Mathematics: "in Context", "Pure", or "with Applications"?*

A contribution to the question of transfer in the learning of mathematics

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> Education is nothing more than polishing of each single link in the great chain that binds humanity together and gives it unity The failings of education and human conduct spring as a rule from our disengaging a single link and giving it special treatment as though it were a unit in itself rather than a part of the chain It is as though we thought the strength and utility of the link came from its being silver-plated, gilded, or even jeweled, rather than from its being joined unraveled to the links close to it, strong and supple enough to share with them the daily stresses and strains of the chain (Johann Heinrich Pestalozzi, *The education of man*)

Discussions about the old educational problem of knowledge transfer have been revived in mathematics education in the wake of the seminal works of Lave [1988] and Walkerdine [1988]. Essentially, Lave and Walkerdine made two points: one, that everyday understandings and mental procedures involving certain elementary mathematical operations and relations are totally different from those that are expected of students in school and in experimental tasks seemingly involving the same operations and relations; two, that these everyday understandings and procedures should not be judged as something of lower quality than those prescribed by the traditional school curriculum. The impossibility of knowledge transfer from school knowledge to everyday problem solving situations is claimed to be a consequence of point one Point two suggests that the very problem of "knowledge transfer" can be misplaced in an epistemology that does not put academic knowledge as a model to follow, above the everyday ways of knowing and solving problems.

Lave [1988, p. xiv] says in the Preface to her book, that her study of routine calculations in tailor shops in Liberia "challenged the importance of learning transfer as a source of knowledge and skill across situations, raised doubts about experimental methods of investigating cognition, and made plain the need for an alternative analytic framework with which to approach the study of everyday practice" Walkerdine [1988], on her side, challenges the epistemology underlying the school-readiness tests based on the Piagetian theory of developmental stages, and, more generally, the modern use of theories to regulate social practices. The Piagetian theory has been regarded as having produced the truth about child development, and has turned into a powerful political tool to discriminate between the "normal" child and the, say, "less normal" one [Walkerdine, ibid., p. 5]. Both Lave and Walkerdine's studies of the discursive practices and reasonings in out-of-school and out-of-experiment natural contexts attempted to rehabilitate the situational, the practical, the commonsensical kinds of knowledge The idea of "good" thinking cannot be shaped, they say, by the hierarchical distinctions between the scientific and the ordinary, between the theoretical and practical reasoning.

The conclusions that some people have drawn from this research, as well as from the so-called "ethnomathematical" research, are that learning at school should be more like learning at home, growing and acquiring a vocation in the local community or in a workshop under the guidance of a master Instruction should be replaced by coaching, learning by apprenticeships Cognition is situated in contexts and if we want students to learn something transferable, these contexts should be close to those in which the students will live and work in their adult lives

This conception of education, as well as Lave's claim of the impossibility of transfer from school to everyday problem solving activities, has already been questioned [Boaler, 1993] or criticized, while proposals of adjustments [e g Heckman & Weissglass, 1994] or even alternative theoretical frameworks [Boero, 1989, 1993] made. Lesh [1985] has noticed that "many mathematical ideas are unlikely to evolve outside of mathematically rich instructional environments" Verstappen [1994], in the wake of curricular decisions made in The Netherlands, has explained how poorly founded is the idea of "mathematics as an organization of a field of [real life] experience".

In this paper I shall add some more arguments to the effect that a curriculum based on mathematics such as it arises in solving everyday problems is unrealistic, and that the problem of transfer cannot be resolved by claiming that it should not exist in the first place.

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> "Learning mathematics in real-life contexts" has become associated with postulates such as: the core of the learning activities should be associated with working on long term tasks allowing for *sustained thinking about a problem* over longer periods of time. [The Cognition and Technology Group at Vanderbilt, 1990]

This is what happens in authentic problem situations as opposed to usual school tasks, which have a well determined and known answer to be obtained in a short time Hence another postulate:

> students should be given an opportunity to engage in *authentic* problem solving activities "such as they occur in everyday situations and not in specially devised school tasks". [ibid.]

An authentic problem is not one that is given by the teacher; it must be *generated by the students* [Silver, 1994]. In order to generate problems, choose their own strategies of solution without being bound to a prescribed set of rules, students must believe in their inner freedom to act, or:

> students should develop into *autonomous* learners

And, as what is aimed at is practical knowledge, not always fully verbalizable,

> learning should occur in settings analogous to apprenticeships and coaching, where imitation and practice form the basis of interaction

Modulo "real-life" these postulates are not something specific to "learning mathematics in real-life contexts", and it will be argued in the sequel that if we think that it is the realization of these postulates that can raise the probability of transfer, there could be other, less problematic, solutions to the transfer problem.

"Learning mathematics in real-life contexts" is an extreme solution: mathematics should be learned directly in those contexts in which it is expected to be used by the student. On the opposite pole there is the "learning of pure mathematics": the more general and decontextualized the knowledge taught the better chances there are of its applicability and therefore of transfer; hence students should be taught pure mathematics, i e. mathematics as a system of knowledge in itself with no regard to possible applications. As a compromise between the two is the "learning of mathematics with applications": it is believed here that the ability to apply mathematics is not a natural consequence of knowing mathematical theories. Therefore, students must be taught the art of applications, besides being taught mathematical theories, methods, and notational systems. Moreover, applications are regarded both as a goal in itself and a pretext for doing more and more elaborate mathematics [Lesh, 1985; Kaput, 1994] It is this third approach that seems to be the most realistic and this is the one that will be supported here

The question of transfer in the "learning of mathematics in contexts"

Transfer requires engaging in authentic problem solving activities

Both Lave and Walkerdine are against making a sharp hierarchical distinction between scientific thinking and everyday thinking. But what is compared, in both Lave's and Walkerdine's experiments and observations is not scientific thinking and everyday thinking but school activities and everyday activities, simulated (or fake) problem situations and authentic problem situations. The conclusion that students should learn mathematics in real-life contexts does not obtain. The conclusion is rather that there can be no transfer from simulated, artificial, or fake problem solving to authentic problem solving because even the notion of problem is different in the two cases

> What constitutes "a problem" in the supermarket or kitchen? What motivates problem solving if not demands for compliance by problem-givers? . Some answers have begun to take shape: quantitative procedures in the supermarket appear to take their character in ongoing activity rather than to imprint canonical forms of problem solving on spaces between segments of grocery shopping. People do not have a math problem unless they have a resolution shape—a sense of an answer and a process for bringing it together with its parts Problem solvers proceed in action, often integrally engaging body, self, common sensibilities and the setting... Impelled and given meaning by conflicts generated out of the contradictions governing social practice, "problems" are dilemmas to be resolved, rarely problems to be solved [Lave, ibid., p. 19-20]

This description of how a problem comes into being and how it is resolved "in the supermarket or kitchen" could very well read as a description of what constitutes a problem in, for example, mathematical research

A school problem is usually *given* to the problem solver, it is not generated by the solver, as normally occurs in authentic problem-solving activities. This is why some researchers suggest that "problem-posing" activities be proposed to students, not just "problem-solving" activities [Silver, 1994] However, such "problem-posing activities" can easily turn into solving the following problem given by the teacher: "can you generate some problems related to this situation?" Knowledge at school functions in a specific way, characteristic of the school institution. School activities are and will remain authentic school activities and not authentic everyday or mathematical activities. Chevallard's [1991] studies of the didactic transposition of mathematical concepts, and Bernstein's analysis of pedagogic discourse [1990] give rather solid ground for this statement

Authenticity of interactions and truth in mathematics

In everyday problem solving situations or conversations it is rare that one of the persons involved knows all the answers, or is accepted by others as having all the answers. There is usually a lot of uncertainty, and therefore more

participation of the engaged parties, more discussion and negotiation. In order to make school situations more authentic, it is proposed that teacher-student interactions should follow a similar model (which is sometimes referred to as "weakened framing", Bernstein, [1977]) It is also claimed that, in mathematics classes, a student's view or conjecture can or should be accepted as "as good as anyone else's" [Nickson, 1992]. However, this proposal contradicts the assumption that the class works according to some curriculum, no matter how "weakly classified" it may be If there is a curriculum, then there is some content to be taught. In this situation, what will the teacher do if a student's belief contradicts one of the facts that the teacher attempts to teach? For example, a student believes that minus times minus cannot be plus because if you have a debt and you multiply it by another debt, you cannot suddenly have all the money back! Will this view be "as good as anyone else's?" Or will the teacher attempt to make the student change this view and thus convey the message that some students have had better views? The teacher can only choose a nice way of saying "you are wrong". Boaler [1993] believes that transfer can be given a chance if students have the opportunity to realize that "there can be more than one answer [to a mathematical problem], that mathematics can involve discussion, negotiation and interpretation", if, in short, a more "process-based view of mathematics" is conveyed to them. But we should be very careful in interpreting such statements. It would be unwise to make the teachers and the students believe that a mathematical result is validated in the same way as a point in a literary interpretation Of course, there are many ways of arriving at an answer of a complex calculation; one can use geometric or algebraic, static or transformational arguments in justifying theorems of geometry; one can, of course, decide to work in non-standard analysis and then be allowed to claim that 1 and 0.9999... are distinct numbers. One can use a "procedural" or a "structural" language in stating a result or posing a question, either introducing the person who is knowing and acting, or just speaking about sets and the conditions defining sets. For example, I can ask a student: "If you perform certain elementary row operations on an augmented matrix, and then solve the corresponding system of equations, will the solutions you get also be the solutions of the initial system, and will these be all the solutions?", or I can ask: "Is it true that if two augmented matrices are row equivalent then the associated systems of linear equations have the same solution set?" [Lay, 1994, p. 34] In the latter case the question is concerned with the possibility of having two different sets of conditions defining the same set of vectors in R^n. These are subtle differences of approach and understanding of a mathematical theory But on the whole these differences do not affect the validity of statements. Once we have agreed to work within one theory, then on the level of the mathematics that we do with our students up to the undergraduate level, most of the time we can say whether a statement is true or false, and if there is any discussion, it is more about the form than about the essence of things. It is important to let the students express their points of view, attitudes, and argumentation, for in so doing they come to a better understanding of what they mean and what the concepts mean, and it is all right for the teacher to listen, to respond, and to negotiate, but it is unfair to the students to make them believe that the truth of mathematical statements can be voted or negotiated on the same basis as, for example, some practical point. Negotiation will normally bear upon the form of speaking about something, the underlying philosophy, the relativity of mathematical statements with respect to the theory to which they belong, the conventional character of mathematical notations, the significance of a mathematical statement, but not on its logical value.

Everyday life contexts, imported to school, lose their authenticity and become school tasks, on which one can fail or pass, and not dilemmas that are resolved as the problem grows Conversations have a definite didactic goal, word problems are disguised mathematical relations and operations. If one wants to make students' activity more authentic and takes children out of the school, into supermarkets and other kinds of environment (like the vacant lots in the neighborhood), their cognitive activity will become more authentically *everyday,* but much less authentically *mathematical.* For mathematics has to do with representation and generalization, and not with solution of concrete or practical problems which can always be solved with ad hoc methods This is a dilemma that is not so easily resolved

Contents: more authentically mathematical or more authentically everyday?

Heckman and Weissglass [1994], in their proposals to involve students in more "everyday authentic" and, at the same time, mathematically richer activities, describe a project in which students engaged in solving a problem of the local community: "management of trash in vacant lots". The mathematical activities involved were, the authors say: looking for patterns, counting, measuring, scaling, interpreting codes, solving systems of inequalities, evaluating costs through calculation of perimeters, and taking into account various factors (such as size of holes in the fence webbing). It is not clear from the story, however, whether students engaged in any kind of generalization, so as to find a *method* for the resolution of this type of problems or whether they were interested only in getting this particular problem resolved by whatever means available In the latter case, what they were doing was not mathematics

To see mathematics in a situation, one must already know some mathematics and be "mathematically tuned". It is hard to imagine that in a situation which would be completely open, not labeled as intended to be mathematized, the activity and curiosity of the children would lead them to uttering mathematical statements and asking mathematical questions. Let us just think of the simple example of children in their first grade shown a picture of three birds sitting on a branch and two flying away, and asked the question: "What can you say about it?" Certainly the statement "5 - 2 = 3" is the least interesting thing to say about birds! If children do say this,

however, it is because they have learned how to behave in a mathematics class.

If one is in a mathematical mood and does try to see the mathematics of a situation, it may happen that one knows too little mathematics to cope with the question without some instructional help and suggestions. This can happen, in fact, quite often, for practical problems are complex; their idealization and mathematization require non-trivial mathematics, or looking at known mathematics in a new way. This is why applications of mathematics usually lead to more mathematics. An interesting example was given by H.P. Pollak in his ICMI talk at the ICM '94 in Zürich.

The example was concerned with the problem of hanging paintings on the wall so that they hang straight, in two cases: when the nail is not meant to show above the painting, and when the nail is far above the painting (as when the painting is very large). You are interested in your string forming an isosceles triangle with its base being the line between the two nails you have inserted in the frame of the picture. It turns out that it is quite difficult: in the first case, for example, a very slight change in the height of the triangle will effect a biggish change in the angle (e.g. it may happen that if you diminish the height by 1.4%, the angle will increase by about 13%), and this makes the painting look decidedly oblique. This leads to the mathematical question of the stability of theorems, in this case to the question of how stable is the well known geometric theorem that opposite equal sides in a triangle you have equal angles? Similar conclusions stem from James Kaput's research on using technology for the creation of "viable, functional connections between the world of authentic human experience and the formal systems of mathematics" [Kaput, 1994]. The environments Kaput proposes lead to thinking in terms of functions, variable magnitudes, change, and relationship between changes, and not in terms of states, unknowns, and equations to be solved. In fact, the students will have to use, implicitly or explicitly, the methods of calculus and numerical analysis in order to "orient themselves" [cf. Jahnke, 1994] in these environments.

When Lave [ibid.] says that the adult student, this mother of four, who, in the supermarket, was able to estimate the number of apples she should buy so that she could satisfy the needs of everyone at home and not exceed the capacity of her refrigerator, but was having problems with solving school arithmetic problems, she looks at two totally different situations from the mathematical point of view. In the supermarket, this woman is not making sums, she is not even solving equations; she is solving a system of inequalities. If she were to mathematize her problem this is what she would end up with. But she is not so advanced yet in her mathematics. To get to this point she must first learn simple school arithmetic. The purpose of this simple arithmetic she is doing at school is not to give her proficiency in calculations—she is very good at that already—but to give her an opportunity to reflect on the properties of the operations that she is performing so quickly in her mind. She is also expected to take the concept of number she only uses as a tool in the supermarket as an object of her reflection. This reflection is what will lead her to mathematical thinking, which is why she has come to school. Otherwise, she could have stayed at home. Of course, this knowledge will not be useful in her supermarket dealings. But then, is it meant to be?

Problems solved in the real-life contexts of shopping, or the understanding of relational terms as used in everyday conversations between a mother and a child, are much more complex and are governed by a different logic [cf. Grice, 1981] compared to the simple calculations and rather trivial comparisons used in school or experimental tasks. However, while these calculations and comparisons are trivial on the contextual level, it is not such a trivial thing to analyze them, to symbolize them in a notational system, to theorize about them, etc. The Piagetian experiments tested children's ability to engage in such activities and to get involved with such concerns. Children are not at school to learn to be efficient in everyday conversations and shopping activities, but to learn to theorize, to analyze language, to see structures. If schools were to teach people how to conduct everyday conversations and how to do shopping, there would be no need for schools at all, because these are things, as Lave and Walkerdine have shown, that are perfectly well learned out-of-school.

Of course, one can question the view of the goal of school education as the preparation of the theorizing mind. One can say that the goal is the preparation for "life" (as if "to live" did not mean, for humans, to theorize a little). But this implies that school is there to replace the job that would normally be done by a community if only its members had time to train the young apprentices. If people had enough time, the child would learn to speak, read and write with the mother; he or she would learn a vocation in the father's business or workshop, or go to other persons' workplaces to learn something that he or she would then do all his or her life. However, because people haven't got the time, there must be teachers and places where the natural interactions and activities could be simulated.

It is only if the school's goal is the preparation for life in a restricted, pragmatic sense, that the problem of the transfer of knowledge from school to everyday life obtains. it does not if the goal is the training of the theorizing mind. The transfer of knowledge is a problem in the first case because the simulation changes the natural contexts.

It may be useful to notice here that while some researchers deny the possibility of transfer from school knowledge to everyday thinking on the individual level, this transfer has been taking place on a different level, the social and historical level: the zero, the decimal system, the formulas for area, the systems of measures, and all these facilities our homes are full of, starting from thermometers and finishing with electronic mail, are products not of the practical mind but of theorizing thought. The "everyday thinking" of a mother of four by the end of the twentieth century, who may be using a calculator and software to plan her budget, is probably a little different from the "everyday thinking" of a pre-Stevin mother of four. There is a danger that, if schools are

reduced to the role of "foster mothers" or "workshop masters", in the long run there will be not enough of the theorizing power around to maintain the influx of those mental and other tools so handy in the everyday lives of individuals

Problem of transfer or problem of inert knowledge?

When we speak of the problem of "knowledge transfer" and claim that students are not using what they learned at school in solving everyday life problems, are we sure that what these students learned was knowledge? Maybe this was just information, or mechanical skill. And these are indeed context-bound. Only knowledge is transferable, and knowledge is something that has been constructed as a solution to a personally generated problem or a problem that has been personalized. [Bachelard, 1938; Brousseau, 1976] Boaler [1993], in a critical article about "learning mathematics in contexts", seems to be saying just that:

> It seems likely that an activity which engages a student and enables her to attain some personal meaning *will* enhance transfer to the extent that it allows deeper understanding of the mathematics involved.

Referring to Wiliam [1988], she adds:

> If students are able to make problems their own in this way, their learning, by virtue of its possession of meaning, will be available for use in real life problems [Boaler, 1993].

This is, maybe, a rather far-fetched conclusion. If this woman in the supermarket could cope with her shopping problems more and more efficiently, it is because, through experience, she has acquired some knowledge. This knowledge was a solution to problems that she had to deal with. Her experiences at school may not have led her to develop knowledge, but even if they did, this would not necessarily be knowledge relevant to solving her problems in the supermarket: this knowledge could answer different types of questions.

Thus the question to which "learning mathematics in contexts" seems to be an answer might be: how do we make students acquire knowledge at school, and not just information or mechanical skills? Is it at all possible?

The question that comes next is, of course: what kind of knowledge? The choice of "real-life" contexts is usually made in the case when students are expected to acquire knowledge that is immediately relevant for the solution of problems in everyday situations. Other contexts are found more appropriate if the goals of learning mathematics at school are less pragmatic, or less directly pragmatic.

Even those who seem to endorse the view that school should prepare children for dealing with real-life problems do not always claim that the "contexts" in mathematics classes should be real-life contexts, especially when very often they are only fake real-life problems. An interesting view is expressed by Roland Fischer [1988] who does not speak explicitly about contexts but proposes something far subtler in order to humanize and personalize the learning of mathematics at school. He is not worried about the distance between the "individual knowledges" that people develop and "official knowledge" that is, for example, written in textbooks. It is even necessary, he says, that they be different, "because otherwise the individual would cease to exist and innovatory knowledge would become impossible" [Fischer, ibid.]. What school mathematics education can do is take advantage of both official and individual knowledges by looking at mathematics as a "tool for representing relations, and thereby a means of communication". "Mathematics", he says, "is a social construct which has its roots not only in individual thinking but also in the interaction of men". If this interaction is allowed in the mathematics classroom, students will soon bring up a variety of interpretations of mathematical notations, as well as question the sense of certain mathematical concepts introduced by the teacher. An example of the latter case is the introduction of the modulus operation ($|a|$) in the context of concrete numbers. Students will not see the point of it, and rightly so, because, as Fischer says, "the modulus sign is of use if general facts are to be expressed, such as the triangle inequality, or $\sqrt{\alpha^2} = |\alpha|$". We, as mathematics educators, thus learn something from students that is useful in our profession. For Fischer, in discussing questions of what makes sense with students, we focus on the relationship between people and mathematics, "Both being of equal importance". He suggests that "this should become the content of mathematics education in the classroom—in addition to the ordinary teaching and learning of the subject matter". Viewing mathematics as a system of mass-communication, and a means of communication, "materialized communication about non-material matters", his overall goal for mathematics education is "to improve communication"

> I believe that the school of the future will have to concentrate more on the relationship between people and knowledge than on knowledge itself. I offer two reasons for this conjecture: (*) Knowledge, especially mathematical knowledge, is increasingly available in applicable materialized form, in books, journals and machines. The competence of people in the flexible handling of this knowledge is important. For this a meta-knowledge of the relationship between people and knowledge is necessary. (**) There is an abundance of knowledge available today. General advice for everybody is problematic, we must stress the responsibility of the individual for the organization of his learning and for choosing the content. But there should be general guidelines, resulting from reflection on the relationship man-knowledge to help the individual to find his way—not only in school, but also afterwards. [Fischer, 1988]

In curricula in which the goals of education have been perceived less pragmatically, there is less concern about providing students with "real-life" contexts in school but there *is* a concern about teaching knowledge and not "inert knowledge" [Whitehead, 1921]. In, for example, Brousseau's "theory of situations" the counterpart of "context" is "milieu" [1977; 1990]: the knowledge students will develop in a particular didactic situation is determined by the constraints exerted by his or her relations with the milieu. It is even considered the main task of research in mathematics education to study and design situations in which the mathematical knowledge

proposed by the curriculum appears as an optimal solution for the students Fischer's example of the operation of the modulus, as making sense only when general statements containing variables are made, and not in the context of concrete numbers, can illustrate what is meant by the constraints of the situation.

> The interactions of the student with the milieu are fundamental. Knowledge exists in a student insofar as it is an optimal and stable solution in a set of constraints in the relations between the student and the milieu . The didactic activity consists in organizing these constraints and in maintaining the conditions of optimal interactions. [Brousseau, 1977]

It is important to note, in the above citation, that the constraints defining a piece of knowledge are not located in the milieu but in the relations of the student to the milieu. As such, they depend also on the student's previous knowledge, on his or her conceptions, beliefs, etc

> The meaning of a piece of mathematical knowledge is defined not only by the set of situations in which this knowledge is realized as a mathematical theory (semantic theory in the sense of Carnap), not only by the collection of situations in which the subject meets it as a means of solution, but also by the set of conceptions, previous choices that this knowledge rejects, the errors that it helps to avoid .. the economy it allows for, the formulations it puts into question. . [Brousseau, 1976]

Let me give an example here from our continuous observation of students learning linear algebra with the help of a tutor. It will illustrate the point that students develop as knowledge only that which appears as a necessary tool in solving a problem; their conceptions will not be general beyond the needs of the problems they are dealing with.

Example: students will develop no more knowledge than is necessary to cope with the problems they solve

Student C was reading the chapter on linear transformations from R^n to R^m in D.C. Lay's book "Linear algebra with applications" [1994], where the structural definition of this concept was given: a mapping from one vector space to another which preserves linear combinations. Next he started to work on an "application example" offered in the book (Section 2.7). One of the examples in which the student engaged relates to the population movement between a city and a suburb In the example the migration data are given in a diagram, and it is hardly necessary to read the text in order to understand the assumptions of the problem: 5% of the city's population moves yearly to the suburbs, and 3% of the suburban population moves yearly to the city. Suppose the populations of the city and the suburb in, say, 1990, are given by the population vector $x_0 = [r_0, s_0]$, and the population vectors of the subsequent years are indexed accordingly, x_1, x_2, etc. The question posed by the book is "to describe mathematically how these vectors might be related". The expected solution consists in writing the "migration matrix" M, and the formula $x_{k+1} = Mx_k$, for $k = 0, 1, 2, \ldots$, where M is

$$\begin{matrix} .95 & .03 \\ .05 & .97 \end{matrix}$$

Student C does not read the solution. He tries to work on the example by himself But he does not find it necessary to write the relations between the old r's and the new r's and s's in the form of a matrix multiplication. He writes:

$$r_0 = T(r_0) = 95\%\ r_0 + 3\%\ s_0.$$
$$s_0 = T(s_0) = 97\%\ s_0 + 5\%\ r_0.$$

He does not see any point in writing this system in a matrix form. Prompted by the Tutor, he just puts two vertical lines, like this, to satisfy him:

$$r_0 = T(r_0) = |95\%\ r_0 + 3\%\ s_0|$$
$$s_0 = T(s_0) = |97\%\ s_0 + 5\%\ r_0|$$

asking: "Isn't this the same thing?" This is not exactly the same thing, but the student is not aware of this. The way he thinks about linear transformations is close to seeing them as something that changes (linearly) one set of variables into another set of variables, which is evocative of the definition of linear transformation still given by the popular *The Penguin Dictionary of Mathematics* edited by J Daintith and R.D. Nelson [1989, p. 201]. So far, this "variables-to-variable" conception of linear transformations has been perfectly sufficient

Eventually, upon request, Student C writes the matrix of the transformation, but he will not spontaneously use it in his attempt to solve the question of the tendency in the changes This question comes up in the conversation between the Tutor and the Student as they muse over what will happen in subsequent years: as more people seem to flee the city than come to live in it, is it possible that at some point the population of the city will become zero? They decide to assume that the number of people living in the city and the suburb is the same, equal to one million, and see what happens over the years Student C does not just multiply the matrix he obtained by the population vector, but instead constructs a table with two columns, one representing the city population, the other the suburban population, and makes consecutive calculations according to his equations. He is thus in conformity with his variables-to-variables view of a transformation He obtains:

City	Suburb
100%	100%
1 000 000	1 000 000
980 000	1 020 000
931 000	1 038 400

At this point he gets bored and says:

> 210. C: Can we make a function that continues that?
> 211. T: What do you want to find?
> 212. C: Can you make such a function?

Now he is ready for a matrix view of the transformation In this interaction, the Student and the Tutor used the computer algebra system *Maple* to calculate the consecutive values of r's and s's. Besides showing that the population of the city is slightly diminishing, this approach did not give a definitive answer. The Tutor then decided to use methods of diagonalization to obtain a general formula

for the population in the kth consecutive year in terms of r_0 and s_0, asking the student to solve, in the meanwhile, an eigenvalue problem (without using this term, of course): is it possible for the population to return to the same state after some time? Find an initial state such that, after a year, the populations will change at most by a scalar factor, so that $M^* [r_0, s_0] = a [r_0, s_0]$ for some number a The Student solved the problem with elementary methods, and with his variables-to-variables notion of linear transformation, he arrived at the conclusion that a must be one and the ratio of r to s must be 3 to 5 The formulas that the Tutor found allowed both to conclude that no matter what the initial populations, the ratio of r to s tends to 3/5. It took them all this work to find out that, instead of looking at absolute values of the populations, it was better to calculate their ratios. While the student was yet unable to find the general formulas for r_k and s_k, he would be able to approach the problem numerically by writing a simple program for calculating the consecutive ratios of r to s and guess what it would tend to in the long run. Methods of diagonalization would then appear as possible tools for the validation of this guess, and this is where the student could develop a more structural view of linear transformations. The structural concept of linear transformation could also come up if the tutor suggested that the student change the data, see what would happen if the migration matrix were singular, what this would mean in terms of the population movement, etc.

Such is the potential richness of this application problem: it does not force a structural concept of linear transformation as it is, but it does bring up the variables-to-variables notion of linear transformation quite naturally, and opens the gate to the more structural understanding through the eigenvalues and diagonalization problems However this potential must be noticed and exploited by the teacher who uses the book. The teacher's role cannot be over-stressed in the regulation of the students' relations to the milieu. Like many others examples in the book, this one can also be read literally, reduced to just the plain question that is explicitly asked. Also here, the teacher can spoil the potential by forcing the student to use the language of matrices from the very beginning, by returning to the definitions, by demanding that "the standard notation" be used, instead of waiting to see what the student can do spontaneously, how far he or she can go, where natural curiosity will lead him or her

Let me note here that if Student C showed interest in solving this application problem, it was not primarily because he felt it was related to some real life situations The Student even complained many times that the assumptions of the population movement problem are completely unrealistic. What he liked was the activity of mathematization, of discovering the mathematics in these situations

The question of transfer in the teaching of "pure mathematics"

The object, mechanisms and conditions of transfer

How is the teaching of "pure mathematics" justified in mass education? The classical argument refers to mathematics as *disciplina mentis,* but the history of education knows other arguments as well. H.N. Jahnke [1994] reminds us of one used in the early nineteenth century in Germany under the influence of philosophers such as von Humboldt, Schelling, Fichte It was a consequence of a kind of holistic world view in the center of which was the category of "organism", opposed to the rationalist category of "mechanism". An organism is a whole which cannot be analyzed into a direct sum of its parts: it has properties that do not belong to any of the elements. Teaching at school was supposed to develop "organic reasoning", a reasoning that attempts to grasp the holistic character of objects while being holistic itself as reasoning, i.e. developing from its own assumptions and rules, and studying an object for its own sake. Mathematics ("pure") was included in the curriculum for exactly the reason that, as a closed system, mathematics promoted reasonings "developing from their own conditions". For Humboldt, education is not just a preparation for the "mundane life"; it has to take into account the possibility of future scientific activity Humboldt opposed himself to applications of arithmetic to everyday situations even in the early grades, and allowed for more scientific applications only by the end of schooling [Jahnke, ibid , p 418].

But "organic reasoning" cannot develop through the solving of isolated simple school exercises:

> In order to become educated human beings persons must, in their own development, have had at least once the experience of getting totally involved with a problem and coping with it productively. Only persons who have seen at least in one particular field that there are things that are holistic and have their own laws will be in a position to assess what it means not just to adhere to a number of rules in their own life, but to have the inner freedom to act [Jahnke, ibid , p. 417]

We have here a belief in two conditions for knowledge transfer: getting "totally involved with a problem", and "having the inner freedom to act", or making decisions on the basis of what one knows and not being bound just to follow the rules given by others These conditions seem to be invariant in proposals of curriculum change deemed to ensure knowledge transfer.

For example, today, the first factor is considered crucial in the arguments of The Cognition and Technology Group at Vanderbilt [1990], a group of researchers who promote the learning of mathematics in contexts. They refer to this aspect when they speak of the necessity of "sustained thinking about specific problems over long periods of time" in order to acquire expertise in some field. They claim that anchoring instruction in complex problem spaces, which they call "macrocontexts", enhances the possibility of such intensive exploration, and helps prevent the build-up of "inert knowledge".

The second factor is often referred to as "autonomy", or the need to educate students as "autonomous learners".

Pure mathematics in integrated curricula

The blame for the lack of transfer has sometimes been put on the organization of the curriculum in a way that turns it into a collection of strongly insulated "subjects", such as

mathematics, literature, science, and others. Therefore, a solution was sought in a different organization: in "integrated type" curricula [Bernstein, 1977], with breaking open the boundaries between subjects, making connections, and, sometimes, abolishing the division of curricular contents into subjects altogether. One would think that, in such a curriculum, the teaching of mathematics as "pure", systematic, knowledge "that develops from its own connections" would be inconceivable. However, it seems that certain kinds of integrated curricula would allow for that, with the provision, however, that the learning of mathematics be accompanied by a meta-level reflection.

There are different possible kinds of integrated curricula. The basic principle of any integrated curriculum is only that connections must be made amongst contents selected as important. Different models are obtained depending on the organization of these connections, as well as on what is considered as important content, which, in its turn, leads back to what is considered as the main goal of education (whether it is the development of a creative member of society, or the preparation of an individual for private life, or the education of a competent worker ready to adapt to changing circumstances and capable of contributing to his country's success in international economic competition, or some other). In less pragmatically oriented models, the integrating network of connections can be such that the teaching of pure mathematics could be justified.

For example, in the project of the Wroclaw (read *Vrotzlav*) School of the Future in Poland, it was proposed to replace the structure of the division of contents into subjects by a structure in which the important contents would be linked to some central "unit" and thus become interconnected among themselves. In the Wroclaw project, this central unit was meant to have a conceptual character, which, branching into subconcepts, would create an "internally coherent, sufficiently comprehensive and stable framework" [Waszkiewicz, 1978]. The category of "society" was chosen, as one possibility among others. Society is related to culture, communication, language, institution, etc. In this framework, mathematics could be studied as part of the society's culture, as a language, as a factor in the creation (and manipulation) of meanings, as a tool for understanding the world, as well as a tool for the production, organization, and management of a society's material environment and institutions (financial, related to health, education, welfare, etc.) This attitude does not preclude the study of mathematics for its own sake, nor for the sake of its applicability, and it even justifies the study of mathematical proof as a specific type of argument, method of validation of statements, and means of communication. Proving is related to communication and so is a social activity. Proving makes sense only within a social context, just as with any question about whether something is true or not. I am always proving something to someone even if this someone is only me. Moreover a society is built on sharing ideas, which requires negotiation, reasoning, convincing others about things. These activities are somehow linked with mathematical proof, but differ from it in some important ways. A discussion about these differences could integrate many ideas from mathematics, philosophy, history, sociology. In this kind of integrated curriculum, organized around the category of society, any kind of mathematics could be studied, even formal mathematics. Otte [1990] reminded us of the humanistic dimension of foundational research in mathematics by quoting Norbert Wiener, for whom the formalists could be seen as those who raised mathematical thinking and experience to the level of an object of study. What they studied were the tools of the mathematician at work: logic and proof.

The central integrating category in the Wroclaw project was "society". There can be other proposals. For example, although not referring explicitly to an integrated curriculum, the proposals of Roland Fischer [1988] concerning mathematical education fit into the schema of a centrally structured curriculum, with the central unit being the concept of "relations between knowledge and men". Fischer stresses the education of an individual who would be responsible for what he learns and how he organizes it. The goal of education in the Wroclaw project was "the development of a creative member of society". But a creative member of society—it was claimed in the discussion documents—is one who believes in internal rather than external control [references were made here to psychological studies of personality, Lefcourt, 1982; Phares, 1976]. All these ideas are related: Fischer's "responsibility" for one's own learning; the concept of "internal control"; *Selbsttätigkeit* and the "belief in having the inner freedom to act" evoked by the ideologists of the early nineteenth century German reform of the Gymnasium; the idea of "autonomous learners" often mentioned by the promoters of "the learning of mathematics in contexts". It is thus common to very different approaches to the teaching of mathematics.

Let us look back at the arguments. What is claimed to be transferred in the learning of "pure mathematics" is (a) discipline of the mind; or (b) organic reasoning; or (c) an awareness of the relations between mathematics and people, their culture, systems of representation. The mechanisms of transfer can be triggered by (a) getting totally involved with a problem, and/or (b) meta-reflection on the mathematics learned. They will be triggered, however, under the condition that the student develops a belief in "having the inner freedom to act", or in "inner control", or in having the responsibility for his or her own learning. The goals of education evoked were: development of disciplined thinking; or of holistic reasoning; the development of a creative member of the society; the improvement of communication, competence, in the flexible handling of mathematical knowledge as available in books, journals, and machines.

The teaching of proof

If we want students to learn mathematics (and not something else) and to learn mathematics as knowledge and not inert knowledge, we shall need to have students learn mathematics in contexts, but these contexts will have to transcend the contexts of practical reasoning. As Boero [1989] says, the "fields of experience" will have to be transcended by "semantic fields" that will allow the

rationalization of the former by means of mathematical concepts and procedures; later, the development of linguistic competencies in rationalization and modeling, together with metacognitive reflection on the relations between the experiences of life and the intellectual tools used to grasp their meaning, will lead to an organization of understanding into "conceptual fields".

An activity that is the most characteristically mathematical is proof. In order to give more authenticity to proof, curricula have assigned time for activities called "exploration", "investigation", "argumentation".

Proof and experimentation

In the U.K. schoolchildren are doing mathematical "investigations" as part of their coursework [cf. Morgan, 1992]. The NCTM Standards recommend that children explore, for example, various triangles on a geoboard to "find the area of any triangle", and then "the area of any trapezoid" [NCTM Assessment Standards, 1993, p. 75-77]. But if one takes seriously the arguments for a distinction between practical and theoretical arguments such as, for example, Bachelard's [1938] or Ryle's [1969], then one can see the difficulty of this task. Ryle claimed that if there were a smooth transition between practical and theoretical reasoning, most of the well known paradoxes, such as *Dichotomy* or *Achilles*, would not even exist [Ryle, ibid., p. 50]. This question of the distinction between practical and theoretical thinking has been very much discussed in philosophy. Not everybody wants to sharply distinguish between the two [e.g. Edgley, 1979], but the arguments of those who do see a large gap between practical and theoretical types of reasoning cannot be ignored.

What is expected of children, in the vision of the UK national curriculum or the NCTM Standards, is indeed, *experimentation*, and this involves (a) the perception of the difference between the experimental and the theoretical in their activities with material objects such as triangles on the geoboard; and (b) being in possession of some theory already, in order to be able to set their experimentation, to structure it somehow. This structure is necessary in view of the need, later, to interpret the results of the experimentation. In fact, an experiment usually tests something, a conjecture, and a conjecture is already an outcome of some theoretical thinking. Chevallard [1991-1992] studies the concept of experimentation in mathematics in some depth. If, he says, an experimentation supposes only that there is a system (the object of experimentation), on which a certain manipulation is performed relative to a research question about this system, and if the behavior of the system under the manipulation is at least partly unknown, then there is potentially quite a lot of experimentation in mathematics. But, Chevallard stresses, understood this way, experimentation presupposes theorization. This theorization is already present in the identification of the object of experimentation as a system, in the formulation of the research question, and in the conception of the manipulation to be performed so that this question can be answered by the experiment. It is only the manipulation part of the whole activity that allows us to label it as experimental. There can be no escape from theorizing in any kind of mathematical activity.

Proof and argumentation

"Real mathematics" contains proofs. "Realistic mathematics" contains argumentation. How close are these things?

As Balacheff [1991] has repeatedly pointed out, referring himself to, among others, philosophical studies on argumentation [e.g. Moeschler, 1985; Perelman, 1984] and studies on practical thinking [e.g. Bourdieu, 1988], that argumentative discourse in a context that is, firstly, a social context of interaction, and, secondly, the institutional context of a classroom, is very likely to have features in conflict with the aims and nature of mathematical proof. Such a discourse may have a polemic character, aimed at demonstrating that the other is wrong, and not at establishing the truth of an assertion. All means are permitted in this case, and these are not confined to the principles of logical deduction. Aristotle made a distinction between true proofs and sophistical proofs and identified several kinds of arguments that sophists use for the sake of competition in discussion alone, like "refutation, fallacy, paradox, solecism, and fifthly, to reduce the opponent in the discussion to babbling" [Aristotle, *On sophistical refutations*, in McKeon's edition, 1941, p. 210]. In argumentative behaviors, ordinary induction (passing from the particular to the general) and deduction will have equal status, depending on whether the speaker aims at explaining something in an accessible way to an interlocutor who is interested in understanding, or at effectively convincing a "contentious" person:

> Induction is the more convincing and clear: it is more readily learnt by the use of the senses and is applicable generally to mass of men, though Reasoning is more forcible and effective for convincing contradictious people. [Aristotle, Topics, BK. I; in McKeon's edition, p. 198]

The conclusion for us here might be that we should be satisfied with induction in mass education, and if we want students to feel the need of deduction we would be wise to put them in a contentious mood.

The context of a "classroom conversation", or any conversation involving a large number of persons for that matter, is not favorable to the construction of a mathematical proof, although it may give ideas and an incentive to individuals to engage in such a construction. A proof is an achievement of an individual mind, someone who can see the main idea of it, and hold on to it as one logical thread leading from the premises to the conclusion. This activity requires silence, a concentration of the mental effort. Listening to others speaking, following their lines of thought—this is distracting. No wonder, for these discussions turn into conversations or an exchange of loosely connected remarks. Of course after a certain proof is thought of, it must be (and usually is) submitted for public criticism. Then, a mistake or a gap can be pointed out to the author, who will try to amend it. But he or she will again need some silence for this. It can be noticed that mathematicians rarely work in groups, although they do

give talks and listen to talks. The silence at the International Congresses of Mathematicians is very telling: there are no questions at the end of talks, people are not seen vividly discussing mathematics in groups; sometimes they converse in pairs, but most often they are seen lost in thought in remote corridors, staring at a few signs on a sheet of paper. Paul Erdös does not even need a remote corridor; he can be seen working alone in the busiest places of the congress.

Proof and apprenticeship

If there is anything in the learning of mathematics that requires apprenticeship, it is proof. The ability to find and write proofs must be acquired through apprenticeship because it involves mostly tacit knowledge and a kind of practice [Polanyi, 1964]. Learning through apprenticeship rather than through direct instruction is thus not specific to "anchored instruction" or "learning mathematics in contexts" or "realistic mathematics" and the like.

Often, "learning through apprenticeship" is mentioned along with the need to foster students' autonomy as learners. Rightly so, because in the context of an apprenticeship autonomy becomes a problem. As follows from Schön's studies of this type of learning and instruction, coaching for autonomy is a difficult art, as is learning from a coach. In fact to enter the learning process as an apprentice the student must accept that, at the beginning, she will have to follow instructions the meaning of which she does not fully understand yet. She must come to the studio "equipped with a capacity for imitation, an ability to do as she sees another person doing". But in order to liberate herself from this initial dependent attitude she will have to be active, she will have to "put herself into a mode of operative attention, intensifying her demands on the coach's descriptions and demonstrations and on her own listening and observation". By reproducing elements of an activity she will experience them, "feel what they are like, and discover in them, by reflection, meanings she had not previously suspected". The student's active response, and efforts at the tasks (proposed, first, by the coach, and not student-generated) will "provide the coach with evidence from which to infer her difficulties and understandings and a basis for the framing of questions, criticisms, and suggestions" [Schön, ibid., p. 117].

Things can easily go wrong in the coach-apprentice relationship. Speaking about apprenticeship in an architecture design studio, Schön explains:

> Some studio masters feel a need to protect their special artistry. Fearing that students may misunderstand, misuse or misappropriate it, these instructors tend, sometimes unconsciously, under the guise of teaching, to actually withhold what they know. Some students feel threatened by the studio master's aura of expertise and respond to their learning predicament by becoming defensive. Under the guise of learning, they actually protect themselves against learning anything new. [Schön, 1987, p. 119]

> The understandings of student and instructor are always initially more or less incongruent. Under these circumstances, miscommunication is highly probable. Its correction depends on students and studio masters being able and willing to search actively for convergence of meaning through a dialogue of reciprocal reflection-in-action. But this depends, in turn, on the creation of a behavioral world conducive to such a dialogue, and several factors may work against its creation. The student's early experience of loss of control, competence, and confidence—always present to some degree—can readily produce a sense of vulnerability that leads the student to become defensive. And the instructor may respond to the student's defensiveness ... by strategies of unilateral control that increase defensiveness and reduce the chances for reciprocal reflection. Then the stage is set for a win/lose game. [ibid., p. 137]

In order to learn to design, or to do, or apply mathematics in this kind of setting, students must learn "operative listening, reflective imitation, reflection on their own knowing-in-action, and the coach's meanings" [Schön, ibid., p. 118]. The coach, in his or her turn, "must learn ways of showing and telling matched to the peculiar qualities of the student before him, learn how to read her particular difficulties and potentials from her efforts at performance, and discover and test what she makes of his interventions" [ibid.]. This is asking a lot.

Example: coaching and apprenticeship in linear algebra

Let me give an example, again from our continuous observation of tutoring linear algebra. In the session on the general concept of vector space, which started badly with the Tutor interrogating Student C, further development led to something that looked like a coaching session. The student had to show that there is only one zero vector in each vector space. The student's attitude was gradually changing, from his refusal to prove something that was obvious for him to a collapse of understanding and to a demand that things be explained to him. Gradually, questions started to be posed by the student, not the tutor. The Tutor had to produce proofs of existence of other vectors than (0,...0) that would also be zero vectors. He proved that the function $f(x) = 0$ is the zero vector of the vector space of real functions. The Student was then left to produce a proof of the uniqueness of the zero vector. He got it wrong. He discussed it with the tutor. Made another attempt, wrong again; another discussion followed. At some point, the tutor guides the writing of a correct proof ("guides the student by the hand"). When the student is asked to repeat it, he does, and formally it is correct but the student does not see the validity of the proof; he thinks the argument is circular. A new discussion is engaged, new evidence is provided, and finally the source of misunderstanding is found by the student who makes a major distinction between "zero vector" and "vector with the zero property". The Tutor gets into speaking about neutral elements with respect to operations, gives examples of identity transformations, unity with respect to multiplication. The Student is asked to write the proof that there is just one unity for multiplication (he is thus practicing on a different material). The proof is correct.

Below are some excerpts from the session:

Session of April 29, 1994

Episode I: Student C refuses to prove

215 STUDENT C [*reads*] "Using only axioms one can show that the zero vector in Axiom 4 [*silence*] is unique". [*silence*] What is there to prove here? [*silence sings.*]

...

220 T: You don't believe that there is exactly one zero vector?
221 C: How many can there be?... 223 C. Wait! I am CONVINCED there is only one zero vector!
226 T: What is a zero vector?
227 C: One that is composed of zeros only, that's what it is!

Episode 2: Student C starts asking for explanations

243 C: Zero is such that added to another vector, it gives you the same vector.
244 T: This is all you know about the zero vector.
245 C: True.
246 T: Suppose now that you have two such vectors.

...

261 C: What do you mean: "two zero vectors"? Explain this to me!
262 T: And each time you add them to any vector u, you get the same vector.
265 C: [*Writes:* $u + \alpha^* = u$, $u + \alpha = u$] And so what?
266 T: You have to deduce from that that alpha and alpha star are the same thing.
267 C: Of course, it must be the same thing! They have the same elements
268 T: I am telling you that you cannot speak about elements!
269 C: How come they cannot have only zeros? That's incredible! We are speaking about the zero vector all the time

...

277 C: Explain it to me, how can there be a zero vector that is not composed of zeros. How can such a thing be at all possible?
270 T: Listen..
271 C: No "listen". Yes or no? .. 283. Yes or no?!!
284 T: Yes.
285 C: Okay Why? What would it look like? Give me an example. If you can't explain this to me then I don't think I'll be able to go on.
286 T: A function that assigns zero to any number, is this "composed of zeros"? Can you say that? Take a function f that, for every real number x, is equal to zero.

Episode 3: Student C loses assurance; declares that his understanding has collapsed

323 C: So what do you get? A vector whose elements are not all zeros, right? Is such a vector a zero vector?
324 T: This is the zero of this space.
325 C: This is the zero of this space, but IS THE ZERO OF THIS SPACE A ZERO VECTOR? Is this something I have to accept into my mind?
326 T: I don't know. The zero of this space is the zero function, a function that is equal to zero everywhere.
327 C: So the zero function and zero vector are synonyms at this point?
328 T: Yes. ... We can prove that this satisfies these ten conditions and therefore this is a vector space, and the elements of this space are called vectors.
329 C: (*sighs*)
330 T: These functions are vectors.
331 C: I have just discovered that my world has been built as a house of cards, and someone has just blown it away. That's really coarse. ... 335. I must say that this has really put me down. Realizing that all you've been taught and you thought you have understood, *c'est du bullshit aussi* [*silence*] Okay Let's go on. [*silence*] Okay! Don't feel guilty

Episode 4: Student C takes the problem as his own

340 T: Okay, how can we prove it just using these axioms.
341 C: How can we... Look, I propose to..., look.
342 T: Yes?
343 C: You want me to get it, to understand I have to prove it to myself because I still cannot believe it.

In Episode 5 the Tutor goes on to prove that the zero function is the zero of the space of functions, stressing very much the words "in this space", which ends up by annoying the Student. In this process, the Student starts seeing a little bit through the fog.

Episode 6: Student C writes a proof

443 C: [*sings*] To show that if you have two, then in any case there is just one. So u, you add an alpha to it, and you add a beta to it, so that we can really see they are different, and what we are looking for, *ce qu'on cherche, ... c'est que alfa est égal à beta.*

C is left alone to write the proof. He writes

Donné	*Cherché*
$u + \alpha = u$	
	$\alpha = \beta$

$$u + \alpha = u$$
$$u = u - \alpha$$
$$u - a + \beta = u$$
$$\text{donc, } \beta = \alpha + u - u$$
$$\beta = \alpha$$

In episode 7 the tutor criticizes the proof and points to the implicit assumptions the Student was making. End of the session of April 29

Session of May 1, 1994

After the Student has produced another incorrect proof of the uniqueness of the zero vector, the Tutor helps him to

correct it Then he asks the Student to repeat the proof he has just heard: So how do you prove that there is just one zero vector in a vector space? The Student is annoyed. In reproducing the proof, he feels there is a circular argument in it.

203 C: Jesus! you add alpha to beta and you get alpha You add alpha to beta, you get beta.
204 T: Why?
205 C: You are doing exactly the same operation and you get two different things. So one thing must be equal to the other thing. But does this mean that one of them must be zero?
206 T: No, listen Alpha and beta are zeros
207 C: Alpha and beta. So you assume from the beginning that they are zeros.
208 T: Both are zeros.
209 C: You assume they are zeros?!
210 T: Yes, of course.
211 C: Doesn't the assumption that both are zeros tell us that they are equal to each other?
212 T: But we re only proving this now Listen, to be a zero vector means only this, look, axiom four.
213 C: $a = 5$ and $b = 5$. Doesn't this mean that a is equal b?
214 T: No, you don't understand.
215 C: We are speaking about numbers...
216 T: No, no, no! 218 We speak about properties! Wait, let me say something!
219 C: No, I am saying that we cannot speak about numbers in the same way we speak about, whatever, cars, like that this one is a Mercedes, and this is a Fiat, and both are cars. Numbers simply don't have so many features
220 T: But we are speaking about the feature of a vector, not about numbers. What does it mean "to be a zero vector"?... This is a vector that, added to any other vector, gives you the same vector. This is a property And you are showing that there is just one vector with such a property.
223 C: The problem is that everybody knows, it is somehow imprinted in our minds, that there is just one zero and there are many even numbers.
224 T: You still think about the zero as a number, and not as a property.
225 C: AHA! AHA! Zero as a NUMBER, zero as a PROPERTY!
226 T: Zero is like a neutral element with respect to addition. Look, you can consider other systems, other operations. For example, the system of geometrical transformations in the plane.

In Episode 9 there follow examples of rotation by angle zero, 360°, translation by a zero vector, generalization to an identity mapping. "Unity" as a neutral element with respect to multiplication (in integer numbers) is also discussed The Tutor asks the Student to show that there is just one unity, in the same way that there is just one zero The student shows it easily ($A \cdot a = A$, $A \cdot a = a$, hence $A = a$), but then he argues that one should assume that neither A nor a are zero.

In Episode 10 Student C criticizes the imprecision of the Tutor's language (slowly regains the control of his thought)

282 T: But I am saying they are unities
283 C: "Both are unities". [*sarcastic*]
284 T: Unities, I mean they are such that if I multiply any of them by anything else I get this anything else.
285 C: You should be saying: "they satisfy the UNITY PROPERTY", you cannot say "they are both unities".
286 T: Okay, I'll say "unity property".
287 C: You are asking me to understand properties, and you yourself are using some kind of vulgar mathematical language.
288 T: It's true it is some kind of jargon.

This example documents two points: one, already mentioned above, the necessity and fruitfulness of coaching rather than instructing in the case of proof learning; two, the need to take into account the epistemological genesis of the more general mathematical concepts in designing the teaching of them.

When reading the axiomatics of vector spaces, Student C kept thinking about R^n Words such as "addition", "zero", etc., evoke operations on numbers (or sequences of numbers) rather than operations on functions, with which composition, limits, derivation, and integration are associated most of the time What kinds of problems ("contexts") would make the formulation of the general definition of the vector space necessary, or at least useful? Would it be enough to have students work within vector spaces necessary, or at least useful? Would it be enough to have students work within vector spaces of functions, without speaking explicitly about that, but having them notice, for example, that if certain functions are solutions of a homogeneous differential equation, then their linear combinations are also solutions of this equation? Or would this lead rather to the concept of a set closed under linear combinations? In fact, it seems that the concept of subspace (in this sense), together with the operation of taking linear combinations, is more fundamental epistemologically, if not logically, than the concept of vector space. The concept of being closed under linear combinations also yields important and meaningful results about particular vector spaces. Contrary to this, the general concept of vector space can only lead to trivial logical questions, such as that of the uniqueness of zero, or verifying that certain artificially constructed systems are vector spaces. These questions seem to be a most deplorable product of the didactic transposition—indeed, a didactic inversion of the historical genesis of concepts. It is this kind of teaching that leads to the feeling of loss of sense in students who then say that they can see no point to linear algebra [Sierpinska, 1994a]

There seems indeed to be no point in training students to prove statements that are obvious to them. In the excerpt above, the hardest thing for the tutor was to convince the student that there is indeed something to prove.

Conclusion

Let me claim here that if people care about "learning mathematics in contexts" it is because they believe it guarantees that students will engage in authentic cognitive activities, and will learn something valuable for their lives in society and in the world. But it could be that the restriction to "real-life contexts" hinders the attainment of these goals. For school would then be only reproducing the community's knowledge, it would not provide sufficient ground for the creation of new knowledge.

This is not how society sees the role of schools. Schools have always been viewed as springboards for promotion, advance: children are expected to learn more at school than their parents and the street can teach them [cf. Chevallard, 1991]. "Teaching mathematics in everyday life contexts" contradicts the way in which school is expected to function *in a society*.

There are other ways to both preserve the role of the school in society and ensure that students learn something that is not just a stack of information. We need "contexts", but only in the sense of problems that give meaning and sense to what students learn: *knowledge is always an answer to a question*. Of course, we want students to ask themselves important questions. We all agree with that; where we part is in our ideas about what is an important question.

Some say the important questions are those that lead to "useful knowledge" and opt for, if not "realistic mathematics", then at least "mathematics with applications". Others, though much less openly these days, prefer to say that "usefulness" is no reason to learn mathematics. The controversy is old and well known; let me quote two classical opponents in the discussion on this matter, L. Hogben and G.H. Hardy.

Here is the opinion of Lancelot Hogben, from his popular book *Mathematics for the million*, first published in 1937 and republished many times since then in revised version.

> In the time of Diderot the lives and happiness of individuals might still depend on holding correct beliefs about religion. Today, the lives and happiness of people depend more than most of us realize upon correct interpretation of public statistics kept by Government offices. Atomic power depends on calculations which may destroy us or may guarantee worldwide freedom from want... Without some understanding of mathematics, we lack the language in which to talk intelligently about the forces that now fashion the future of our species, if any... We live in a welter of figures: cookery recipes..., unemployment aggregates..., bank rates..., motoring records..., lotteries..., ... Ratios, limits, acceleration, are no longer remote abstractions, dimly apprehended by the solitary genius. They are photographed upon every page of our existence. [Hogben, 1993, p. 10]

And here is Hardy's reaction. He, in a direct confrontation with Hogben, calls the mathematics justified by his utilitarian arguments "trivial mathematics".

> Hogben... says that "without a knowledge of mathematics, the grammar of size and order, we cannot plan the rational society in which there will be leisure for all and poverty for none". There are then two mathematics. There is the real mathematics of the real mathematicians, and there is what I will call the "trivial" mathematics, for want of a better word. The trivial mathematics may be justified by arguments that would appeal to Hogben, or other writers of his school, but there is no such defense for the real mathematics, which must be justified as art if it can be justified at all. [Hardy, 1992, p. 136, 139]

Most of "real mathematics", "modern geometry and algebra, the theory of numbers, the theory of aggregates and functions, relativity, quantum mechanics", Hardy says, cannot be justified by their practical usefulness or contribution "to the material comfort of mankind".

> There is no real mathematician whose life can be justified on this ground. If this be the test, then Abel, Riemann, and Poincaré wasted their lives; their contribution to human comfort was negligible, and the world would have been as happy a place without them. [Hardy, 1992, p. 136]

What makes the importance—or rather, the seriousness, as Hardy prefers to say—of a mathematical theorem, is connections; but these connections need not be of an external, and certainly not of practical, nature. Paradoxically it is serious mathematics in this internal sense that seems most likely to be externally useful. [Hardy, 1992, p. 89]

We may agree with Hardy or with Hogben. We may say that important mathematics is that which has significant internal connections to other mathematics, or that important mathematics is that which has significant connections with the "real" world. Whatever we choose, however, our main preoccupation as teachers will be to obtain a more genuine interest and motivation of students in engaging in the mathematical activities proposed to them at school. This means we have to look for good questions and "contexts" in which these questions will arise.

> Not every problem, nor every thesis should be examined, but only one which might puzzle one of those who need argument not punishment or perception... The subjects should not border too closely upon the sphere of demonstration, nor yet be too far removed from it: for the former cases admit of no doubt, while the latter involve difficulties too great for the art of the trainer. [Aristotle, *Topics*, Bk. I, in McKeon's edition, 1941, p. 198]

Maybe, also, we should not worry so much about the authenticity of school tasks. Authentic activities will come later on, after school, or out-of-school. A studio master who guides an architecture student does not expect him or her to engage in authentic designing activities, which would naturally imply the responsibility for seeing that the building was built, that it stood instead of falling down. But in the studio the student gets prepared to endorse this responsibility later on. This responsibility is his own responsibility, not the master's. Likewise, the mathematics teacher at school is not responsible for what his or her students will do with the knowledge they have learned in his or her class. The students' activity will become more authentic when they accept this responsibility as their own.

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