THE PEDAGOGIC USE OF LANGUAGE: INTERPLAY BETWEEN THE TWO MODES OF WHAT IS SAID AND WHAT IS SEEN

DAVE HEWITT

> We name things for reference, and hopefully for ease of reference, to draw attention to the thing named. But naming also classifies and hence causes us to look at the named thing in particular ways, the chosen symbol stresses some and ignoring other attributes of the named object. (Pimm, 1987, p. 127)

This article looks at some of the dynamics between "the thing" and the naming of that thing, and how this dynamic plays a role in students' learning and meaning making of what they hear and see. I will consider the interplay between the two modes of what is said and what is seen. In doing so, I want to highlight ways in which language can be used as a pedagogic tool in the teaching and learning of mathematics.

Language has been the focus of much research in mathematics education (for example, Barwell, 2014; Morgan, 2013) but relatively little attention has been given to its use as a pedagogic tool in relation to the two modes of what is said/heard and what is seen. I am proposing that the language used within teaching can be carefully chosen so that there is opportunity within the language of what is said for a learner to make mathematically appropriate connections and abstractions with what they are seeing or doing at that moment in time.

The introduction of conventional mathematical notation is not an easy one for many learners. Farrugia (2017) gives an example of introducing arithmetic notation, such as $6 - 4 = 2$, to five-year-olds through an activity of a frog jumping along a physical representation of a number line. The frog's movements, along with accompanying gestures, were linked with the new mathematical symbols. This is an example of what Arzarello (2006) calls a *semiotic bundle,* where gestures used within a mathematical context are accompanied by other words or signs. A teacher has a role to act as a cultural mediator (Mariotti, 2009), helping connect personal and idiosyncratic ways of expressing mathematical activity to the socially constructed mode of conventional mathematical notation. Farrugia (2017) talks about one aim of a teacher being "to support student appropriation of culturally shared, or 'institutional' meaning of signs" (p. 3). Signs and symbols are socially agreed upon and consequently are, in that sense, arbitrary (Hewitt, 1999). Thus, learners will need to be informed of these signs and symbols in some way or another. This means that a teacher has the role of looking after the symbols (Tahta, 1989), managing their introduction and helping students use and adopt them. I offer a series of examples of interactions with learners of varying ages to highlight the interplay between what is said and what is seen. Language, as a teaching resource, is just like any other resource; it has its own affordances and constraints (Gibson, 1977). What is said or not said, and the context within which that happens, is a pedagogic tool for a teacher.

The two modes: differences and difficulties

I start with two questions. The first question is spoken, the second is written:

> Spoken – "What is next in this sequence: *one, ten, hundred, thousand,…*"

> Written – What is next in this sequence: *1, 10, 100, 1000,…*

My experience is that many students respond by saying "million" to the first question. Such a response is understandable given that the next new name is "million". With the second question, I do not get *1 000 000* as an answer, I get a *10 000*. The questions are, essentially, the same, just asked in a different form. The nature of the form can trigger different associations and also offers different affordances for noticing patterns. I observed a Year 8 student (12–13 years old) trying to solve the equation $4m^2 = 100$. They said, "Squared, times by itself. Yeah. And then a hundred divided by four is twenty-five. And divided by itself would be one". If squaring is *times by itself,* then why would the inverse process of square-rooting not be *divides by itself*? The language can seduce someone into thinking that this would be the case.

This can happen at all levels of mathematical learning. A colleague of mine reported to me about an undergraduate student who, on hearing "the fifth power", wrote $x^{1/5}$. The fact that in English the same word is used for the ordinal number and fraction creates a possible association in the spoken mode. In the written mode things can be different, with the x^5 notation sometimes being associated with multiplication due to $5x$ also having a number and a letter alongside each other. Each mode offers its own set of possible associations.

How we say what we see can change the way in which we view something. I have worked with 11–12 year olds with a short activity where I claim to be amazingly good at division. I challenge someone to tell me a really hard division to do and I say I will write the answer on the board within three seconds.

They do not believe me. One of them says "Do four hundred and thirty-one divided by six". My face grimaces for a couple of seconds and then I quickly write $^{431}/_{6}$, saying "Four hundred and thirty-one sixths" and ask for the next division. They laugh. But this is often followed by some comments along the lines of "Ohhh… does that mean division?" It is the first time for many of them that they associate a fraction with division. It usually only takes one or two of these for a student to come up to the board and show everyone that they are just as amazing at division as I am. Several students come up and have a go. The students often continue laughing but I ensure, amongst this laughter, that the question is asked as a division and the answer is stated as a fraction. The form of $^{431}/_{6}$ is simultaneously the question and the answer. Although what is written remains the same, what is said changes from a question to an answer; between a process (division) to be carried out, to a single object (fraction). The ability to have a proceptual view (Gray & Tall, 1994) of expressions is important, seeing an expression both as a process to be carried out and as an object in its own right.

I now consider how the language choices a teacher makes can change the affordances available for students to make mathematical connections and abstractions.

Teacher choices

I observed a student teacher showing the rest of their cohort a metre ruler which was divided equally into ten sections; the boundary between each being marked with a line. This student teacher said that one end was zero and the other end was one. They then pointed to various marks and their fellow students called out "nought point four", "nought point two", "nought point seven" accordingly. The right-hand end of the ruler was then pointed to and there was a discussion about the fact that some school students might say "nought point ten". One of the student teachers commented that this might be a correct statement; it is just a matter of how that gets written. Since the previous numbers, such as "four" in "nought point four", are placed in the tenths column, then the zero of the "ten" should start there as well just as you would respect columns when writing 4 and 10 (see Figure 1).

The language of "nought point ten" might offer possibilities for learning but it is not a conventional way of saying the number. With this same metre ruler activity, there is another choice that a teacher could make regarding the language used. Naming can be done using "tenths" rather than decimals; in which case the end point on the ruler becomes "ten tenths". Teacher choices about what is said, in this case fractions or decimals, can affect what mathematics is stressed or avoided. Watson, Jones and Pratt (2013) discuss the issue of how the gradient of a straight line might be expressed as either $\frac{3}{2}$, 1.5 or $1\frac{1}{2}$. They point out that only the first option carries with it a sense of the structure of "up over along", or "rise over run". The form matters, whether that be the written form or the choice of how something is said. Pimm (1995) makes the point that "the belief that 'a rose by any other name would smell as sweet' misses a central part of the experience of 'doing' mathematics" (p. 5). The form matters and, in this case, the form that is used to express the gradient of a line, matters.

The language of number names can either support the learning of number work or create some potential obstacles

0		"nought"
•		"point"
	4	"four"

0 • 4	Resultant number

0		"nought"
•		"point"
1	0	"ten"

1 • 0	Resultant number

Figure 1. In which place value column is the number "ten" written?

(Cankaya, LeFevre & Dunbar, 2014; Miller, Smith, Zhu & Zhang, 1995). The language of "twenty plus twelve" offers few clues that the answer will be "thirty-two". Pedagogic choices can be made, however, to utilise what regularity there is within the language to try to make the structure of our number system more transparent for learners. I have argued elsewhere (Hewitt, 2017) that after learning the number names of "one", "two", "three", …, "nine", the next name to be taught might not be "ten". The language of "ten", "eleven", "twelve" and into the *-teens* is irregular and does not offer a learner the opportunity to see the more general pattern of *name-ty name*. If the basis for choosing the order of introducing number names was a pedagogic one of allowing learners to see the general structure of how numbers are named, then a choice of "hundred" might be more appropriate than "ten". The Tens Chart (Figure 2), which was introduced by Gattegno (1974), puts numbers into a structured array which is based upon what I call the digit names ("one", "two", "three", …, "nine"), horizontally, and the value names ("-ty", "hundred", "thousand", "tenths", "hundredths", "thousandths"), vertically. If all the numbers were to be said regularly, then every number from 0.001 to 9999.999 can be said using only 15 words. Why not give learners this powerful sense of generality first before dealing with the exceptions?

There are pedagogic ways in which language, and the structure of the Tens Chart, can be utilised in order to assist learning. For example, when introducing the names for decimal numbers, I might do the following:

1. Tap on 6 and say "six".
2. Tap *up* two rows to 600 and say "six hundred".
3. Tap on the 6 again and say "six".
4. Tap *down* two rows and say "six hundredths" (with an emphasis on *-ths*).

This choice allows a relationship to be established not only between the similar sounding words of *hundred* and *hundredths* but also accounts for why they might be connected in terms of their relationship to the initial digit row. Pimm (1991) states that "teachers, in order to teach, need to acquire linguistic strategies" (p. 167). He was referring to how the use of language can direct student attention to certain things which have been said, or the nature of what has been said. In this case, I am attempting to make selective choices about language in order to stress mathematical structure. The use of language can be a premeditated strategy which forms part of the planning of activities. Taking this further, a subtle change in how things are said in the example of six hundred above, can have a significant shift in mathematical activity. If I make

1000	2000	3000	4000	5000	6000	7000	8000	9000
100	200	300	400	500	600	700	800	900
10	20	30	40	50	60	70	80	90
1	2	3	4	5	6	7	8	9
.1	.2	.3	.4	.5	.6	.7	.8	.9
.01	.02	.03	.04	.05	.06	.07	.08	.09
.001	.002	.003	.004	.005	.006	.007	.008	.009

Figure 2. The Tens Chart.

the movement of two rows up and say "six hundred" whilst pointing to 600, then the words will appear to name the sign 600. However, saying "six" when pointing at the 6, and "hundred" during the act of tapping up two rows, has the potential of making *hundred* an operation—something that I do to six (see Hewitt, 1998, for more on this). Likewise, the other value names can be seen as different operations of moving up and down the chart. Numbers can be seen to have a variety of possible names, each of which can reveal something different about the number. For example, the number 600 can be arrived at by starting at 6000 and moving one row down. In this case, the downwards movement is the operation *tenths*, and so 600 becomes *six thousand tenths.* The name reveals how many tenths there are in 600. The dynamic between what the two modes of what is said and what is seen is highly significant in the potential mathematics which can develop from the activity.

Language as a pedagogical tool

Language also offers the possibility of getting a lot from a little. I was talking with a child who had just turned five years old one evening when I asked them what five times three was. After getting "fifteen" as a response, I then asked what was five thousand times three, to which they replied "fifteen thousand". The questions of what was five million times three and five trillion times three were also answered correctly. I was quite sure that this was just a word game, but one which took them into a realm of being able to answer many mathematical questions from knowing just one such answer. Foster (2020) warns about the possible dangers of just following patterns, including language patterns like this. However, I argue that there can be merit in learners being taken on a journey and finding themselves in a new place as long as there is a time for making sense of where that new place is in relation to what was previously solid ground for them. In this case, there is an everyday reality which can act as a check of what they find themselves saying: if I have five of something and I have three lots of those, then I know I have fifteen of them. It is of no concern what the "them" are. They could be cars, pencils, chairs, umbrellas, eighteenths, cups, million billions, iguanas, houses, or the supposedly tricky x. I do not have to know anything about iguanas, or any other of those things, to know *for sure* that if I had three lots of five of them, I would have fifteen of them. To treat 5×3 in any other way is a disservice to the statement.

There is an issue for me here about what is considered to be *difficult* for a learner. Too often, the size of a number is considered to be a deciding factor in what is deemed to be appropriate or not appropriate for learners of certain ages. Many curricula are constructed with the size of number determining the level of difficulty. However, seeing "five times three" as a statement of generality means that "five thousand times three" or "five million times three" is no more challenging than applying "five times three" to a "real-life" context such as having a set of five pencils three times.

Choices about language when planning tasks and activities are important and so are choices made during interactions with learners. I will consider an example related to an algebraic equation. Below is an interchange I had with a Year 8 (12–13 year old) student whilst they were trying to solve the equation $2m - 2 = m + 8$.

Student Okay. I'm thinking if get rid of the *m* on this side [*the right hand side*]; put the one, the one inside. So, you do minus *m*. So, you don't have to leave minus *m* from that. So, it just be *m* [*on the left hand side*]. And then, it'll be *m* minus 2. Just write *m* minus 2 equals 8. So, then, I think it would minus the 2, and then, so that you'd minus the 8, which should be 6. So, then that *m* would equals to 6.

DH So, you just talk me through that last bit that you did?

Student I did [*pause*] take away the 2. So, there's no minus 2 on that side. But then, you also just have to take it away from the 8 as well, which should equal to 6. Yeah. So, then, *m* would equal to 6, which is *m* is the same as 6. So, I moved back.

DH So, is there any way we can check this by the way?

Student I'm not sure. You could. Yeah. You could try *m* as 6. So, you do 6 times 2, 12. That should take away 2 equals 10. And then 6 plus 8. That's not right.

During the next few interchanges, the student realised that the last stage, where they subtracted two, might be where something was not right. However, they could not think of any other step to do. We then continued as below.

DH [*pause*] If I say to you, I'm thinking of a number, I take away 2, and I get 8. Could you tell me what my number was?

Student Oh 10. Yeah.

The student then looked back at the equation and saw that they added to the 8 rather than subtracted.

This interchange has different aspects to it. Firstly, there is a shift of mode from what is written to what is said. Secondly, there is the introduction of a narrative where there is now a person involved who is talking about doing things with numbers. Both are significant. The first helps take attention away from what is written and with which they were stuck. However, they may still carry with them the same, incorrect, thoughts. The second brings in a narrative which can help the student start afresh by listening to this story as if it were something new, rather than remaining stuck with previous thoughts. What struck me about this interchange was that whilst we were talking, we had been looking at the written notation. However, when I said "If I say to you…", the attention went to the verbal language and the new narrative, which helped them sort out what the right answer would be. They then returned to the visual notation to "see" this new awareness within the written notation.

Figure 3 shows this dynamic between the two modes where being stuck in one can lead to a pedagogic decision for the teacher to shift attention into the other mode. The new awareness gained is brought back to the first mode where, in this case, they see their new awareness within the written notation.

The spoken mode can be used to assist interpretation of the visual mode. Speech is said in time, and the temporal nature of speech can be utilised as a pedagogic tool. There is a difference between saying "three times *x*… plus four" and "three times… *x*plusfour". The three dots are meant to indicate a pause and the final "*x*plusfour" would be said quickly as if it were one word. Some languages, such as Finnish, have a habit of combining two or more words into a single word. An example is "omakotitaloalue", which is the name given to an area where there are separate private houses (as opposed to apartments). This is a combination of "oma" (private), "koti" (home), "talo" (house) and "alue" (area), all put together into a single word, a single entity. Similarly, "*x*plusfour" can be considered a single entity upon which the operation of *three times* is performed. In this way, the use of language can help a learner develop a more proceptual view of the expression, as an object as well as a process. The expression does not have to be said in the left-to-right order but can, instead, be said in the order of operations whilst pointing at the appropriate signs. Speech allows time to be utilised as a pedagogic tool, which cannot be so easily achieved with written expressions. The attribute of time from the spoken mode can be taken into the written mode. Just seeing $3(x + 4)$ as a static expression makes it tempting to read in the left-to-right order. However, if the expression were to be written in the order of operations, rather than the usual left-to-right, then a learner can see the order of operations in the way in which it was written. Start with $x + 4$ and then add brackets, saying "multiplied by", and only then write the 3.

The dynamic between these modes and the "loose" nature of notation (Pimm, 1995) can be pedagogically exploited. A six-year-old, who was confident with subtracting certain numbers, came to me saying that they did not know what to do with 10 – 100. I decided not to start a conversation about negative numbers. Instead, I asked how much they *could* take away from ten, and they said ten. I then asked how much more they still needed to take away and they replied ninety. I then said that I would write down –90, whilst saying that we still need to take away ninety. The six-year-old seemed perfectly happy with this and proceeded to do other subtractions resulting in negative number answers successfully. The language I used stayed with the "take away" context and so presented the written notation of –90 as something which was yet to be done. Thus, there was a sense

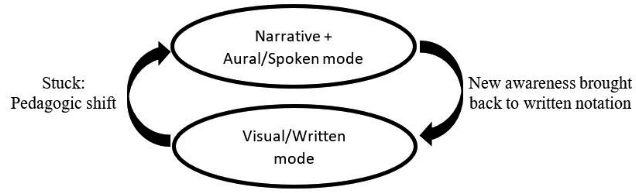

Figure 3. Shift from one mode to another and back again.

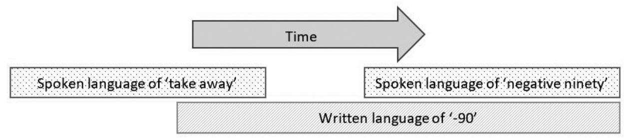

Figure 4. The temporal overlap of modes used for pedagogic purposes.

of –90 not being an answer as much as it being a placeholder for something which still needed to be carried out. There was no sense in this exchange of –90 being a number. There will be a time later when the notation –90 can be renamed as "negative ninety" and developed into being known as a number in its own right. Meanwhile the way in which the *take away* language was used enabled them to accept –90 as a notational answer to the question. Carrying the language of what is yet to be done from the spoken mode, whilst writing what might be considered by others to be a numerical "answer" in the written mode, allows the establishment of new notation without the need to engage in conceptual issues about negative numbers. As time continues, the increasingly established notation can be carried on whilst a new name is given to that, now familiar, notation (Figure 4). At each point when an overlap begins, the familiarity of one mode allows the acceptance of a change in the other mode. What keeps the modes joined is the use of association between the new and what is already familiar. Initially, it is the spoken *take away* which is familiar and later it is the newly establish written notation of –90 which is familiar and for which a new spoken label of "negative ninety" can be introduced. The way in which it is spoken, "negativeninety" rather than "negative… ninety", can enhance the sense of this being an object; a name for a number rather than an operation.

How does a learner express what they do know about subtraction in the new situation where the subtrahend is greater than that minuend? An initial response of "I cannot do it" brings that matter to a premature stop. A teacher might choose not to introduce such subtractions because they feel the learner "cannot do those yet". This slippage in language can allow learners to use what they do know about numbers and subtraction in this new situation without the need to learn a whole lot of new mathematics beforehand. It makes 10 – 100 accessible within their existing conceptual mode. Bruner (1966) stated that "any idea or problem or body of knowledge can be presented in a form simple enough so that any particular learner can understand it in a recognizable form" (p. 44). The pedagogic challenge is one of making 10 – 100 accessible to a learner who does not know about negative numbers, and doing so without having to teach negative numbers first.

A similar situation occurred when a seven-year-old wanted some help with divisions. When doing 27 ÷ 3, they counted up in threes and made a note on paper of how many threes they had counted. They then asked me for a hard one to do. I decided to show them how I would do something like 28 ÷ 3 and copied what they had done earlier until I got to 27 and said that I had counted nine 3s so far and wrote 9. I then asked what I had left, and they could reply that there was one left. So, I wrote 1 a little smaller and to the right of the 9 (thus making 9^{1}) and asked them what I was dividing by. When they said “three”, I finished off by drawing a division line (thus making $9^{1/}$), saying “divided by”, and then 3 underneath, (thus making $9^{1/3}$), saying “three”. I explained that this is how we write this when we have not been able to finish the division and we still have that to do. I then gave some other divisions to do, such as 28 ÷ 5, and they were successful at doing these, writing the answer in this standard notational way. As with the subtraction case, the notation was written as a way to express something which was yet to be done. As such it was a verb rather than a noun. It is another matter for this notation to become associated with a single number rather than an unfinished process. This can be addressed when the time seems appropriate. I would argue that introducing this standard notation can be done with appropriate use of language and avoids the unnecessary, in my opinion, use of the “r” notation for “remainder”.

Summary

Through discussion of a series of anecdotal episodes from interactions with learners, I have sought to bring out the following themes concerning the two modes of what is said and what is seen:

- Each mode brings with it the potential for a learner to make associations and connections, not all of which are mathematically appropriate;
- How something is said can change the mathematical meaning given to what is seen;
- Different language choices made by a teacher can change what is available for learners to make mathematical connections and abstractions;
- The form of what is said or written is important;
- Careful choice of language and associated actions can help learners perceive mathematical structure within what they are seeing;
- Exploit the generality within a statement: get a lot from a little;
- Patterns within language can be used to enable learners to find themselves in new mathematical terrain (but be sure to help them relate this new terrain to what is solid ground for them);
- Question the received wisdom of what is considered to be too difficult for learners;
- Shifting from one mode to the other can bring insights which can then be brought back to the original mode;
- The use of time can be a powerful pedagogic resource;
- The loose nature of notation can be exploited when signs are introduced. Meanings for the signs can shift over time;
- Some notation can be viewed both as a process (yet to be done) and also as an object (the result of that process). A change in language can shift whether the notation is viewed as a process or an object;
- Language is an important pedagogic tool.

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