ANALYZING PROOF TEACHING AT THE TERTIARY LEVEL USING PERELMAN'S NEW RHETORIC

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In this article we propose and motivate the use of 'The New Rhetoric' of Perelman and Olbrechts-Tyteca (1969) for designing and analyzing the teaching of proof. This may seem a surprising choice, if one accepts Russell's (1917) claim that "Mathematics, rightly viewed, possesses not only truth, but supreme beauty […] like that of sculpture, without appeal to any part of our weaker nature […] sublimely pure, and capable of a stern perfection such as only the greatest art can show" (p. 60). This perception of mathematics as an almost divine art seems to stand in contrast to rhetoric, the ancient art of persuasion that over the centuries became mostly related to the study of style and expression, and perceived as dealing with the ornamental and artificial aspects of discourse (Perelman, 1974). In order to see that this contrast is only apparent, we briefly consider the origins of rhetoric and its relation to dialectic.

Rhetoric and dialectic—illusion of a conflict

For the ancients, dialectic was the technique of discussion and debate, and rhetoric was the theory of persuasive discourse, aimed to provide ways to conduct a debate with various points of view, leaving the decision to the audience. Dialectic follows the norms of logical rationality whereas rhetorical argumentation considers situations and follows suitable social norms. Dialectic employs plain, technical language whereas rhetoric accommodates language for persuasive purposes (Leff, 2000). These distinctions hint why dialectic came to be considered superior to rhetoric, with the consequence that, in the course of history, rhetoric became a symbol of outdated education and has been viewed as ostentatious and artificial (Perelman, 1974).

Presumably, opposite to rhetoric stands mathematics. The conception of mathematics as a discipline in which every claim needs to be proved, started in the era of Greek mathematicians and shaped mathematics as dealing with abstractions based on deductive reasoning. Practicing mathematics can almost be identified with formulating and proving mathematical statements, but surprisingly, no globally accepted definition of mathematical proof exists.

A common definition is "a formal and logical line of reasoning that begins with a set of axioms and moves through logical steps to a conclusion. A proof, once given, is permanent" (Griffiths, 2000, p. 1). However, quite different accounts of proof have been given by mathematicians and philosophers, some of whom have focused on the social validity of proof. For Hersh (1993) "a proof is just a convincing argument as judged by competent judges" (p. 389), namely experts recognized by the current mathematical community. This invalidates Griffiths' idea of permanence. Hardy (1929) characterized proofs as "rhetorical flourishes designed to affect psychology, pictures on the board in the lecture, devices to stimulate the imagination of pupils" (p. 18). Finally, Krantz (2007) considered proof as a "psychological convincing device" (p. ix) emphasizing the contextual properties of a proof and referring to it as a communication device that should be evaluated by its impact on the intended audience.

For Griffiths (2000) the purpose of proof is establishing certainty. However, Avigad (2006) argues that if the sole purpose of a proof were establishing certainty, mathematicians would not provide different proofs for the same theorem and Netz (2005) suggests treating mathematical texts as stories, relating to their inherent rhetorical elements which influence the reader's appreciation and have a pedagogical value. The views expressed by Hersh, Hardy, Krantz, Avigad, and Netz connect proof to rhetoric as they relate to the context, the audience and literate ways of communicating a proof. Indeed, over the last few decades, scholars questioned the separation between rhetoric and mathematics, and discuss its intellectual, pedagogical, and sociocultural consequences.

A pioneering effort to show the role of rhetoric in mathematics was made by Davis and Hersh (1987). They claimed "that mathematics is not really the antithesis of rhetoric, but rather that rhetoric may sometimes be mathematical, and that mathematics may sometimes be rhetorical" (p. 54) and explored the use of rhetorical modes of argument during a proof presentation in a college mathematics lesson. The wide range of use of concepts from rhetoric in mathematics education research is further elaborated by Gabel (2019).

In summary, an exhaustive account of mathematical argumentation should consider rhetorical aspects. This is particularly true for proof and its teaching. Toulmin's (1958) model is often used in mathematics education to efficiently account for the structure of the argumentation, but it ignores other aspects, such as factors that influence students' acceptance of the proof or rhetorical features of the presentation.

The New Rhetoric (PNR) and its adaptation to the analysis of teaching proof

'The New Rhetoric' of Perelman and Olbrechts-Tyteca (1969), henceforth PNR, is a comprehensive argumentation theory that relates to important aspects such as audience, basis of agreement and forms of presentation. Perelman's aim was to show how choices, decisions, and actions can be justified on rational grounds. Thus, he turned to the classical disciplines of dialectic and rhetoric, which he combined with ideas from formal logic. Like Toulmin's work, PNR extends the range of argumentational practice, emphasizes jurisprudential approaches to reasoning, and has become a key factor in the development of argumentation theory as an independent discipline (van Eemeren *et al.,* 2013). However, unlike Toulmin, it has not yet been used in mathematics education.

PNR assumes the existence of a person, an arguer, who addresses an audience by speech or in writing. Perelman (1974) describes PNR as

> the study of discursive techniques that aim to provoke or to increase the adherence of men's minds to the theses that are presented for their assent [...] the conditions that allow argumentation to begin and to be developed, as well as the effects produced by this development [...] it is not concerned with the forms of discourse for their ornamental or aesthetic value but solely insofar as they are means of persuasion [...] through the techniques of presentation (para. 1–3).

The 'starting point of an argument' is a central notion related to establishing a shared basis of agreement with the audience, which is mandatory for successful argumentation. The presentation of the arguments is associated with their scope and organization, and with ways of creating presence to arguments. These are tied to a conception of what the arguer believes that the audience will accept.

In the three sub-sections of this section, we discuss central PNR notions together with their adaptation to the context of proof teaching.

Audience

Whereas in formal logic a valid deduction is supposed to be compelling to anyone who accepts the formal system concerned, according to PNR argumentation must be designed to achieve a particular effect on those for whom it is intended and therefore needs to be adjusted to the audience's previous knowledge, experience, expectations, opinions and norms (van Eemeren *et al.,* 2013). An arguer should constantly construct arguments that will persuade two types of audience: particular and universal. The particular audience is a specific group of people with specific characteristics that the arguer explicitly addresses. However, an arguer may use argumentation that transcends such particularity, and address a universal audience consisting of all people that they consider competent.

Like particular audiences, the universal audience is not absolute but is a subjective construct of the arguer, taking into account the goal of the argument. In the context of the mathematics classroom, the particular audience is comprised of the students attending the lesson. The universal audience may include expert mathematicians and other colleagues who share the methods, conventions and socio-mathematical norms of the lecturer.

Basis of agreement

According to PNR all argumentation must be based on premises accepted by the audience. Perelman and Olbrechts-Tyteca call these premises 'the points of departure' or 'objects of agreement'. An arguer who relies on premises not accepted by the audience commits an argumentative fallacy, though not necessarily a mistake in formal logic. Two classes of premises are defined in PNR: premises relating to the real and premises relating to the preferable.

Premises relating to the real are assumed as accepted by the universal audience and include: *facts, truths* and *presumptions* [1]. *Facts* and *truths* are considered to require no further justification. *Truths* are connections between *facts*. A premise is considered as a *fact* only if it expresses a universal agreement. However, if a *fact* is questioned later (*e.g.,* if the audience raises doubts) it might lose its "privileged status" and can no longer be used as a starting point of argumentation. Everything just said about *facts* is applicable to *truths*; still, the distinction between *facts* and *truths* is helpful since *truths* represent wider systems. Following Perelman we consider newly established mathematical results as *truths*, as they express connections between well-established prior results that may be considered as *facts*. *Presumptions* concern the usual course of events and are usually "admitted straight away as a starting point for argumentation [and] connected with what is normal and likely" (1969, p. 71). However, what is 'normal and likely' may depend on beliefs and conventions shared by members of a particular society (*e.g.,* scientific, juridical).

Premises relating to the preferable concern the preferences of a particular audience and include *values* and *value hierarchies. Values* relate to the preference of one particular audience as opposed to another and serve as guidelines for making choices, but are not binding. *Values* determine what a certain audience will be inclined to accept. *Values* are normally arranged in *value hierarchies,* which are often characteristic of different audiences that may possess the same set of *values* arranged in different *hierarchies*. *Value hierarchies,* like *values*, generally remain implicit. We adapted PNR's classification of types of premises to the analysis of proof presentation as described in Table 1.

There may be a lack of agreement, or gaps, between the lecturer and the audience of students concerning certain premises. For example, the lecturer may consider as a *fact* something that the audience expects to be proved. The lecturer may use a premise considered irrelevant by the audience. Even if they have the same values (for example, (1) every claim in a proof should be justified; (2) cumbersome technical calculations may be removed from proofs) the lecturer's value hierarchy may place (2) over (1) while the students' hierarchy may place (1) over (2). In this situation there is gap between the lecturer's and students' *value hierarchies.*

Presentation

In PNR, the form and substance of the discourse are not separated. On the contrary, stylistic structures and figures are

Table 1. Adapting PNR types of premises to a proof teaching context.

Adaptation to the analysis of proof presentation	Premise type	
Axioms, definitions, givens, accepted results, established notations	*Facts*	**Premises relating to the real**
Lemmas, theorems, new results	*Truths*	
Judgements about using previous knowledge, such as appropriate examples, useful techniques and proving methods, meta-proof features (*e.g.,* structure, main ideas)	*Presumptions*	
The adaptability to a particular audience of a certain proving method or presentation; beliefs about mathematics and its teaching and learning	Mathematical and didactical *Values*	**Premises relating to the preferable**
The order of preference of a given set of mathematical and didactical values	*Value hierarchies*	

studied with relation to their purpose in the argumentation. However, form is subordinated to content, to the effort to persuade and to convince (Perelman, 1974).

Scope and organization A discourse that seeks to persuade or convince requires an organization of the selected arguments in an order that will strengthen them. PNR studies challenges raised by the scope of the argumentation, the choice of the arguments and their order. While formal proof is most admired when it is brief, argumentation with more arguments can be expected to be more effective in reinforcing the adherence to a thesis. However, in practice, there are bounds to the scope of an argumentation, such as limits of time and space and limits to the capacity of an audience to pay attention (Perelman, 1974). Thus, the arguer must consider that arguments do not have equal strength, that arguments that appeal a larger part of an audience are superior, and that the order of arguments affects the persuasiveness of the discourse.

Creating presence The arguer may select certain elements and focus attention on them by endowing them with ‘presence’. Presence is a product of style and delivery, an outcome of the persuasive strategies which make the audience perceive, discriminate and remember the main ideas and lines of an argument (Karon, 1989). According to PNR, presence can serve as a significant psychological element in rhetoric. Perelman (1982) affirms the usefulness of presence, especially in the first stages of argumentation, as one way to establish a shared basis of agreement. If premises are not adequately accepted by the audience, the arguer should reinforce them by endowing them with presence before using them as points of departure. Elements may be endowed with presence through rhetorical figures, such as a vivid picturesque description, metaphors and analogies and repetition. Other means for creating presence are building an argument through a shared arguer-audience effort, claims with multiple warrants (possibly of different types) and humor.

Elsewhere, we describe how we have used the theoretical framework of PNR to analyze rhetorical aspects of mathematical proof presentation at the tertiary level. In particular, we analyzed: (i) the scope and organization of the argumentation and the presence of different types of elements of the proof presentation; (ii) the establishment of a shared basis of agreement between the lecturer and the students and potential gaps in this basis, and (iii) the inherent tension resulting from the need to present a proof that is both, convincing to the universal audience and persuasive to the particular audience, in our context the students.

Using PNR to analyze the flow of a proof—an example

In this section we present an example for a PNR analysis of a short episode from a lesson in set theory taught by Rachel, a highly experienced mathematics lecturer, to prospective secondary mathematics teachers. The analysis of the proof presentation uses a concept we call ‘the flow of a proof’. The ‘flow of a proof’ consists of:

(i) presentation of the logical structure of the proof (arranging the proof of the theorem into claims, which are proved in a specific order);

(ii) the way informal features and considerations of the proof and proving process (such as examples, intuitions, diagrams, analogies, motivations) are incorporated into the proof presentation.

Aspects (i) and (ii) are an outcome of the choices made by the lecturer, taking into account mathematical and instructional contextual factors (such as students’ previous knowledge and background, curricular requirements). Gabel (2019) offers a more complete explanation and elaborates a methodology to analyze rhetorical aspects of the flow of a proof within the theoretical framework of PNR.

Here we discuss an example from the lesson Rachel taught, about De Morgan’s laws for sets:

$$(1)\ (A\cap B)^c = A^c \cup B^c\ ;\ (2)\ (A\cup B)^c = A^c \cap B^c$$

where A^c denotes the complement of A. The analysis presented here relates to a discussion of the generality of Euler diagrams (frequently also called ‘Venn’ diagrams).

Rachel asked the students to sketch all possible reciprocal situations between two sets *A, B,* represented by Euler diagrams, and conducted a back and forth discussion, encouraging them to raise ideas until they had proposed the five diagrams depicted in Figure 1.

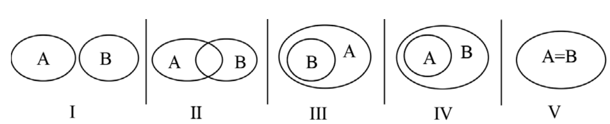

Figure 1. Euler diagrams of the five reciprocal situations between two sets.

This type of discussion was characteristic of Rachel's lessons. According to a post lesson interview, she regularly tried to elicit proofs from the students after preparing suitable components: "I set the ground for these things *[...]* I definitely prepare all the elements". Consequently, there was no evidence for gaps between lecturer and student premises in this part of the lesson, an issue that will be revisited below. Rachel's ways to promote elaborate student participation are exemplified in the following lesson transcript:

245 *Rachel* Previously, when we discussed the example *[...]* I said that *A* is actually the collection of all the points that are inside *A*, but there is another way to relate to this and now I want to give you a slightly different approach *[...]* I will first give a concrete example and then we will return to our previous example. Suppose I walk around and I surround some area in the center of Tel-Aviv by a fence, declaring it as mine. Am I allowed to do that?

246 *Students* Yes/No *[...]* *[Laughing]*

247 *Rachel* Well, it is controversial. OK, we will not provoke this 'legal' issue, we'll just take a different example. Suppose I am in the Sahara desert and I surround some area with a fence, will someone mind?

248 *Students* Yes/No *[...]* *[some]* people will mind *[...]*

249 *Rachel* However, if there will be some palm trees in the midst of the area I have encircled, and I place my fence around these trees, then someone will come and say: Hey, lady, that's not yours, right? Now, in the example of this desert, the size of the area I circled really doesn't matter, big or small, what matters is if there are trees inside or not. So drawing a frame around a set does not necessarily imply that the set includes all the points inside; when we circle a set it means that there are maybe points inside that belong to the set, maybe we have a couple of palm trees. Therefore I can also represent the empty set by a circle surrounding some area and declare there is nothing inside *[...]*

Rachel explicitly stated her intention to return to the abstract situation of the Euler diagrams, this time with the aid of a concrete example (Turn 245). She then used metaphors regarding Euler diagrams as representations of sets (Turns 247–249). Rachel planted the image of sets as fenced areas with or without trees in the minds of the students, intending to help them discuss the abstract question: 'Which of the five diagrams on the blackboard can represent all possible situations?'

This metaphor helped to foster a fruitful argumentative discussion, during which the metaphor was repeatedly and effectively used. It provides evidence of the shared basis of agreement between Rachel and her students, for example in Turns 273–276, in which representing the empty set is discussed.

273 *Rachel* If the lens is empty then this diagram *[Diagram II]* represents this diagram *[Diagram I]*. Does anyone have questions about this?

274 *Student* Can you demonstrate it with the palm trees?

275 *Rachel* Yes, I can *[...]* if I place palm trees only here and here *[drawing x's in the sets on the left and the right, leaving the 'lens' in the middle empty]*

276 *Student* So this *[the lens]* is the empty set

277 *Rachel* So that's the empty set, I have no trees here. Therefore these two sets are disjoint. Or, instead of trees, so it will be more comfortable, let's place here numbers. Say we write here 1,2 and here the numbers 7,8,10. These two sets are disjoint, no element is common. If they had a common element, say 7, where would I put it?

278 *Student* In the middle.

Rachel demonstrated the abstract situation using the palm tree metaphor and then moved on to sets of numbers. This reflects another one of her aims, stated in the post lesson interview, namely gradually building students' knowledge in order to enable their involvement.

Rachel summarized this part of the lesson as follows:

299 *Rachel* So what is the result of this whole story? That if I want to give a mathematical proof about properties of sets *[...]* then this diagram is sufficient *[Diagram II]* as a representative diagram. Why? Because it includes all other diagrams. So if I want to prove something, I don't need to say: if this happens, or this happens, or this happens *[...]* it is enough to treat this *[Diagram II],* always keeping in mind that one of these areas may be empty *[...]*

Rachel repeats the conclusion of this module and the advantage of having a representative Venn diagram. Interestingly, she addressed this theorem as a story (Turn 299) that she wishes to convey to the students.

Our PNR interpretation is that the palm metaphor is a rhetorical figure used by Rachel to share premises with the students, in this case: Euler diagrams being representatives of sets (*truth*), the generality of Diagram II (*truth*), the suitability of Diagram II to prove claims about properties of two sets (*presumption*). The use of this powerful metaphor was enhanced by other rhetorical figures: the second example using sets of numbers (Turn 277), repetition (Turn 299), and the shared classroom discussion (exemplified by the many student warrants in Figure 2) which was conducted with humor. As we will shortly see, Rachel's rhetoric had the intended effect and the presence endowed to these premises created a shared basis of agreement with the students with no apparent gaps.

To summarize, through the use of rhetorical figures the lecturer endowed presence to mathematical elements and established a shared basis of agreement. The assertion that Diagram II may be used to prove claims about any two sets was present and shared with the students. This is exemplified by an excerpt, taken from the next part of the lesson, where the students used a Venn diagram to prove that $(A \cup B) \cup C = A \cup (B \cup C)$.

333 *Student*	Is proof by drawing a legitimate proof? Under the assumption of what we proved earlier *[…]*
334 *Rachel*	Yes. This is why I first did all the drawings, and claimed that this is a situation that is a general representation. Once it is a general representation then the proof is legitimate.

The student is aware that there might be a problem with a visual proof but then recalls the generality of Diagram II which is affirmed by the lecturer. It seems that there is no gap regarding the premise of using Venn diagrams to prove theorems about sets.

Conclusions

Perelman did not intend 'The New Rhetoric' to relate to mathematics, which he considered a discipline whose argumentation forms are covered by formal logic (Dufour, 2013). Similarly, Toulmin (1958) initially considered his model as fit to describe only non-mathematical argument. Yet the effectiveness of using Toulmin's model to study argumentation in the mathematics classroom at all levels has been well recognized. We suggest that similarly, PNR is an important theoretical and methodological perspective that can extend mathematics education research to account for significant rhetorical aspects of mathematical argumentation carried in the mathematics classroom, in particular proof teaching.

In principle, lecturers could just write on the blackboard the proofs they need to teach—but they do not. Lecturers adopt a variety of pedagogical considerations and use different types of informal arguments. As we have demonstrated above, the lecturer invested much time and effort in emphasizing proof features that go beyond the definitions, assumptions and the chain of deductions that constitute the formal proof; and hence the use of PNR adds meaningful layers of analysis beyond the analysis of the structure of argumentation that is achieved through the use of Toulmin's model.

We have also demonstrated that the existence of a lecturer-student dialogue is important in creating a basis of agreement between lecturer and students. Classroom discussion is significant in endowing presence to important elements and a willing lecturer can foster classroom discussion and generate communication with the students. PNR is a comprehensive theory that provides a concrete interpretation to what may be considered such 'good communication', explains what obstructs and what improves it and provides means to discuss and evaluate reasons for communication gaps between the lecturer and the students. Thus, lecturers who consciously use rhetorical figures may improve and enrich their proof teaching and support the ideas that they consider important to convey to the students.

Note

[1] Henceforth, when the terms 'fact', 'truth', etc. are used as PNR premises, they are printed in italics.

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