EXPERIMENTATION AND PROOF IN A SOLID GEOMETRY TEACHING SITUATION

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> La déduction est à l'aveugle ce que l'intuition est au paralytique : l'une avance et ne voit pas, l'autre voit et n'avance pas. – René Thom [2]

Celia Hoyles (1997) has urged the community of mathematics educators to better consider curriculum organisation and sequencing when grappling with the crucial issue of teaching and learning proof. She characterized the evolution of the curricular approach to proof in the UK as a pendulum's swing [3], the extremes of which could be caricatured as follows:

1) At one extreme, its teaching being confined to Euclidean geometry, proof is considered as the mathematically ultimate way of validating a result, to be produced according to a specific two-column format in isolation from exploring and constructing activities. Statements to be proven are declared to be true ('Show that …') or are quasi-obvious from the figure.
2) The other extreme relies heavily on the works of Pólya and Mason among others. In reaction to the product-oriented approach of the first, a 'process-oriented' approach to proof is adopted. Emphasis is placed on enunciating and validating conjectures brought forward by students themselves through exploration and experimentation.

Based on numerous studies (Arsac, 1998; Chazan, 1993; Wu, 1996; Stylianides, 2008; Vinner, 1983; *etc.*), we tentatively propose that the first of the two approaches supports a ritualistic conception (Harel & Sowder, 1998) in which proof seems meaningless and purposeless to students. Tanguay (2005, §3) diagnoses this ritualistic scheme as the psychological result of the student being vaguely conscious that she or he remains unable to unravel the terms of the contract: the student believes that proof is about truth of the called in reasons (as in an argumentation, in Duval's sense, 1991), while it is in fact about validity of the chains of deductions. In Tanguay (2007, §1.3), we argue that the underlying refocusing, from truth towards validity, requires a radical shift in level, from the pragmatic to the theoretical (Balacheff, 1987), and we explain from a psychological perspective why this adherence to *theory* is in no way spontaneous or natural for children or young students.

Hoyles (1997) also warns of possible side effects from the second approach, against a too drastic shift from the first to the second, with social argumentation leaving no room for reasoning and scaffolding genuinely deductive in structure:

> Students … are deficient in ways not observed before the reforms: [they] have little sense of mathematics; they think it is about measuring, estimating, induction from individual cases, rather than rational scientific process. … Given that there are so few definitions in the [recent UK] curriculum, it would hardly be surprising if students are unable to distinguish premises and to reason from these to any conclusion. (*op. cit.,* p. 10)

This article is developed from our contribution to ICMI-19 (Tanguay & Grenier, 2009), and partly supports this warning. Through the experiment presented here, we show how experimentation and construction, in conjunction with some usual classroom contracts about proof, may sometimes contribute to move students away from the relevant reasoning and proving.

A didactical assumption

For the design of the activity, our starting assumption was that understanding the process of proving in its entirety requires that students regularly be placed in the situation of experimenting (Chevallard, 1992), studying examples and researching possible counterexamples (Lakatos, 1976), defining (de Villiers, 1998), modelling (Freudenthal, 1973), formulating conjectures and proving. Ideally, the situation would be designed so that proof appears to students as a necessity for establishing the truth of the proposed conjectures. For years, these didactical assumptions have been the basis for the work of the *Maths-à-Modeler* team at the Université Joseph Fourier (UJF) in Grenoble, a team that creates and studies the devolution and management of 'research situations for the classroom' (*Situations de recherche pour la classe,* or SiRC) at different levels of schooling (Godot & Grenier, 2004; Grenier & Payan, 1998; Grenier, 2001, 2006). This work is being continued in collaboration with researchers at the Université du Québec à Montréal (UQAM).

The situation we present here fits within the framework of the collaboration between these teams. Solid geometry, a field where basic properties are not obvious, strikes us as a source of tasks in which the conditions mentioned above may be combined. Indeed, the proposed situation relates to the activities of defining, exploring and experimenting via concrete constructions and manipulation, and to the necessity of resorting to proof in order to ascertain that no other constructions are possible.

Stating the problem

The following three tasks were given to students:

Phase 1. Define and describe regular polyhedra.
Phase 2. Produce them with specific given materials.
Phase 3. Prove that the list established in phases 1 and 2 is complete.

Mathematical and didactical *a priori* analyses of the problem

The definition

Recall that the number of faces (or equivalently the number of edges) adjacent at a given vertex is called the *degree* of this vertex. Two adjacent faces form a *dihedral angle,* which is the angle whose vertex is on the common edge, whose sides are perpendicular to this edge and are each in one of the planes including the two faces. We will call *face angle* the angle formed by two adjacent edges included in a face. According to the most standard definition, a *regular polyhedron* is a convex solid in space, delimited by faces which are all congruent to the same regular polygon, and having all vertices of the same degree. This last property can be replaced by either one of properties b, c, and d, below. As a matter of fact, for a convex polyhedron whose faces are congruent to the same regular polygon, the four statements are equivalent:

- a: The vertices have the same degree.
- b: All the dihedral angles are congruent.
- c: The polyhedron is inscribable in a sphere.
- d: The group of direct symmetries of the polyhedron acts transitively on the vertices.

A good way of intuitively grasping the link between these statements is to see each vertex of the regular (convex) polyhedron as the apex of a right regular pyramid fitting exactly into the polyhedron, each pyramid being the image of each other by a suitable rotation in space. Recall that it can be shown (see the *Proof* section below) that there are only five regular (convex) polyhedra, the so-called *Platonic solids* (fig. 1).

The condition that all faces must be identical is, without a doubt, the regularity property which spontaneously comes to the fore. One might expect students to consider the criterion 'the same regular polygon for the faces' to be sufficient. One possible reason is that non convex polyhedra or polyhedra whose vertices are not all of the same degree are not specifically studied at the primary or secondary levels. Yet, one can easily construct convex polyhedra all faces of which are equilateral triangles and whose vertices are not all of the same degree: glue together along their pentagonal bases two (right regular) pyramids or even more simply, use two tetrahedra. The analogous construction with two square-based pyramids results in the octahedron, which satisfies all the regularity criteria. Another family of counterexamples can be obtained from each of Platonic solids by gluing a pyramid on each face. One can thus construct non-convex star polyhedra, all faces of which are congruent equilateral triangles.

Constructions

How can one guarantee that the polygonal faces of a polyhedron – the skeleton of which having been assembled with Plasticine and woodsticks – are not 'distorted' and are indeed regular polygons? Even with what seems a regular polyhedron constructed from rigid material such as *Polydron* (jointed plastic polygons), one might question whether all the dihedral angles are the same. As legitimate as these questions may be, we expect *a priori* that for most of the students having materially constructed a polyhedron with congruent regular faces and equal degrees, the question of deciding whether the solid is valid or not won't be at stake: in a student's mind, the material existence of the object might warrant its theoretical existence.

A formal treatment of this 'mathematical existence' can be found in Hartshorne (2000, chap. 8). Hartshorne mathematically constructs the five Platonic solids by considering the five possible 'configurations at the vertices' (see next section) and shows that they verify the four properties a, b, c and d stated above. He then shows that a convex polyhedron whose faces are the same regular polygon and which moreover verifies a), is necessarily similar to one of the constructed solids.

Proof

As discussed previously, in the proving phase (Phase 3), we don't expect students to validate whether their constructed polyhedra are truly 'regular'. Rather, we expect them to seek to establish that no other regular polyhedron exists, apart from those constructed and set out in the collective discussion ending Phase 2.

For this purpose, one needs first to be convinced that there must be at least three faces sharing each vertex: with only two faces, there is no way of adding to these faces to create a solid. Next, one must establish that the sum of the face angles sharing a vertex be less than 360°. If this sum equals or is superior to 360 – which would be the case if, for example, three regular hexagons or six equilateral triangles shared a vertex – then either the polygons are coplanar and one cannot enclose a volume, or there is a fold towards the 'outside' in the vicinity of the vertex and the polyhedron cannot be convex. Next, it is sufficient to consider the finite number of possible situations, having ascertained that by virtue of these statements, one cannot make a regular polyhedron with polygons having six or more sides, nor can more than five equilateral triangles, more than three squares or three pentagons share a vertex. The only possible configurations thus consists of 3, 4, or 5 equilateral triangles, 3 squares or 3 pentagons sharing each vertex.

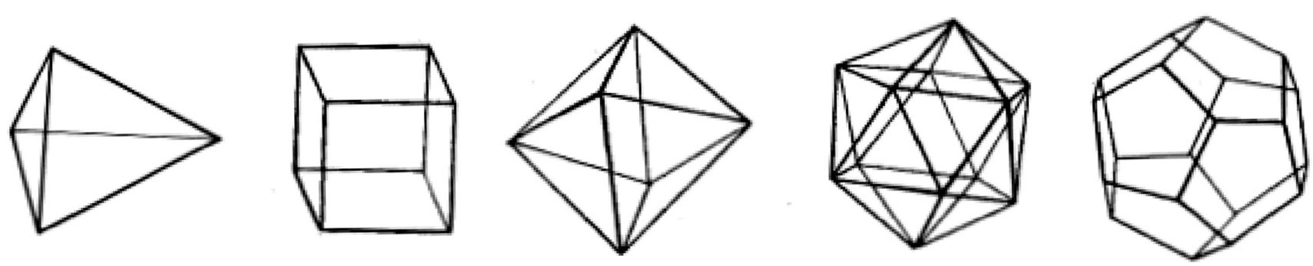

Figure 1. The five Platonic solids: tetrahedron, cube, octahedron, icosahedron, dodecahedron

The experiment

We explored the situation in an experiment with two groups [4]. Members of the first group were in the third year of a four-year teacher-training programme at UQAM, studying to become high school math teachers. Members of the second group were pre-service math teachers in their third year of the *Licence de mathématiques* at UJF.

We taught the courses which included these sessions. Both of us were present at the UQAM session while in Grenoble, only the researcher who gave the course there was present. In Grenoble, the experiment took place in one three-hour session while in Montreal, it took place in two sessions, one of three-hours and the second, a one and a half hour session two days later. In each case, the experiment provided an opportunity to introduce the students to didactical problems in solid geometry with which they had little prior experience. The students worked in teams of three: eleven teams at UQAM and eight at UJF. The researchers tried to keep from intervening, but would eventually give hints when they evaluated that the work was not making progress. An example is presented in the next section. Each of the three phases concluded with a collective debate conducted by the teacher-researchers, where the correct definitions and results would be made explicit. At UQAM, two teams were filmed. They were made up of 'average' students chosen because of being likely to discuss a lot. We were able to collect working notes from teams at both universities. A more complete report on the experiment is available through Grenier and Tanguay (2008).

Description of the students' attempts and analyses

Definitions

The specific materials were to be given only for Phase 2. However, in defining regularity, the students spontaneously tried to construct shapes using their usual materials (rulers, pens, pieces of paper), their aim being to have their own representations and to agree on the characteristics to be retained. The general definitions that resulted were very 'broad'. Every team mentioned congruence and regularity of the faces: "The faces are all the same", "It's everywhere the same regular polygon", and so on. Some teams added other criteria: convexity, closure (in the sense that it encloses a finite volume), inscribability in a sphere (one team). No team mentioned congruence of dihedral angles or equal number of edges (or of faces) at each vertex.

It is striking to note the extent to which these students had difficulty conceptualizing the dihedral angle. As we will see, this difficulty has consequences in construction and proof phases. We observed a lingering confusion between the face angles and the dihedral angles, likely due to the fact that the only 'visibly represented' angles are the face angles. Evidence of this confusion is the following exchange, from Filmed-team 1, and their comments following the teacher's intervention:

Student B: Is there anything else? [*i.e.,* to add as elements to the definition; the three team-members have already selected and written down "closed", "all faces are the same regular polygon" and "the more sides it has, the more it looks like a ball."]

Student A: [summing up] All the faces …, … no, no I mean, we're talking about faces, …let's not even discuss angles because, because it's, … *it's the polygons that form the angles.* [emphasis added]

In the filmed lesson (UQAM), still in Phase 1, the question of knowing whether other criteria should be added would eventually be raised by the teacher (one of the authors): "Does this ensure it [the polyhedron] enough regularity?" To put students on the track of dihedral angles, the teacher used a piece of paper to represent two faces and varies the angle, saying, "At first glance, the angle between the two faces seems not so much dependent of the face angles". Later, the teacher asked students whether the following solids should be kept as 'regular': the star polyhedron formed by gluing square-based pyramids on the faces of a cube, the polyhedron formed by gluing two tetrahedra. Mention of the star polyhedron elicited the following thoughts from the filmed team quoted earlier: "I don't get the impression that the angles between the faces are all the same", "The angle between the faces and the angle between the sides, it's not all that connected", "There are reflex angles, it's not convex anymore". Then the team-members added the criterion 'No reflex angle' to their definition. Finally, in the group debate ending Phase 1, the definition that was discussed and selected for Phase 2 would be the same as the one which we had hoped to arrive at, and the question of the dihedral angles would be discussed and somewhat clarified.

Constructions

It should be mentioned from the start that, according to the presentation document, students were not to stay confined to construction in Phase 2, but were directed towards arguments relevant to the proving phase, which was to be next:

> Which geometrical properties (number of edges, of faces, type of faces, angles) should be verified to ensure having a regular polyhedron? For instance, what occurs at a given vertex? Justify your answers. Try to construct as much regular polyhedra as possible… (Presentation document)

The discussion regarding the necessity of adding equality of degrees to the regularity criteria, which took place following presentation of the family of counterexamples produced by gluing two congruent pyramids together, is perhaps the source of the following observation, one that came as a surprise to the researchers and contradicted their assumption that constructed polyhedra wouldn't be questioned (see our *a priori* analysis of the construction phase). Several teams did in fact build the octahedron but then rejected it, claiming that it was not regular. What was suggested most often was that its dihedral angles are not congruent, a perception most likely created by the fact that all its vertices would not have the same status, and that the polyhedron would not be perfectly symmetrical. The mis-

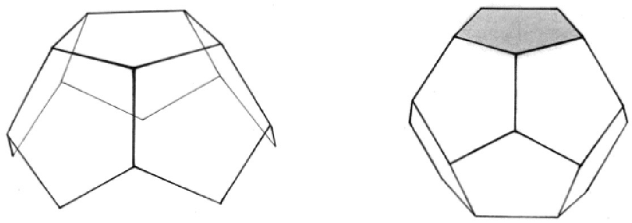

Figure 2: The 'upper layer' of the dodecahedron and its 'cap'

conception of 'one polyhedron per polygon' (see next) may have reinforced this perceived irregularity of the octahedron in some students' mind. In the collective discussion ending Phase 2, Student F of filmed Team 2 did clearly express that among the polyhedra constructed with triangles, the tetrahedron should be 'kept' and the octahedron rejected.

Indeed, Phase 2 revealed the most striking misconception, observed from at least six of the eleven teams at UQAM [5], that regular polyhedra constitute an infinite family, with one polyhedron per type of (regular) polygon for the faces, with the number of faces increasing with n, the number of sides of these faces, and a resulting polyhedron closer and closer to the sphere, as n goes to infinity. This conception brought several teams into attempts to construct a polyhedron with hexagons. Filmed Team 1, for instance, quickly constructed the tetrahedron and the cube with Plasticine and woodsticks, and then managed to construct the dodecahedron, despite the instability of the construction. After being supplied with jointed plastic hexagons the three teammates assembled about ten of these – all lying flatly on the desktop from the start – and then tried to raise the ones on the fringe. They blamed the rigidity of the material for not being able to 'fold', as they put it. They had in mind a 'layer' decomposition ('décomposition en étage', in French), according to which the tetrahedron has no layer, the cube one, the dodecahedron two (1 + 5 + 5 + 1: one 'base', two layers and one 'cap'). They formulated the conjectures that the polyhedron with hexagonal faces will have three layers and 26 faces (1 + 6 + 12 + 6 + 1) and that more generally, a layer should be added each time n is increased by one (the octahedron had neither been constructed nor considered by this team).

Having constructed the tetrahedron, the cube and the dodecahedron with magnetic balls and metal sticks, Filmed Team 2 searched for a formula that would allow them to compute the number of edges knowing the number n of sides for each face. One of the three students proposed $n + n(n - 2)$, while pointing on the cube to what each term of the expression corresponded to. She noticed that the formula also worked for the tetrahedron, but not for the dodecahedron. She then proposed $n + n(n - 2) + n(n - 3)$, and none of the teammates seemed to realize that the formula did not work with the cube. Next, they assembled some hexagons and complained to each other about the material not being sufficiently rigid. They nevertheless tried to understand the decomposition in rows (the French 'rangées') of the hypothesized polyhedron, and figured out that it has 20 hexagonal faces (1 + 6 + 6 + 6 + 1: one base, three rows and one cap). They then conjectured that regular polyhedra have two rows when n is odd and three rows when n is even.

Student E: Yes! It's a matter of parity! When n is odd, you've got two rows, one over, one below, and the two fit [showing on the dodecahedron]. But when n is even, there is a row right in the middle.

Student F: That's it, then you can't do 'times two' [when n is even].

Student E: [explaining to Student D, with gesture] At the bottom, you've got a base, plus a row of six hexagons, one for each side. Same thing at the top, but the bottom and the top, they don't fit [still referring to the dodecahedron, where the two bottom and top rows do fit], you need one more row in the middle.

Student D: [doubtful] But when you've got seven of these [polygons making a row when $n = 7$], will they still, …'fit well'? [She makes the gesture of bringing together her two hands, curved like cups, one below and one over.]

Student F: [confident] I think they will.

Without any more manipulation, a formula pursuit ensued, bringing the team to propose the following: the number of edges is given by $n + n(n - 2) + n(n - 3)$ and to get the number of faces, one must divide by n (each face has n edges) and multiply by two (each edge touches two faces). The teammates thus arrived at 16 as the number of faces for the polyhedron with hexagonal faces, but seemed unperturbed that this contradicted the number 20 given by their previous row decomposition.

In the collective episode ending Phase 2, the two researchers (recall that one was in charge of the course) submitted to debate the conjectures they previously gathered from students' work and discussions. The first conjecture stated that the only possible polygons for the faces are those that tessellate the plane, and was rejected by showing a constructed dodecahedron. The second conjecture asserted that there can't be more than three faces sharing each vertex "because we are in a 3-dimensional space" (as students themselves stated it). It was rejected by teams having constructed the octahedron and the icosahedron. The third conjecture was presented by the researcher as "… in a way, the inverse of the previous conjecture", and stated that "you can place, at each vertex, as many triangles as you like, and the more triangles there are the closer to the sphere the polyhedron will be". The students were hesitant at first, but two among them, students G and H, from two different teams, finally brought forward that "it can't work if you have 360 degrees covered":

Student G: If you put six triangles, you will have 360 degrees and then, it's impossible to lift up the triangles. It's going to be the same thing with hexagons. As soon as you put them together, it makes 360, in the same way.

Student H: The maximum you can have is 5 triangles.

The researcher added, "That's it! If there are 360 degrees of

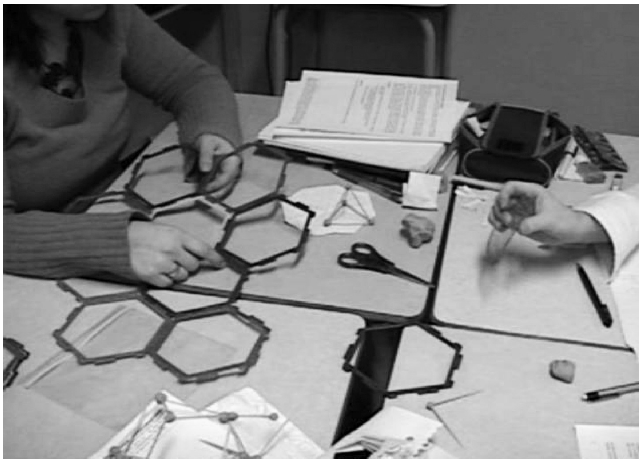

Figure 3. "That's why we were not able to fold!"

angle covered, then it lays flat, you won't be able to enclose a volume", but the students were not convinced and we could clearly hear Student C mumbling, "I don't understand". While students G and H insisted on the impossibility of constructing a polyhedron with hexagons, repeating "it lays flat, you can't fold it", Student A seemed struck by lightning. She had in one hand nine assembled plastic hexagons (see fig. 3) and pointed the angles with her free hand, exclaiming: "It makes 360, it doesn't work, it can't work! That's why we were not able to fold, it's precisely because of that! If it would have been pentagons, we would have been able!" As for members of Filmed Team 2, they were still wondering about the octahedron, Student F having rapidly reconstructed it to allow her and her teammates a closer examination.

Proofs

From a research perspective, the proof phase is not conclusive because the presentation document (Grenier & Tanguay, 2008, Appendix 2) proposed steps based on the production of graphs (Schlegel's diagrams), and this approach didn't appear to students as proceeding from the definition and construction activities. Rather they saw it as a *new problem.* A new experiment, with different guidelines for Phase 3 and leading to a more standard proof that there exist only five regular polyhedra, is in process. Our concern here is about meaning and purpose allocated to the experimental process by the students, about understanding why the potentiality offered by exploration and construction have contributed so little to the emergence of the appropriate reasoning and argument, accessible right from Phase 2. That's what we want to investigate in the next sections. We evaluate that there is here a confluence of factors to be considered.

Sequences, regularities and formulae

For a majority of the observed students, including the members of the two filmed teams, the quest for regularity and formulae has overshadowed any other form of reasoning or judgement; to the point that, for example, members of Team 2 would admit, without batting an eyelid, two distinct and incompatible formulae to 'justify' the terms of a sequence of three integers! Why such a focus on sequences and formulae?

Evaluating to what extent it could be an effect of curricular trends, such as those flagged by Hoyles in the earlier quote (above), is difficult. Most of the observed students learned their secondary level maths according to curricula of the 1990s. The systematic work on Euclidean geometry had then been replaced by more informal approaches, based on the study of transformations. The emphasis on exploration, experimentation and conjecture formulation, as it pertains in the recent ministerial programs from France and Quebec, was not as prominent in the 1990s, but precursory elements of it could certainly be traced back to that period. Besides, it should be mentioned that being preservice teachers, the observed students know the new curriculum, having studied and dissected it in some of their university courses.

Despite the difficulty of evaluating the precise role of curricula, we suggest that in the actual schooling context the way these students tackled the task may have been influenced by too great an emphasis on conjecture formulation – the fact that conjectures are most usually treated in an empirical way, rather than as a trigger to seek structural explanations – and the too frequent and almost systematic association between proof and algebraic verifications – an instance of the more general tendency according to which everything is too often 'true and computable' in school mathematics.

Laborde describes what she calls "dichotomy between conjecture and explanation or proof" as follows: the observed teachers in Laborde (2001) exploited Cabri as a dynamic visualisation tool to conjecture properties, but "the teachers did not mention the possibility of using Cabri to find reason or to elaborate a proof. It is as if there was no interaction between visualisation and proving" (p. 306). These concerns are certainly related to the above observations, and will be discussed further in the conclusion.

Intuitions, metaphors and associations

In a 1994 article that has garnered considerable comment from mathematics education researchers (*e.g.,* Hanna, 1995), Thurston sought to lessen the role of formalism in math-research. In drawing up the list of "brain and mind facilities" researchers use conjointly, Thurston set "logic and deduction" side by side with "intuition, association and metaphor", the importance of which he highlighted:

> Personally, I put a lot of effort into "listening" to my intuitions and associations, and building them into metaphors and connections. This involves a kind of simultaneous quieting and focusing of my mind. Words, logic, and detailed pictures rattling around can inhibit intuitions and associations. (p. 165)

On the basis of Thurston's observations, some researchers in mathematics education (*e.g.,* Douek, 1999) questioned instruction in proof focused on the precise deductive mechanisms, aiming at learning logical and formal structures. Instead, they insist on the proving process and argue that it entails an intense and essential argumentative activity, largely resting on intuitions, analogies and metaphors, these being hindered by a too exclusive focus on formalism.

In Tanguay (2002, p. 375), we argued that the reflections of a math researcher such as Thurston should be put in perspective when proof teaching and learning at an undergraduate level is at stake. The present experiment supports our concern. We are indeed convinced that setting formal logic and deduction on one side, against intuitions, associations and metaphors on the other, is too sketchily simplifying the general problem of students' access to proof. It is clear for us that in the instance of this experiment and in the observed students' mind, the existence of an infinite sequence of regular polyhedra (associated with the infinite sequence of regular polygons), the decomposition in 'layers' extrapolated from the dodecahedron, the quasi-metaphorical meaning ascribed to words such as 'base', 'layers', 'rows' … , stemmed from analogical reasoning and from a too confident and uncontrolled resort to intuition. Yet, these elements did contribute *to move* students *away* from the deductive arguments, or even from relevant ones.

Local truth versus global validity

In our analysis of students' attempts to define regular polyhedra, we have stressed students' difficulties in conceptualizing the dihedral angle. At the very core of the targeted proof stands the understanding of the link *face angle – dihedral angle,* and of the three statements:

i) There must be at least three faces sharing each vertex.

ii) At a given vertex, the dihedral angles are less than 180° only if the sum of face angles at this vertex is less than 360°.

iii) A polyhedron is convex if and only if none of its dihedral angles is greater than 180°.

We expected that manipulations and constructions would instil the link between face angles and dihedral angles. As a matter of fact, the statements did come to the fore during the discussion ending Phase 2, but had not been the matter of a thorough discussion and explicitation – that is, of an 'institutionalization', in Brousseau's (1998) sense. We already mentioned (cf. introduction or Tanguay, 2007) that access to proof requires the student to refocus from the truth of the statements at stake towards the validity of the reasoning through a process that is fundamentally that of *building local theories,* as it is described in Jahnke (2010, p. 30). We formulate the hypothesis that the observed students' apprehension of statements i, ii and iii was built through a relationship to *truth,* essentially isolated in nature (isolated with respect to time, as a quickly forgotten truth, and with respect to linking with other ideas involved), instead of a relationship to *validity,* with respect to controlled argumentations and consistency of a theoretical scaffolding. A renewed experiment, where statements i, ii and iii are the object of a discussion at the end of Phase 2 and their theoretical status – as hypotheses, in Jahnke's sense – explicitly settled in the guidelines for Phase 3, is still in process of experimentation.

Conclusion

Let us go back to Hoyles' request, raised at the start of this article. In our view, current curricular trends, promulgating proving processes based on experimentations and conjectures, will lead to an efficient learning of proof, with proof attaining its full meaning in the learner's understanding, only if these processes are set within a genuine process of building 'small theories', resorting to hypotheses in Jahnke's sense (*i.e.,* axioms that "are neither given by a higher being nor expressions of eternal ideas, [but] simply man made" – Jahnke, 2010, p. 31). From these axioms/hypotheses would be elaborated hypothetico-deductive networks, which would then be confronted with the initial experimentations and conjectures. However, in this perspective, we don't agree with Lakatos (1976) who evaluates such processes as belonging to scientific discovery and quasi-empirism, in opposition to formalism. In our view, the relevant proving processes, guided by a constrained and controlled intuition, call for a minimum of theorization and formalization, with statements whose theoretical status are examined, discussed and 'institutionalized'.

But it does not go as far as dealing with theoretical objects unrelated to their representations. In agreement with Laborde (2001, p. 306), we are convinced that the role played by interactions and mutual control between non-ostensive (or theoretical) and ostensive objects (in the sense of Bosch & Chevallard: expressions, formulae, diagrams, symbolic, graphical or material representations ...) is fundamental. Under what mathematical and didactical conditions students will recognize their work on ostensive objects as possibly leading to structural explanations or proofs? This question stems from our experiment, is still open, and should be addressed by math education research.

More specifically we speculate that curricular tendencies to ask for proof, in contexts where it is reduced to algebraic computation, have detrimental effects on its learning process. In this respect, we are concerned about the constant decline of synthetic geometry in favour of analytic geometry (in coordinates). We are also concerned by the fact that at the secondary level, validations in solid geometry are done almost exclusively on an empirical-perceptual basis, to such an extent that simultaneous works in plane and solid geometry are not in phase, as they are not being done according to the same paradigm (Houdement & Kuzniak, 2006; Furtuna, 2008).

We conclude with the following reflection : in our view, the study confirms – if such a confirmation were needed – that coordination between meaning and functioning of proof, between intuition and deduction, experimentation and formalization, heuristic and logical organization, should be at the very core of educational research on teaching and learning proof, and that any line of arguments too strongly inclining towards one side of the spectrum is likely to be a counterproductive over-simplification of the question.

Notes

[1] The support of the *Fonds québécois de recherche sur la société et la culture* (Grant # 2007-NP-116155) is gratefully recognized.

[2] Deduction is to the blind as intuition to the paralytic: the former moves forward without seeing, the latter sees but does not move.

[3] A similar evolution could be traced out in the curricula of France and Quebec, though perhaps through different year-by-year accounts.

[4] The situation was described in a document, where the three questions given above were specified to ensure 'good' devolution. See Grenier and Tanguay, 2008, Appendix 2, accessible on line at http://www.math.uqam.ca/~tanguay_d)

[5] The same phenomenon has been observed at UJF but from how many teams, we are not sure. Note also that attempts from primary-school teachers at constructing a regular polygon with hexagons is reported in Dias &

Durand-Guerrier (2005). In fact, the same situation is regularly, year after year, submitted to pre-service and in-service teachers in Grenoble, and gives rise to the same observations, about students trying to construct a polyhedron with (regular) hexagons and being persuaded of the existence of an infinite family of polyhedra. We don't report on these realisations here because they were not observed and controlled as systematic experiments.

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A path laid …

Stephen and Beryl recently repaved the metre-wide path to the front door of their new house. They used 50 cm × 50 cm square slabs in two colours, grey and brown.

Starting with two grey slabs, they arranged the pieces so that that the pattern of the four slabs in any square metre was not repeated (as would be seen by someone taking the path toward the front door). This would not have been possible had the path been any longer. How long is it? And what colours are the last two slabs?

(source unknown)