CAS-INDUCED DIFFICULTIES IN LEARNING MATHEMATICS?

UFFE THOMAS JANKVIST, MORTEN MISFELDT

In Danish upper secondary school, Computer Algebra Systems (CAS) are now an integrated part of the teaching of mathematics. As university mathematics educators, we often hear upper secondary school mathematics teachers complain that CAS has not brought the promised wonders of deep mathematical understanding. Of course, this is probably a timeless complaint and one might expect that similar things were said when handheld calculators found their way into mathematics classrooms in the 1970s, or even earlier when tables replaced algorithms for calculating square roots. However, we recently encountered a different kind of concern from a mathematics teacher at one of the top upper secondary schools in the Copenhagen area. Having witnessed much of the introduction of CAS (*e.g.,* TI Nspire) in the upper secondary school mathematics program, this teacher told us that he had recently begun to see students have certain kinds of difficulties that he did not see when he first began as a teacher 15 years ago, and that he suspected that CAS was somehow responsible for them.

Scanning the mathematics education literature on using CAS in mathematics education, one soon finds that focus is mainly on the potentially positive impact that such use has in terms of mathematics teaching and learning [1]. A couple of decades ago, expectations were particularly high about what could be accomplished (*e.g.,* Dreyfus, 1994). Since then, research has increasingly focused on characterizing learning processes (*e.g.,* Artigue, 2002; Drijvers, 2003; Guin, Ruthven & Trouche, 2005) and, more recently, on teaching processes (*e.g.,* Gueudet, Pepin & Trouche, 2012; Tabach, Hershkowitz & Dreyfus, 2013), including the way in which these processes are different when CAS is used. It is well known that a pragmatic approach to using CAS may cause students to lose some of their arithmetic skills (*e.g.,* Artigue, 2010). However, what the upper secondary school teacher's experiences indicate is that there might be something more substantial at stake. It raises the following questions: what might be the other side of the coin of a heavy reliance on CAS in the teaching and learning of mathematics? To what extent may we speak of CAS-induced stumbling blocks, misconceptions, or learning difficulties in relation to mathematics and, in particular, to mathematical concept formation and development?

We have chosen to address these questions from a somewhat theoretical point of view, drawing on a selection of the available literature and theories of mathematics education, to see if the claim of CAS-induced learning difficulties holds water. But before we reach this point, there are some initial steps that need to be taken, involving, among other things, a clarification of what we take to be a learning difficulty. Along the way, we also examine some existing studies on CAS use in mathematics education to support our claims. First, however, we look at an example provided to us by the upper secondary school teacher—an example which, together with the teacher's experiences of something qualitatively different taking place, provides a concrete starting point for our discussion.

An example of a potential difficulty

CAS entered the Danish upper secondary school program with the reforms of 2005. Use of CAS is not mandatory, but part of the written exams are constructed assuming that students have CAS tools at their disposal. One consequence of this policy is that the introduction of CAS in textbooks, as well as in teaching, is quite diverse, and different schools or even teachers may have their own policies about students' use of CAS. If Danish upper secondary school students take mathematics at the highest level, meaning that they have mathematics for all three years of upper secondary school, they will be introduced to differential equations in their third year, and hence also to the "desolve" command in CAS.

In the teacher's example, he gave his class a mock exam, including the task:

> Given $dN/dt = -16N + 32$ with initial value condition $N(10) = 1$, find an expression for $N(t)$ and account for N being an increasing function.

Now, before the reform of 2005, students would solve such a task in one of two ways. In the first way, they would rely on separation of the variables, here N and t. In order to make our example consistent with the CAS notation, we first rename N to y and t to x. Thus, we have $y' = -16y + 32$ or $y' = -16(y - 2)$. Setting $q(y) = -16(y - 2)$ and $p(x) = 1$ we rewrite our original differential equation:

$$dy/dx = q(y)\, p(x),$$

as

$$1/q(y)\, dy = p(x)\, dx,$$

i.e. by separating the variables. This equation is solved by taking the integral with respect to y on the left hand side, and with respect to x on the right hand side:

$$\int 1/q(y)\, dy = \int p(x)\, dx.$$

Inserting the expressions for q and p into this equation gives:

$$-1/16 \int 1/(y - 2)\, dy = \int 1\, dx,$$

which is equivalent to:

$$-1/16 \ln |y - 2| + c_1 = x + c_2.$$

Hence:

$$\ln|y - 2| = -16x + c_1$$

and

$$|y - 2| = e^{-16x}\ c_2.$$

Taking into consideration the three situations of $y < 2$, $y > 2$, and $y = 2$, students are able to find the complete solution to the differential equation, as well as the particular solution with the initial value given in the problem. Although the students should, in principle, be able to do all of the above, for this specific problem they may also rely on a theorem—this method being the second way. This theorem, for which the students are also presented with a proof, says that the complete solution to

$$dy/dx = ay + b \text{ with } a, b \in \mathbf{R} \text{ and } a \neq 0$$

is

$$y = -b/a + c \cdot e^{ax} \text{ for } x \in \mathbf{R}.$$

The usual approach, the teacher explained, is to go through the above methods with the students, and only afterwards introduce them to the "desolve" command in CAS. When having to solve such tasks, however, it is entirely up to the students if they do it by paper and pencil or by using CAS.

One particular student mentioned by the teacher used CAS to solve the exam task. This approach first involves the translation of the problem into the usual variable names of the system, typically x and y. The student, however, erroneously translated the original problem into $y' = -16x + 32$, leading her to the solution $y = c - 8x^2 + 32x$ (see Figure 1). This was then the expression she relied on when solving the initial value problem, also using CAS, as well as later on in the task when having to argue that y is an ever-increasing function (!). The teacher argued that if the student had relied on a paper and pencil approach, she would not have made this error. "She does so, because CAS blurs the picture," he said [2], and continued:

> In my mind there is no doubt that if you force students to [manually] solve a differential equation of this [the above] kind, then through the algebra they learn to work towards an understanding, which will enable them to distinguish between x and y. Understanding of what is going on comes gradually by doing it over and over again. This sense vanishes completely when solving with CAS. To solve the differential equation in CAS, you just enter: desolve ($y' = -16 \cdot y + 32, x, y$) and out comes the general solution.

The teacher's point is that if this student had chosen one of the two paper and pencil approaches, she would have had the opportunity to discover her error, because these methods do not make much sense for her translation of the problem. CAS does not really provide this opportunity, since the desolve command does not require students to distinguish between different types of differential equations. And with the CAS function for finding the particular solution, another opportunity for discovering the error is potentially missed. The student only has to enter:

desolve ($y' = -16 \cdot x + 32$ and $y(10) = 1, x, y$),

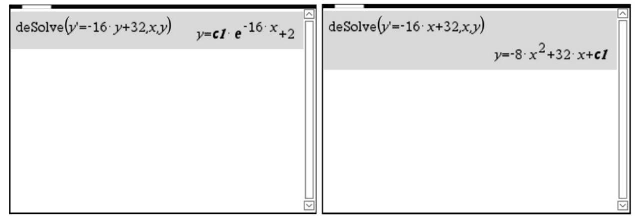

Figure 1. Screenshots from TI Nspire. The image to the left shows the correct formulation of the differential equation (and the correct solution). The image to the right shows the student's misguided translation from variables N and t to y and x.

which immediately reveals:

$$y = -8x^2 + 32x + 481.$$

Realizing that students can use software packages to solve specific problems and hence perform a mathematical activity (in this case solving a differential equation) without actually thinking or working with the concepts that they need to learn is not new. However, the problem has often been scoped locally; when a student uses a package to investigate the monotony of a function, find sides and angles in a triangle, perform long division or to solve equations or differential equations, some of the underlying routines are skipped and the student develops a different CAS-based technique [3] to address a certain problem, taking advantage of the new tool. This new, CAS-based technique may change the students' concept formation (Artigue, 2002; Trouche, 2005). In this article, we look at the potentially global effects of such changes by considering them as a source for difficulties in learning mathematics: CAS-induced difficulties.

Learning difficulties and concept formation in mathematics

We regard "genuine difficulties in learning mathematics" to be those obstacles and impediments, or stumbling blocks, which some students experience in their attempt to learn the subject (Jankvist & Niss, under review). They include a wide range of misconceptions, misinterpretations, misguided procedures, *etc.* with regard to established notions of mathematics. Such difficulties can, for example, be related to mathematical concepts and concept formation, to mathematical notation and conventions, to problem solving, to mathematical reasoning and proof, or to mathematical models and modeling. We do not include general learning disabilities, such as those of a neurological kind, cognitive or affective disorders and the like. Nor is the lack of motivation on the students' behalf regarded as part of such genuine difficulties.

In this sense, the error of the student discussed above qualifies as a difficulty in learning mathematics, since it appears to involve a potential misconception as well as misguided procedure and a misinterpretation. The fact that the student, according to the teacher, does not even seem aware of or able to understand the nature of the error being made underpins this interpretation. Of course, some might say that it is merely a "translation error" (from N to y, mistakenly translating an N to an x), but the fact that the student carries on to argue that the resulting expression is an ever-increasing function suggests otherwise. If, in fact, this were only a translation error, the student should have discovered it when attempting to interpret the CAS solution. Like the teacher, we find that there is something deeper going on here. Hence, we are inclined to regard it as a potential CAS-induced difficulty in this particular case. We continue this line of thought from a more theoretical stance.

Sfard (1991) described the development of a mathematical concept as a hierarchy of three stages: interiorization; condensation; and reification. *Interiorization* is where the learner gets acquainted with the processes which will eventually give rise to a new concept. These processes are operations on lower-level objects and, with reference to Piaget, Sfard defines a process to be interiorized if it can be carried out through acquired mental representations. For example, the concept of function operates on the lower-level objects of variable and number (value). Interiorization of the concept of function is "when the idea of variable is learned and the ability of using a formula to find values of the 'dependent' variable is acquired" (Sfard, 1991, p. 19). As another example, the concept of derivative and that of a differential equation both involve operations on functions, which in this case make up the lower-level object.

Condensation is where the learner becomes capable of thinking about a given process as a whole, without the need to go into detail. For example, thinking about the process of solving a differential equation as a whole, or as an "autonomous procedure", is when the concept of a differential equation is officially born. According to Sfard, progress in condensation will be shown by a "growing easiness to alternate between different representations of the concept" (p. 19), which is similar to Dreyfus's (1991) idea of success in mathematics as the ability to switch between several mental representations of a concept. For instance, we may be able to easily alternate between representations of a function as an expression, say $y = x + 5$, as a table of y-values resulting from x-values, as a graphical representation of this linear function in a coordinate system, and as a verbal version, say "y is 5 bigger than x". For differential equations, the alternation might involve being able to switch between algebraic representations and graphic representations such as, for example, compartment models, or to be able to switch between different ways of solving differential equations (*e.g.,* analytical, numerical, or qualitative).

The stage of *reification* is entered when a concept is detached from its related processes, and the learner is capable of grasping it as a full-fledged object in itself. "In the case of function," Sfard (1991) writes, "reification may be evidenced by proficiency in solving equations in which 'unknowns' are functions (differential and functional equations, equations with parameters), by ability to talk about general properties of different processes performed on functions (such as composition or inversion), and by ultimate recognition that computability is not a necessary characteristic of the sets of ordered pairs which are to be regarded as functions" (p. 20). Hence, the perception of a differential equation as a mathematical object in itself, builds on previous reification of the concepts of derivative, function, variable, and number (at least). Furthermore, as pointed to by Sfard, there is an inherent difficulty in reification, which she refers to as a *vicious circle,* since "on one hand, without an attempt at the higher-level interiorization, the reification will not occur; on the other hand, existence of objects on which the higher-level processes are performed seems indispensable for the interiorization" (p. 31).

From a more general point of view of understanding, that is, not specifically related to mathematical concept formation, Skemp (1976), following Mellin-Olsen's notion, distinguished between *instrumental understanding* and *relational understanding:* the first having to do with knowing what to do (hence including also procedural knowledge as well as knowledge about which procedures to apply), and the latter with "knowing both what to do and why" (Skemp, 1976, p. 20). The point made by Mellin-Olsen and Skemp is that even procedural knowledge, which consists only of "rules without reason" (Skemp, 1976, p. 20), is still a kind of understanding. It is difficult to imagine relational understanding as not involving some instrumental understanding. On the contrary, instrumental understanding without relational understanding is not difficult to imagine, particularly in a CAS-oriented teaching environment. In order to deepen this point we introduce a few relevant concepts and distinctions from the mathematics education literature: specifically, *the lever potential* and *black-box phenomenon,* as well as the distinction between epistemic and pragmatic use of technology.

CAS can perform many of the mathematical tasks that students may be expected to do. Hence, in a CAS environment, it is possible to focus the students' attention on the most relevant activity. This *lever potential* can save time, increase the mathematical capacity of each student, and focus activities in the classroom (Dreyfus, 1994). The lever potential works by outsourcing (black-boxing) certain mathematical processes, and thus directing attention away from these processes. But black-boxing is hard to control and can easily lead to problems with understanding what is actually going on. This negative black-box effect is well described in the literature on CAS and mathematics learning (Buchberger, 2002; Lagrange, 2005), and so are the results of students who are able to perform CAS-based mathematical activities, but unable to understand the underlying processes. Thus, black-boxing leaves students dependent on certain tools and with little experience of performing the low level mathematical processes that are necessary without the tool (Nabb, 2010). Also, CAS, as with any other tool, is applied for specific purposes. The distinction between pragmatic and epistemic purposes is particularly useful here (Artigue, 2002; Lagrange, 2005; Trouche, 2005). A *pragmatic* use of tools is directed towards something external to the user, the tool is used to create a difference in the external world: an example is that of using a hammer to hit nails into a wall. By contrast, an *epistemic* use is directed towards the user's cognitive system: the tool is used to create a different understanding or to support learning, such as, for example, using a magnifying glass to learn about something small (Artigue, 2002; Lagrange, 2005). CAS serves both pragmatic and epistemic purposes. Of course, any use that is only, or mainly, pragmatic is of little (or even negative) educational value (Artigue, 2010).

Problems with CAS

Following the notions of instrumental and relational understanding, as presented by Skemp (1976), what may be taking place for the student in our example is that the ill-considered use of CAS seems to deprive her of any relational understanding related to finding the solution to the differential equation. Whether or not there is any instrumental understanding present is difficult to say, since instrumental understanding in Skemp's sense does not, of course, take into account the heavy use of CAS that mathematics education is experiencing today. Nevertheless, distinguishing between relational understanding, still as knowing what to do and why (and also *when* to do what), and instrumental understanding, as rules without reason or as knowing what to do, *does* make sense in a CAS-based teaching environment. We can refer to *CAS-instrumental understanding* as an understanding of how to follow a procedure using CAS, but not knowing with what purpose or how the procedure fits into a larger mathematical landscape. Of course, a paper and pencil approach may focus only on how to follow procedures as well, but acknowledging the experience of the teacher, there seem to be new ways of working with mathematics that lead to instrumental understandings in a CAS environment. And these new ways may be responsible for learning difficulties.

Although the kind of understanding of which Skemp speaks is of a much more general kind, and not directly related to the formation of mathematical concepts, we may still find a parallel to the framework of Sfard (1991): the discussion of *process* and *object* going hand-in-hand as part of mathematical conception. Does the black-boxing of several mathematical procedures by CAS deprive students of the necessary process? And what does this mean for the objectification of mathematical concepts? Or, to use Sfard's term, for the reification? If the underlying processes are only performed rarely, then reification will be challenged. This is a simple consequence of Sfard's framework and, furthermore, it is easy to imagine that this situation will lead to a variation of Sfard's vicious circle: more and more abstract concepts are built from processes acting on lower level objects that have not been reified. Hence, some of the most classical mathematics education theories allow us to foresee some potential problems with a strong dependence on CAS.

We acknowledge that if used in a thoughtful way, CAS has a lot to offer in terms of differential equations, not least in relation to numerical or qualitative solutions. That is, they may serve as a means for providing different representations of a solution in the condensation stage. On the other hand, if the use of CAS black-boxes the analytical approach to solving differential equations, as appears to be the case for the student discussed above, then the differential equations may never even be interiorized, because students never really get acquainted with the processes supposed to give rise to the concept (such as putting in a function and checking if the equation is true). Hence, students never get close to a reification of the lower level concepts that these processes act upon, such as function, which may be one reason for the student claiming that the "solution" is an increasing function. Following Sfard (1991), this means that the differential equations cannot be condensed and most certainly not reified. If a student only possesses what we have referred to as CAS-instrumental understanding, then it appears that the conditions for concept formation and development are not present. Of course, ill-considered use of CAS is not the only example of problems with missing reification and instrumental understanding (mathematics teaching that solely focusses on algorithms, formulas, and procedures without any attention to concepts and justification are other examples), but the wide use of CAS in upper secondary school calls for an awareness of the potential problems that can follow.

In the remaining sections, we discuss two potential reasons for CAS-induced difficulties by means of two small examples. In the first example, the main problem is that the concept of the derivative, previously central to answering questions about monotony, tends to disappear from the students' work, and hence is neither condensed nor reified. The second example addresses students' tendency to consider differential equations as similar to equations.

Black-boxing fundamental concepts: the derivative

The mathematics teacher who initially directed us towards the CAS-related problems of upper secondary school described another example [4]. When students are to sketch the graph of a function by finding maxima and minima, they can just use CAS to draw the function for them and locate maxima and minima by mere inspection. In that sense students are, on the one hand, able to do the maxima and minima analysis of a given function, but, on the other hand, deprived of the insight that the derivative of a function at a given point tells you something about the original function at this point.

Using a standard CAS system, it is possible to ask for local minima and maxima without working with the derivative. Hence, a student does not need to connect to the derivative or even the concept of variable in order to answer typical questions about monotony. For example, a question like "what is the greatest value of the function (locally/globally)" can be answered by graphical inspection or with commands like max or smartplot. The corresponding x-value can be found by inspection or by using a solve command. Such techniques, we claim, are rather efficient for producing the output required from the typical student in Danish upper secondary school, and hence some students will apply them often or even always. The result of such behavior might be that the derivative is black-boxed when looking for monotony of a function. In the terminology of Sfard (1991), reification of the derivative is no longer supported when finding monotony.

Of course, questions such as "can you express the x that leads to the maximum value of $f(x)$ in terms of the derivative of f" will, once in a while, disturb a student applying such a pragmatic CAS technique. But in most cases, such questions are only steps on the way to obtaining the more global knowledge of a function (a global knowledge that CAS provides so easily). Hence, we can conclude from the monotony example that the most efficient CAS-based techniques to find monotony do not support reification of the derivative. Furthermore, tasks designed to let the students work with the derivative in an epistemic fashion (when they find monotony) no longer seem pragmatically healthy, since they are often mere steps on the way to obtaining knowledge to which the student already has access via CAS. The black-boxing of central concepts, and the resulting lack of reification, is one possible explanation for CAS-induced learning difficulties. But the example with the differential equation also tells a different story.

Blurring the difference: solve and desolve

The CAS-based technique used to address differential equations, which the student of our initial example employed, is very similar to the CAS-based technique used to solve equations (linear, quadratic, *etc.*). The underlying algebraic techniques, and the lower level objects (variable and function) they operate on, are, however, quite different. Algebraic paper and pencil techniques explicate these differences and therefore to a large extent support reification. If students mainly use CAS-based techniques to solve such tasks, then reification is potentially challenged. Of course, several other activities do support formation of the concepts of function and variable (some of these are even CAS-based, such as graphing and the use of sliders). But, in the context of solving equations and differential equations, it is a fair assumption that if a student takes a pragmatic approach, the use of CAS challenges reification and hence concept formation for function and the derivative.

Let us take a closer look at the difference between the techniques for solving differential and algebraic equations. Assume that the student in the initial example takes a pragmatic approach while having weak conceptual understandings of variable and function (at least in the context of solving these equations). Using solve and desolve is, from a procedural perspective, almost the same thing: solve (your problem, x) is a procedural and pragmatic solution strategy. The x at the end signifies the variable with respect to which you solve the equation. This is, of course, important from an epistemic and relational point of view, but we suggest that considering this x as just a part of the algorithm is a viable pragmatic approach that more often than not leads to the right solution. Hence, the student is able to solve most equations without any relational understanding of the concept of variable and without even knowing what input and output the solve algorithm needs in order to perform a mathematically meaningful task. The "algorithm" thus becomes: "if the problem involves something like an equation, then write solving (your problem, x) and hope for a solution that may be related to the problem formulation." Without much concern, this algorithm can be applied to differential equations as well, the only difference being that you have to use the desolve command and specify the function (f or y) as the unknown. The pragmatic (but very efficient) approach to solve differential equations thus becomes: "if the problem involves something like a differential equation then write desolve (your problem, f or y), where f or y is the name used in the problem formulation and hope for a solution which may be related to the problem formulation." In a sense, desolve becomes a special case of solve.

If the above line of thinking does reflect that which the student in our example has applied, then mixing up x and y is now easy to understand, since x and y are just markers that need to be placed in the right way, so that the solve or desolve "algorithm" works in the proper fashion. Hence, these pragmatically oriented techniques blur the distinction between equations and differential equations. This similarity alone does not, of course, explain how the student is able to apply a process without any consideration of the difference between equations and differential equations or the concepts of variable and function, on which they build. But the weakening of concept reification that we, with reference to Sfard (1991), suggest as part of CAS-induced learning difficulties makes such a suggestion more legitimate. Without reified objects of function and variable, our student's mistaking of x for y, can be explained by collecting the y's on the left hand side and the x's on the right hand side of the expression, since this is custom for functions, and then applying the desolve command to the resulting expression, without being entirely aware if it is an equation with a function as the unknown, or only an expression for a function. The combination of weak reification and CAS-based instrumental understanding can explain the student's error as an indication of a deeper conceptual problem than merely mistaking x and y and hence, we believe, articulate the teacher's concern about how CAS may induce new or increase already existing difficulties for students in learning mathematics.

Conclusions

This article is inspired by conversations and correspondences with teachers relating to their real world experiences with CAS in the Danish upper secondary mathematics program. Our contribution is to view these experiences through a selection of mainstream mathematics education theoretical lenses (*e.g.,* Sfard, 1991; Skemp, 1976). Using these lenses has led us to conclude that a strong reliance on CAS in upper secondary school provides new and efficient procedural solution strategies to classical problems (*e.g.,* finding the monotony). On the other hand, these new strategies sometimes black-box central concepts that are the focus of the teaching (such as the derivative). This means that students who are trying to cope with mathematics might hang on to such procedural and pragmatic approaches, since they allow them to "deliver" and participate in class. On the down side, such students may never develop good perceptions of central concepts and topics, and are not even motivated to do so if they mainly meet tasks easily handled by CAS. As expressed by the teacher [5]: "differential equations has gone from being one of the most difficult topics to now being one of the easiest", which of course is related to the desolve command in CAS. Students are also enabled to act in ways that do not suggest even the slightest understanding of the mathematical concepts involved (exemplified by variable and function above). The use of CAS and, in particular, the solve and desolve commands, suggests new classes of "similar" problems, such as equations and differential equations. Although this kind of high-level relational thinking is important and generally valuable to nurture, it can easily lead to such similarities being entirely detached from actual consideration of the concepts involved, thus breaking the hierarchical structure and logical nature of concept formation described by Sfard (1991) [6]. As previously insinuated, the conclusion seems straightforward; if deprived of an understanding of the rationale behind the involved processes, the concepts and resulting objects in question can never be truly reified. Or, in the more general notion of Skemp (1976), if students merely possess what we have termed CAS-instrumental understanding of a given topic, the road towards any relational understanding of this topic is even longer than it is via the usual paper and pencil instrumental understanding, due to the black-boxing of arithmetical, algebraic or functional operations. In a sense, CAS enables Sfard's vicious circle to turn into a kind of *vicious spiral,* where more and more higher-level concepts are built on lower-level concepts that have not been reified. This spiral of course makes it very challenging for a teacher to observe and identify a student's problem or difficulty and address it.

To draw a final conclusion, we believe that the teacher was right in his observation that CAS has brought new kinds of difficulties with it into the teaching and learning of mathematics. Furthermore, we believe that we can explain this by a combination of students' pragmatic solution strategies with non-reified mathematical concepts, leading to what we have called CAS-instrumental understanding. This type of understanding brings with it the potential difficulties observed by the teacher: CAS-induced difficulties in learning mathematics.

Epilogue

Although our conclusion suggests that CAS introduces a number of potential difficulties to students' learning of mathematics, we do not intend to say that CAS cannot support the learning of mathematics. Numerous examples of didactical interventions supporting mathematics learning exist in the literature. However, CAS-induced difficulties can result from a combination of: (i) political pressure for the use of technology in upper secondary school; (ii) the significant pragmatic value of using CAS for the individual students; and (iii) the insufficient level of adaptation of CAS in curriculum, tasks, and textbooks. So, to end where we began; if students appear to be having new kinds of difficulties, it is not the students who are to blame. Neither is it the teachers, nor necessarily CAS itself. The problem is, as far as we can tell, *how* CAS is used as a result of the way curriculum and policy documents prepare the grounds for its use and the way it is then implemented in textbooks.

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Notes

[1] Examples are: Hoyles and Lagrange (2010); Laborde and Strässer (2010); and Trouche *et al.* (2013).

[2] The quote is taken from email correspondence with the teacher.

[3] In the instrumental approach (Artigue, 2002; Trouche, 2005), the standard terminology for techniques involving digital tools is "instrumented technique". We do not adopt this terminology, since we use the word "instrumental" with reference to Skemp's disitinction between instrumental and relational understanding.

[4] The concern that finding max and min can be done without learning about the derivative may sound upside down: of course, the concepts of maximum and minimum can easily be understood without working with the derivative. However, the Danish upper secondary curriculum, assessment tradition, and textbooks, have the investigation of a function using analytical and algebraic means as a central task. When working with such a task, the students find the zeros of the function and investigate where the function is positive/negative. They find the derivative, and the zeros for the derivative, and investigate where the function is increasing and decreasing, and hence are able to sketch the graph of the function with correct zeros, maximum, and minimum.

[5] The quote is taken from a follow-up conversation with the teacher.

[6] Of course, the technological development always has some effect on the importance of various techniques and elements in a mathematics curriculum. As an example, it is generally acknowledged that the widespread availability of calculators has dramatically decreased the importance of training arithmetic algorithms: and of course CAS can have similar effects in a completely healthy manner.

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