, Stellenbosch, South Africa, University of Stellenbosch, pp. 112-119.

Lakatos, I. (1976) *Proofs and Refutations: the Logic of Mathematical Discovery*, Cambridge, Cambridge University Press

Webb, N. M. (1989) 'Peer interaction and learning in small groups', *International Journal of Education* **13**(1) 21-39.