Between Platonism and Constructivism: Is There a Mathematics Acquisition Device?

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1. Two models of mathematical knowledge

Mathematics plays a rather special role in what might loosely be termed the study (or philosophy, or theory) of knowledge, both as regards the quality or nature of knowledge, and how it is acquired. An early and famous example is found in Plato's dialogue *Menon* whose thesis is that knowledge relates to an eternal world of ideas, to which humans have access by way of the memory of their immortal soul. [1] This is allegedly demonstrated through Socrates' mathematical conversation with an uneducated slave boy, and such a view of knowledge and learning is usually called *Platonic realism* or *Platonism*.

Certainly, philosophers since the medieval nominalists have largely disagreed with Plato, and in particular have denounced the adequacy of the mentioned 'experiment' to support his claim: but his viewpoint has neither been forgotten, nor dismissed as a simple mistake. In particular, his use of mathematical knowledge as a central case for epistemology (not least due to its special ontological nature) is continued by most later philosophers, including such as Aristotle, Descartes and Kant

However, while Plato finds the general nature of knowledge *exemplified* in mathematical knowledge, most of his successors insist that mathematics - or, at times, arithmetic or logic - represents knowledge of a very special quality, and as such helps to identify different forms and qualities of knowledge. More recently, mathematics has served a similar (and clearly related) purpose in the field of education, notably in educational psychology

Constructivism is currently the most popular class of theories of knowledge and learning, and is also in rather marked opposition to Plato's realism. In the field of education, the influence of the Geneva school (the doctrine of which I shall refer to as *classical constructivism*) is overwhelming; in particular, theories of cognitive stages and schemas have achieved folklore status among educators at all levels, not least due to massive dissemination through the initial and in-service education of teachers. Later developments, such as *radical, operative* and *social* constructivism, are presently defining themselves and their implications for education (see e.g. Steffe and Gale, 1995).

It is interesting to note (and important for the points of this article) that these later developments seem less rooted in studies of the nature of mathematical knowledge, and more based on attempts to challenge and replace traditional 'objectivist' views of science, the overall tendency being to focus on the individual or socially situated knowing subject (researcher, teacher, learner) as a crucial factor in the constitution of knowledge Certainly, attempts have been made to accommodate mathematics within these views (e.g. Ernest, 1998), and indeed the resulting focus on the social side of mathematical cognition and knowledge has proved a refreshing change from the dead-ends of foundationalism and absolutism [2]

However, the feeling seems to be widespread among mathematicians that one cannot explain the (say, common experience of a relatively high) certainty and permanence of mathematical knowledge without appealing to some sort of 'common object' status of mathematical entities (see e.g. Sfard (1994) for a collection of testimonies); this could indeed be seen as important empirical evidence that there are key characteristics of mathematical knowledge which cannot be explained within the framework of constructivism, and even less within more recent brands involving quasi-empiricism.

Mathematicians' partial (and to some extent contradictory [3]) inclination towards realism and nominalism should not simply have to be rejected as ignorance of the constructivist *lumières*, as a condition to avoid a return to the old, decisively invalidated positions of orthodoxy: these inclinations, alive and persistent as they are, should be explained and in a sense be compatible with a correct model of mathematical knowledge. This is particularly important for the sake of mathematics education. The current search for independence and professional integrity of the field *vis-à-vis* academic mathematics, and the effort to adjust the field to the latest trends in the sociology of science, may cause it to base its research and practice on a partially foreign and defective theory of mathematical knowledge. This constitutes, I believe, an immediate and potentially devastating danger for mathematics education, both as a field of practice and (not least) as a field of research

This article is an attempt to suggest a way between Scylla and Charybdis: between the old monsters of Platonism, absolutism and foundationalism on the one side and the dangerous streams of relativism, historicism and solipsism on the other. In particular, a main purpose of my article is to propose an entirely different perspective, [4] and thereby to challenge both the current para-dogmatic character of constructivism in some quarters of mathematics education and the dismissive attitude towards 'humanist non-sense' in others.

2. Knowledge and learning - some innocent remarks

About knowledge as an abstract phenomenon, I believe that

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one may say the following from the mere linguistic form of the term: that is, without committing oneself to any particular philosophical position [5]: *knowledge is a state of subject–object relation among (in general, unalike) entities*. In other words, it is referred to by propositions of the form *A knows B*. At the grammatical level, thus, we have a subject *A* and an object *B* related by the transitive English verb *to know*. [6] At the semantic level, A refers to one or more animate beings (for my purpose, *human* beings). The philosophical discussion concerns:

- the exact semantic nature of the verb and its relation to *B*;
- the external semantic relation (reference) of *B*;
- the establishment of the proposition and its reference (the state it expresses).

The first two points are the concern of the traditional areas of epistemology and ontology, while the last point is the domain of educational studies. As the last point is my main concern here, I elaborate further on the two aspects it contains. *Establishing* the proposition means determining whether or not, or in what partial sense, the proposition *A knows B* (for given *A* and *B*) is true. The *establishment of the reference* of the proposition means the procedure by which the state it expresses comes into being; this procedure has a simple linguistic expression: *A learns B*. In educational practice, establishing the proposition (evaluation of learning) is temporally preceded by establishing or attempting to establish its reference (learning), but in educational theory and design, the two are usually inseparable.

Notice that *no object status* (in the philosophical sense) is attributed to *B* in the above, and that *A knows B* may in many interesting cases not be decidable in the 'true–false' sense. Also, *A learns B* may then express the movement towards a goal, rather than the reaching of it.

These initial (and philosophically fairly innocent) remarks should justify the claim that no coherent theory of learning can be established independently of any theory of knowledge. On the contrary, the basic proposition *A learns B* that I wish to study is, formally speaking, the semantic bridge between the first proposition *A knows B* and its negation, so that a solid position regarding the former proposition is full of implications for the latter.

I should mention here the distinction between *acquisition* and *participation* metaphors in the theory of learning, discussed in Sfard (1998). About the former, she claims that:

> the language of knowledge acquisition [...] makes us think about the human mind as a container to be filled with certain materials, and about the learner as becoming an owner of these materials. (p. 5)

And about the latter:

> learning a subject is now conceived as a process of becoming a member of a certain community. This entails, above all, the ability to communicate in the language of this community and act according to its particular norms. (p. 6)

So maybe a caution is in order here: when I use the word 'acquisition' in the following, in particular in the context of mathematics, I do not mean to invoke the container image. On the contrary, the analogy between language learning and mathematics learning will become central in this article. And in virtually any language acquisition theory, participation (in discourse) is a fundamental pre-requisite for learning language – 'acquiring' is, in this context, standard terminology for this 'learning by participation' process.

Of course, it is nevertheless possible to focus a study on the mechanisms of language acquisition, with minimal assumptions on the actual context of participation. This could be called a 'basic scientific approach' to educational issues, but would clearly not suffice in itself for addressing educational issues as they arise in practice.

3. Remarks on classical constructivism

I want to take a step back and look at the more basic assumptions and claims of constructivism. My point of departure is the Cartesian observation that knowledge is meaningful only in the context of a knowing individual - essentially, the acknowledgement of the necessity of the subject A in the proposition that I considered in section 2.

Two basic categories of knowledge have been identified (Piatelli-Palmarini, 1979, p. 56): *empirical abstractions*, derived from sensory experience of the world surrounding the knowing subject; *reflective abstractions* derived from mental operations performed by the subject, e.g. upon empirical abstractions. Empirical abstractions can be said to precede, or at least underlie, reflective abstractions; reflective abstractions are further organised in a hierarchy of rising complexity and abstractness, as built up during Piaget's cognitive stages (cf. e.g. Piaget, 1953, Chap. II).

A main point of this classification of knowledge structures is to provide substance for the claim that only a very small portion of knowledge comes close to the classical account of knowledge as an image of factual entities existing in the outer world (be it a physical or ideal one). Most, and indeed the most interesting part, of knowledge is *constructed from other knowledge structures* by the knowing subject. This construction of knowledge, then, is the main road to understand the nature of knowledge; as construction is a dynamical process, so is knowledge, according to the constructivist account.

While I do not wish to recall the technical details of the dynamics of knowledge proposed by Piaget and his school, I note the keywords *schema, assimilation, accommodation* and *equilibration* as the main analytic tools involved. In short, the proposition *A knows B* refers to a construction of *B* in the mind of *A*, and the dynamics of this construction may be further detailed in the technical language of cognitive psychology.

Probably one main merit of constructivism is its insistence that *knowledge and learning are structured phenomena in the human mind which are amenable to scientific research*. Knowledge, as referred to by propositions of the form *A knows B*, may be studied systematically as mental representations in the mind of *A*, which relate not only to *B*, but in general to a structural representation of an entire cognitive domain in which *B* is situated. The ontological problem, about the external semantic references of the object *B*, is relegated to the small class of such propositions that represent empirical abstractions, and in this case, a rather conventional explanation in terms of the human sensori-motor systems is given

However, in the special case of mathematical knowledge, we find another classical school, *intuitionism* This shares the basic assumption of general constructivism, that (in this case, mathematical) knowledge is essentially a mental construction of the knowing individual. It has a little more to say about the ontological issue: namely, the natural numbers (not in their mathematical entirety, rather in what is called their 'potential infinity') are given a special status Intuitionists simply take the *intuition of two-oneness* (Brouwer, 1913) as given *a priori* (much in the sense of Kant), and with it the intuition of the numerals 1, 2, ..., *n*, where *n* is any numeral. Then all the rest of mathematical knowledge is constructed from this basis of intuitive knowledge.

The difference between this and classical constructivism, referring the knowledge of small numerals (10 rather than 2) to empirical abstraction, is in effect not very important; the alchemy of mental constructions remains the central model of explanation, both in epistemology and in ontology.

4. Plato's problem and constructivism.

Chomsky (1986) defines what he calls 'Plato's problem' as follows:

> to explain how we know so much, given that the evidence available to us is so sparse. (p. xxvii)

Within a constructivist model of knowledge as sketched above, the evidential basis for mathematics is sparse indeed: a few sensual or *a priori* given notions of numerals, which are to support the potentially vast structures of mathematical knowledge, which, in turn, are even applicable to describe the most intricate phenomena in social and physical contexts (that is, underlying highly non-trivial non-mathematical knowledge as well).

Two related problems, arising from the last remark, are the following (Barnes, 1997):

> Why does reality march to a mathematical tune? (p. 210)

and, as this problem is easily solved by accepting the Platonic ideal world, not least the following:

> [Is] all the mathematics needed to describe the physical Universe within the reach of the constructivist[?] (p 214)

Another facet of Plato's problem in the case of mathematical knowledge is that, mentioned earlier, of the (felt, relative) certainty and permanence of mathematics, if we are explain knowledge as mental constructions: why do idiosyncrasy and individual background not play a greater role than they apparently do?

If the construction of mathematical knowledge simply follows the general procedures of individual knowledge building, then the sparse evidential basis on which it rests ought to produce further divergence and variation than it seemingly does Consider an example to support this claim: in the domain of matrimonial ethics and legislation, there is little consistency or permanence to observe across cultures and historic eras, although the basic sensori-motor knowledge underlying it could be said to be invariant: Euclid's argument concerning the infinity of prime numbers remains essentially convincing to our day. [7] Even regarding mathematical knowledge as dependent on culture in addition to the purely mental construction - as most contemporary constructivists do - the striking difference between the two examples mentioned remains hard to explain.

A much more serious problem arises in the study of linguistic knowledge and learning, which is the setting in which Plato's problem arose for Chomsky:

> the native speaker has acquired a grammar on the basis of very restricted and degenerate evidence; the grammar has empirical consequences that extend far beyond the evidence (1972, p. 27)

The real depth of this discrepancy comes from the observation, rigorously argued e.g. in Chomsky (1957, Chap. 3), that the grammar of a natural language (such as English or Danish) does not only contain a potential infinity of correct phrases, but is not itself finite in the sense that it may reduce the formation of sentences [8] to a finite state Markov process. In more everyday terms, no matter how intricate the rules we devise, playing chess is in an essential way less complicated than speaking a language. Constructivism cannot, at face value, explain how the native speaker builds up a system of knowledge of this complexity based on a finite number of examples of its application

The Chomskyan revolution consists, roughly speaking, in pointing out that this is an irreparable gap in any attempt to reduce language acquisition to mere 'induction from experience' (cf. Bruner, 1983), and in providing an intriguing alternative (to be further discussed in section 5) Many of the basic ideas and technical concepts of constructivist theories of cognition remain meaningful in this altered picture, although the idea of an initial *tabula rasa* must be abandoned.

Incidentally, Chomsky (1988) also touches on the aspect of the problem pertaining to what he calls the 'number faculty':

> Children have the capacity to acquire the number system [...] if a child did not already know that it is possible to add one indefinitely, it could never learn this fact (p 167)

Furthermore, the existence of an innate number faculty [9] is not only analogous to the innate 'universal grammar' (see e.g. Chomsky, 1986, Chap. 3), but is claimed to be derived from it:

> There could not be a mathematical capacity without a language capacity [...] If you think about the history of mathematics, say from Euclid to fairly recently, there are really two basic ideas One idea is numbers; the other idea is the structure of three-dimensional visual space [...] we can have relevant thoughts about geometrical space only because we have language [...] The other notion, of number, probably comes from our language capacity directly. (Chomsky, 1988, p. 184f)

This last quotation comes from a discussion following one of Chomsky's Managua lectures, and may indeed look like an oversimplification of his viewpoint. [10] Yet the point to retain is that, according to Chomsky, a key to Plato's problem as concerns mathematical knowledge may be available in the relation between it and linguistic knowledge.

Elsewhere, I have argued that the characteristics of mathematical communication (or, more precisely, its syntax and discourse) may be fruitfully studied using concepts and ideas from linguistics (Winsløw, 1998). These considerations motivate taking a closer look at some main ideas from the field of language acquisition

5. LAD, LASS, and so on

The basic Chomskyan proposal to solve Plato's problem for language acquisition is the hypothesis that humans possess a *language acquisition device* (LAD) - an innate mental structure, common to all human beings. It takes, as input, primary linguistic data - a finite number of more or less 'correct' utterances in the language to be acquired - and produces a 'grammar' of this language, i e a system of rules which allows the learner to speak the language creatively - to 'perform' linguistically.

The original LAD model (Chomsky, 1965) is rather explicit concerning the detailed 'mechanics' of the device [11]; as a reconstruction, I have represented the whole mechanism in process-diagrammatic form in Figure 1 For a detailed explanation, the reader should consult Chomsky (1965, §1.6-1.8); the basic idea is that, given the finite input, the LAD chooses among an infinity of in-built 'possible grammars' the one which is in optimal consistency with the input Hence, roughly speaking, the phrase *A learns B* (where *B* is a natural language) refers to the device being active, and in *A knows B*, the semantic reference of *B* is properly speaking the current value of the evaluation measure (perceived to be a 'correct grammar').

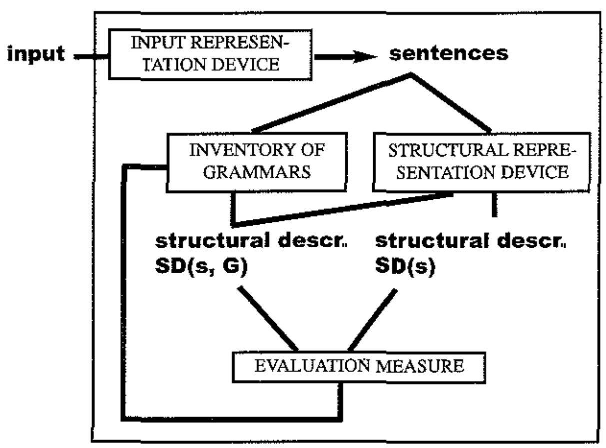

Figure 1 LAD

The scientific status of (this version of) the LAD is, so far, unclear; at best, it is an unverified hypothesis about the innate language competence of the human mind, but as it is even quite unclear how it could be scientifically rejected, a Popperian critique might claim that:

> No substantially stronger claim to scientific status can be made for it than for Homer's collected stories from Olympus (Popper, 1963, p. 38)

Of course, the point of bringing up the idea of a LAD is not to reject it in this way (let alone that Popper's falsification criteria are widely acknowledged to be impracticable in any science). The aim is, first, to point out the LAD model as an outstanding example of an unverified hypothesis which has inspired much fruitful research effort, in this case not only in linguistics, but also in educational psychology, as evidenced e.g. by Bruner (1983) and Karmiloff-Smith (1992)

Second, and more importantly, I wish to emphasise that similar hypotheses form the lifeblood of research regarding the mechanisms of cognition, where measurable evidence in a direct physical sense is still very sparse, in spite of recent progress (to which I return below). A parallel may be found e.g. in nineteenth-century chemistry, which used notions such as valence and molecule as "convenient myths devised to help organize experience" (Chomsky, 1986, p 7).

Later works on language acquisition have provided various auxiliary hypotheses and modulations, such as pointing out the necessity of taking into account the interaction between input and output in a language community setting, e.g. the language acquisition support system (LASS) discussed in Bruner (1983), or proposing domain-specific cognitive 'modules' as discussed (and questioned) in Karmiloff-Smith (1992) Recently, the notion of *metaphor* has been used as a central explanatory model for language-related cognitive mechanisms, including natural language, e g. in Lakoff (1987); the idea is also exemplified in the context of mathematics in Lakoff and Núñez (1997).

Until fairly recently, the study of cognition had to be based solely on evidence of performance, with no possibility of gaining direct evidence of cognitive structures [12] The interest of theories (systems of hypotheses about cognitive structure) derived from analysis of performance is increased by current developments in cognitive neuroscience which suggest that measurable evidence for such theories may become available; see Dehaene (1997) for a leisurely introduction to this exciting area as it applies to mathematics As in any experimental science, techniques used to gain evidence have to be complemented by theories suggesting what to look for; only explicit (and, at least initially, simple) theories will do here

An overall tendency of many recent developments in language acquisition studies seems to be that theories become technically *weaker and more complicated*, and offer conceptually *vaguer* models of explanation As a result, it is substantially harder to see how these theories could be imagined to appear, with the hindsight of an enlightened future, as 'convenient myths' which turned out to reflect palpable reality. The technical complication can eventually be excused, as the mechanisms of cognition, not least as regards its linguistic component, are most likely more complicated than the mechanisms of chemistry [13]

The conceptual vagueness is a hindrance to be vindicated. An important contributor to the problem seems to be the current tendency to try to deal with too large structures at a time, fearing being accused of viewing a smaller structure as totally independent of the larger. Language acquisition studies are tormented by the basic, inevitable interdependence of various levels of meaning and reference, and it may be that conceptual transparency is reachable only at the price of ignoring most of them.

If the domain of mathematical knowledge and learning is also becoming a school example of this phenomenon, it would be less forced by scientific shortfalls than by the attempts, mentioned earlier, to 'mimic' mainstream trends. At the very least, one does not encounter a problem of reference which is nearly as serious as in the case of linguistic knowledge, due to the semantic closedness of mathematical sign systems (Rotman, 1988) Needless to say, learners of mathematics do not always experience this absence of referential complexity as an unconditional facility in the learning process; but it might be one of the central aims of mathematics instruction to reveal it.

6. Linguistic aspects of mathematical knowledge

To start from scratch, I wish to examine again what is meant by the assertion *A knows B*. In the context of mathematical knowledge, *B* will have to be rather specifically determined in order for the assertion to be evaluated, just as for linguistic knowledge The phrases *Anna knows English* and *Anna knows mathematics* are equally diffuse, while *Anna knows the passive transformation* and *Anna knows long division* are certainly verifiable with respect to some requirements of performance (and always relative to a limited amount of such). Here, performance does not only mean response to some input, but could also be meant to be more or less spontaneous. But, in both cases, performance refers to discourse produced by *A*, which is evaluated in terms of rules pertaining to the domain of *B*.

The striking similarity between discourse in natural language and mathematical discourse (in the sense of Winsløw, 1998) has at least the following aspects, which form together a strong indication that Plato's problem arises for mathematical knowledge much in the same way as for language [14]:

- the complexity of the rules governing discourse;
- the extent to which consensus may be reached (within a community of discourse) about the rules;
- the implicit nature of the vast majority of those rules.

I have already mentioned one (rather weak) sense of the first point in the case of natural language (even when only considering syntax), and Chomsky's argument is actually to demonstrate the possibility of embedding a non-finite state mathematical structure into English. [15] The third point is in fact an immediate corollary of the first, and although the relevance of explicit rules may be felt (by users) to be higher in the case of foreign language or mathematical discourse, the difference is inessential when compared with rule-bound performance of finite state type, such as playing chess

The second aspect is not theoretical but empirical in nature, but is theoretically essential to counterbalance the other two: without it, neither natural languages nor mathematics would represent useful means of communication. The parenthetic remark about consistency referring to a community is essential, though; it marks the difference between *languages* and *registers*.

A *language* is a system of structured knowledge applicable to express human thoughts, while a *register* of language is a systematic way of using language in some specifiable setting [16] Thus, the second aspect is about registers (within natural languages, and in the setting of mathematical discourse) Besides natural languages, my definition of language admits forms of expression related to e.g. music, dance, fashion, programming and logic; corresponding registers could be, for example, those of jazz, classical ballet, evening dress, C++ and propositional logic.

An important point, which is immediately visible from these examples, is that registers are not mutually isolated, and may not only overlap, but also use - and even require the use of - more than one language at a time, such as classical ballet, in which dance is integrated with music, dress and scenery. This interaction of language forms is of course a main reason for the analytic value of the concept of register.

To penetrate deeper into the connection between natural language use and mathematical discourse, I look more closely how they interact As suggested in Winsløw (1998) and further elaborated below, mathematical registers integrate natural language use with the use of certain symbolic languages This is true in any form of mathematical communication, [17] but it is most obviously and easily demonstrated in formal mathematical *writing* as found in textbooks and articles about mathematics [18]

This may in itself be a reason to analyse this setting first, but a more important one is that formal writing may be thought of as the main medium of 'adult language' in the context of mathematics, as opposed to informal oral communication that prevails at early stages of learning the register (cf. Pimm, 1991). Just as language acquisition theories (such as LAD) are based in a thorough understanding of adult compentence that is the final goal of language learning, whether or not all learners reach this final stage in full, so must a study of mathematics acquisition be enlightened from the beginning by an understanding of the basic principles of the mathematical register in all its breadth and depth

However, none of the principles covered below are irrelevant to mathematics as used and learned at elementary levels (say, the first three years in school): in fact, each of them is crucial to being able to understand or produce the mathematical sentence 'To multiply a number by 2 is to add it to itself, for example 6 x 2 = 6 + 6 = 12', whether conveyed in writing or otherwise.

Notice that a symbol string may refer to geometrical notions which are often (even better) represented by geometrical figures In fact, the notion of symbol string should be understood to encompass geometrical drawings to the extent that they occur as an integrated part of mathematical discourse. This is indeed the case in many important actual contexts of mathematical communication, from school geometry in the plane to surgery in algebraic topology.

The integration of natural and symbolic language is hence the characteristic procedure for the creation of the mathematical register. It takes place as follows

1 Parts of the word and sentence inventory of natural language are *semantically defined or redefined* for use in the mathematical register. For example, in the register of mainstream calculus, which is a sub-register of mainstream mathematics, special meanings are assigned to the noun 'function', to the verb 'integrate', to the adjective 'real', to the complement 'vanish everywhere', and so on These elements are *redefined* since they exist already in the supporting natural language

   Some of the verbal inventory used in the mathematical register is rarely or never used in other registers, as the noun 'integer', the verb 'subtract', and so on; but since the difference between *defining* and *redefining* depends on the status of other registers as mastered by each language user, and since registers are not mutually exclusive, the main general phenomenon to retain here is that the mathematical register generates its own semantics even for parts of its verbal inventory

2 Symbol strings may replace word strings in phrases of natural language, whose syntax remains otherwise unchanged Replacement does, in this context, not mean that an ordinary meaningful phrase in verbal language is transformed into a phrase of the mathematical register by simply replacing some syntactic elements (e.g , a noun) by a symbol string, but rather that the phrase in the mathematical register can be *formally derived* from a syntactically correct (but possibly meaningless) phrase in verbal language by such a replacement. [19]

   Several examples are given in the list below, which is ordered according to the class of the replaced syntactic elements. [20]

   (a) Replacement of *declarative phrases*, as in 'Therefore, $f \neq 0$ almost everywhere'. In this case, symbolic expressions usually contain relation symbols (symbols indicating relations among surrounding symbols, such as =, <, >, ∈, ⇒, etc.). The only exception seems to be symbol strings representing declarative phrases or strings with relations, as in 'if A, then B''

   (b) Replacement of *proper names* in any grammatical function: 'Let f be a function' (This phrase could also be derived by replacement of *article + noun*, but since no article is present in the phrase to be analysed, this is irrelevant.) Symbol strings may only contain relation signs in the following cases: (i) in order to create a description of the symbol on the left-hand side of the relation, as in 'Take $t > 0$ such that . .', which is equivalent to 'Take a positive number t such that . '; (ii) encapsulated symbolically, e.g. by set brackets, as in 'Clearly, $\{t \in A : t \,?\, 0\}$ is open'; (iii) encapsulated semantically in the sense that the relation represents the semantic reference of the noun, as in 'Hence the inclusion $A \subseteq B$ is proper'.

   (c) Replacement of *nouns* after the articles [21] 'some', 'no', 'every', 'any', 'all', 'each', 'such', as in: 'We see that every n is even'. In this case, relation symbols may occur only in cases (i) and (iii) of (b), as in: 'For every $A \subseteq B$, the claim is clear'. Replacement of nouns after other articles, or after adjectives, occurs only in colloquial style: 'Now let's look at our t ', 'This is the $t > 0$ we were looking for', 'Give me a small ε', etc.

   (d) Except for the cases considered in (b) and (c), it is usually considered bad mathematical style to mix words and symbols in the kernel [22] of a single declarative phrase, such as the student-type shorthand 'Hence the function → 0' (but not outside the kernel, as in 'Hence $x_k \to 0$ as $k \to \infty$); in particular, *single verbs* or *verb phrases* are not subject to replacement

3. Finally, the symbolic inventory (producing symbol strings) has its own syntax, as explained in Winsløw (1998a, Sec. 3.2). The basic concepts (called *universal syntactic features*) here are those of *object, relation* and *operator*, and for syntactic relations among symbol strings the derived concept of *transformation*. Notice in particular that there is no subject-object distinction in the function of noun-replacing symbol strings (that is, this function is called 'object' with no implication of such a distinction). The minimal symbol string contains a single object, and the simplest string containing a relation is of form object-relation-object'

Notice how, in 2, the presence or absence of relations places some restrictions on the replacements in which a string may serve, roughly because relations are of a 'verb-like' or 'preposition-like' nature (or both). The syntax of symbol strings (at a deeper level than the universal syntactic features: for instance, the construction of objects), and not least their *semantics*, can be given in detail only for very specific sub-registers, as it is highly context dependent (cf. e.g. Woodrow, 1982).

It is an important characteristic of the mathematical register that symbolic strings may be assigned arbitrary meanings as one of the three types of single mathematical elements: object, relation and operator This assignment is typically more local to context than the semantic redefinition of verbal language terms. In both cases, I draw attention to the extraordinary semantic flexibility of terms (verbal or symbolic) compared with terms in a register of natural language.

To sum up, the mathematical register is derived from natural languages through *semantic (re)definition* of certain terms, and through *replacement by symbol strings* of some of its syntactic elements; the symbolic language inventory has its own syntax and semantics, which interact in systematic ways with the syntax and semantics of the verbal inventory

The point of the above sketch is that it shows the *similarity*, the *relation* and the *difference* between knowledge of natural language registers and knowledge of mathematical registers The similarity has already been discussed at the beginning of the section, but my discussion of the mathematical register also aims to demonstrate that many of the descriptive notions from linguistics are useful to describe (the communication of) mathematical knowledge. The difference is, on the other hand, quite striking, especially the presence of a symbolic component, and the special features (of linguistic type) related to it.

I notice that the difference is primarily one of *inclusion*, in the sense that all the syntactic features of natural language are present in its use in the mathematical registers, along with additional rules On the other hand, as emphasised in Winsløw (1998b), this is not at all true at the semantic level, as there are many important types of meaning (e.g. feeling and opinion) which may be expressed in natural language registers, but not in mathematical registers, and *vice versa*.

Finally, the most important point here is the crucial *relation* between natural language and mathematics - the fact that mathematical knowledge, understood as knowledge of the mathematical register, is to a large extent *dependent on* knowledge of natural language. The impact of children's language backgrounds on their performance in mathematics learning contexts has indeed been the subject of numerous studies, [23] and it has been shown beyond reasonable doubt that this influence is both strong and many-sided, even when discounting external factors that correlate with natural language capacities [24]

To advance from the recognition of significant data correlation to the study of causal relations, we need hypotheses - convenient myths - regarding the source of those correlations As was argued in this section, such a model of mathematics acquisition will have to relate to (or even contain) a model of language acquisition. For the reasons put forward in sections 4 and 5, one may consider the original model of a LAD as a simple yet significant model of the last sort, and I use it as a working hypothesis for natural language acquisition in the following - the main point being the possibility of *inference* from a model of a LAD to a model of a MAD

7. Contours of a minimal MAD

My task now is to determine a minimal set of competencies for acquiring mathematical knowledge which are not themselves acquired, but which are part of the human cognition apparatus. It is obvious that minimality is most desirable, as I am not willing to settle for the easy but mystifying Platonic solution (claiming *all* mathematical knowledge to be 'built in').

On the other hand, my arguments so far are intended to demonstrate that if the overall aim is a model *accounting* for all mathematical knowledge - and not just the chess-type parts - then this minimal set is not empty. In fact, language acquisition competencies are necessary but not sufficient. The existence and importance of such a minimal solution is a main point of this article. Notice that *uniqueness* is neither implied nor claimed to be implied from the present understanding of the issue. In fact, as already mentioned, the below discussion is of 'if-then' type, where the assumption ('if') is chosen as Chomsky's (1965) model of a LAD

The discussion in section 6 does suggest some elements of the cognitive faculties which the human capacity to learn and foster mathematical knowledge seems to necessitate The most basic one is the ability to perceive symbolic language and to distinguish it from natural language. This it not in itself a faculty which is solely related to mathematics in a strong sense (like mathematical registers as considered here); the human use of symbolic inscription and signification is likely to be as old and broad in scope as natural languages.

We may think of this as a supplementary feature belonging to the 'input representation device': the capacity to perceive and distinguish symbol-type input in different surface forms (audible and visual). This faculty is related to 'mental imagery' as studied in cognitive psychology (Paivio, 1971) and some of its effects are (in a more direct way) studied in iconography. Where the properly mathematical enters the scene is in the ability to represent and manipulate structures involving states and processes of symbolic entities; as suggested in Figure 2, this amounts to recognising and distinguishing the four ground categories of object (objectified symbol), relation, operator and transformation.

	Representation	Structure
State	Object	Relation
Process	Operator	Transormation

Figure 2 Semantic interpretation of universal semantic features

This faculty clearly pertains to the structural description of represented symbol input: that is, it belongs to what is called the structural description device in Chomsky's LAD. It accounts for the universal syntactic features of mathematical knowledge: some symbols are perceived as distinguished *objects* which may be found in the state of *relation* to other objects, while other symbols represent processes by which objects are changed (*operators*); and finally that such processes induce new relations when related objects are changed by the same operator (the state structure is changed by what I call *transformation*).

However, the concrete transformational rule relates different structural descriptions and hence it belongs to the domain of 'grammar', just as in the case of natural language. Assuming further the capacity of evaluating different transformational grammars of symbol language against structural descriptions of represented input, I have situated all the elements of a MAD (construed in analogy and even in addition to the LAD) required to account for the universal syntactic features of mathematical symbol language.

To accomplish the task set out at the beginning, reconsider the phenomena of (re)naming and replacement (point 1 and 2 in section 6) by which the mathematical register is brought together as a whole. Relating these to the symbol processing parts of the MAD described above seems to be a problem of considerable difficulty and, I believe, one of crucial importance to the whole issue. I must confess that none of the solutions suggested below seem to be entirely satisfactory. However, it seems clear from section 6 that a realistic theory cannot describe the MAD as an autonomous cognitive 'module', and instead requires it to be construed in tight interaction with a LAD; naming and replacement are crucial, and difficult to account for, exactly because they are central to this interaction

Naming, which effectuates semantic (re)definition, carries greater weight in mathematics than the mere assignment of lexical signifiers. Viewed semantically, conceptual 'encapsulation' (or 'reification' in the sense of Sfard, 1991) is a distinguished feature of many instances of naming, bringing mathematical elements into existence as linguistic entities in their own right

For example, words like 'function' or 'converge' do more than label in the learning of elementary calculus It seems clear though that a similar naming procedure is as crucial in other domains of communication as well, and that its use in the mathematical register is distinguished only by its role in the interaction of symbolic and natural language, this being perhaps the source of its relatively high potential for structured abstraction I believe it is important to distinguish two different naming procedures here, as suggested by the following examples.

(a) 'A function G from A to B is a subset of $A \times B$ such that for every $a \in A$, there is precisely one $b \in B$ such that $(a, b) \in G$.'

(b) 'Let G be a function ' (Equivalent: 'Let a function be given; call it G ')

The first kind assigns a name to a mathematical structure, and this relation of name to structure becomes part of the 'lexicon', much as in the case of natural language If we are, at a later point, told that a function is a non-commutative group, this leads to conflict By contrast, the second kind of naming feeds a short-memory name to a symbol lexicon which can easily be changed the same way by later namings. The last naming of a symbol prevails and must often be available in the lexicon *before* applying the syntactic parts of the structural representation device to the whole sentence (as in 'Let G be a function with $G' > 0$ ...')

Any revision of the structure-to-name lexicon must be based on the structured description of the whole sentence, as in the feeding of the natural language lexicon. To sum up: the structural representation device contains two semantic components handling two kinds of naming, one which feeds a symbol-name lexicon directly, and one which marks (parts of) the sentence as a naming of structure, which (via the evaluation measure) causes a relation name-structure to be activated in the corresponding lexicon Both lexicons are part of the grammar used in (and affected by) the structural description of a sentence.

The simplest way to accommodate the phenomenon of *replacement* at this final stage seems to be twofold: an 'inverse replacement' component of the structural representation device, adding to the structural description a base phrase from natural language from which the sentence can be derived by lexical insertions; and an addition to the grammar, consisting of rules for replacement by symbol strings (just as rules of replacements are built into the grammar of natural language, cf Harris, 1965, §2)

This part of the grammar relates the syntax built into the LAD and the syntax of symbol language of the additions described in this section Notice that the rules of replacement are, like the rest of the grammar inventory, only 'built in' as an inventory of potential rule sets; which one is acquired depends on the input. Hence, the rule set sketched in section 6 (point 2) merely aims to reflect the competence acquired from exposure to a current version of the mathematical register. [25]

The tentative model of a MAD developed above is summarised schematically in Figure 3, which contains (and is built upon) Chomsky's LAD as interpreted in Figure 1

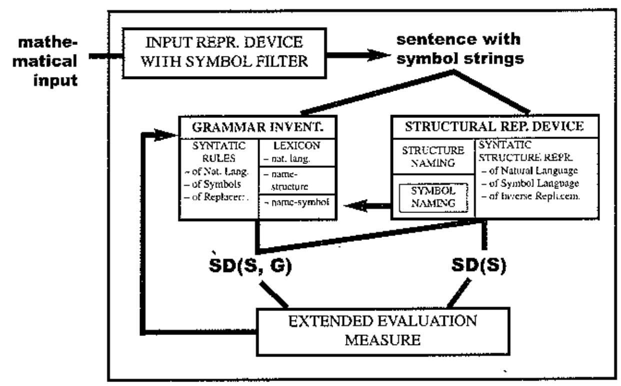

Figure 3 MAD

This picture only aims to reflect the 'acquisitionist' aspects of learning, but it should be clear that output (and hence participation in mathematical discourse) is not ignored because of this. On the contrary, the value of the evaluation measure (specific lexicons and syntactic rules) at any time reflects the linguistic-type knowledge based on which the learner performs mathematically (or, as the LAD is built in, using natural language alone)

What cannot be accounted for by such a model is the actual production of output; the mental structures determining output are, I believe, far too complex and individual that one could hope to make meaningful conjectures about them in anything like the above form. In particular, this includes the trivial remark that genius is *not* amenable to systematic study. However, it is clear that in addition to linguistic-type knowledge, a central item is *memory and the associative use of it*. The common point of constructivism and Platonism it to explain all cognition in the framework of a memory-type structure.

8. Philosophical outlook

At the end of this article, I confront the philosophical issues and positions considered in the initial sections with the view of mathematical knowledge and learning exposed in what followed. I do this by presenting some fragments of a conversation that may have occurred to the reader in the course of reading, as they have to me while writing

PLATONIST: Your account of mathematics acquisition is essentially nominalism in a psychological guise. The stable nature of the subject is not captured by an innate mental framework for learning its 'language'

RESPONSE: Mathematics is neither more nor less stable than natural language. Stability in your sense should maybe rather be called 'accessibility'. We are able to gain access, with more or less effort, to any human language register because of universal acquisition capacities. A MAD makes Euclid's geometry accessible to modern teenagers as it made it accessible to Euclid's contemporaries. However, differences in mathematics register background may result in very different competencies acquired from Euclid

CONSTRUCTIVIST: On the contrary, I feel that all this innate business has been neither motivated nor satisfactorily argued, and that it cannot be. In the example, you simply refer to everyone's *reconstruction* of Euclid, as it may be prompted by the reading of his books

RESPONSE: Right from the start, you need to build a full grammar of the type considered in section 7. Natural languages do not suffice to relate the different sentences, nor do the explicit definitions, axioms and postulates constitute anything like a sufficient set of rules to picture what makes us follow even the first proof. Without the elements of symbol syntax and its relations to natural language, you would not even be able to 'construct' a *wrong* understanding of the first few pages

PLATONIST: Let's leave the academic spectre of Euclid and look at reality. What a strange coincidence that the world continues to reveal new mathematical patterns, from particle physics to cosmology, if we are to believe that the 'universal syntactic nonsense' of object-operator-etc. is all we are given *a priori*!

RESPONSE: Phenomena are 'discovered' only to the extent that we have a language to describe them. Kepler's laws of planetary motion were the best possible description in the mathematical register of his day, while some of their inaccuracy has become apparent more recently in terms of chaos theory. Truly, many types of mathematical registers are potential in the MAD - but as registers belong to the social domain, they are neither subject to arbitrary or abrupt change, nor developed independently of their experienced capacity to 'describe' phenomena

CONSTRUCTIVIST: How does mathematical creativity and invention come in? Your grammar seems only to be changed in automatic ways, by input.

RESPONSE: To create or revise the grammar is not a significant feature of mathematical creativity, and acquisition models do essentially nothing to enlighten us as regards the creative use of language, except that of course knowing (hence learning) language is a prerequisite. Ramanujan, the famous Indian mathematician, was able to create deep and new mathematics after solitary reading of a couple of textbooks (Dehaene, 1997, pp. 160ff). But these initial works were clearly limited to the narrow domain of mathematical knowledge he had acquired from his readings. Ramanujan's story is surprising in two ways: the absence of (ordinary, many-sided) social interaction in the acquisition process, and the level of creativity demonstrated in performance. Only the first is relevant (as an exceptional case) to the issues addressed by the MAD

PLATONIST: Whatever they are, I see any distinction between learning and discovery as an undue promotion of the principles of arbitrary circumstance. Ramanujan 'discovered' many things which were also written in books he, due to particular but uninteresting circumstances, had not read

CONSTRUCTIVIST: He simply arrived at the same constructions, among others which were his own and which he created for the first time

PLATONIST: Create or discover - let's not get trapped by language. Let's trap this MAD theory instead What does it have to say about the natural numbers?

RESPONSE: Their potential infinity arises from the absence of limits on string length, allowing e g the strings 'I', 'I I', 'I I I' etc.; operators (increments, and so on) and relations (e g inequalities) are easily built from naming procedures, but how - and whether - this happens, depends crucially on received input.

PLATONIST: Then who did it first? - and from what input? And what about the reals?

CONSTRUCTIVIST: There are no reals, or natural numbers, except in our minds.

RESPONSE: I strongly disagree with the last claim Numbers are also represented in the discourse of all kinds of text: [26] indeed, they were developed in text, applying mathematical registers Their relative permanence is due to the transmission of those texts, which in turn is made possible by our language capacities. As to the Platonist question of origins, I shall answer by another question: 'Who pronounced the first word?'.

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Notes

[1] For yet another famous exposition of this viewpoint (the so-called allegory of the cave), see Plato's Republic, Book VII

[2] See e g Hersh (1979) for the manifesto of the movement in question

[3] Davis and Hersh (1980) expresses a common attitude of mathematicians thus:

> the typical working mathematician is a Platonist on weekdays and a formalist on Sundays (p. 321)

[4] It may not be entirely new For instance, Usiskin (1996) notes that:

> Chomsky's characterization [of language] could apply as well to mathematics (p 232)

although the subject is not further elaborated there

[5] Maybe a slight reservation is in order here, as the mere attempt to seek logic and coherence in terminology might be viewed by some as a vague form of structuralism

[6] Of course, transitivity is here in the linguistic sense (requiring a grammatical object), which is totally different from the mathematical notion when it makes sense to compare them (such as for *know*, understood as a relation among individuals ("to be aquainted with someone"), which is of course not the sense in which the word is used here)

[7] One might, as a curiosity, notice here that this seems to be true even as regards 'constructivist mathematics', despite its attempts to recover or reformulate mathematics related to indirect proof and 'infinity':

> Euclid, in the 20th proposition of Book IX [ ] knew what he spoke about (Heyting, 1956, p. 7).

[8] In the usual sense: all, and only, correct sentences

[9] The mere concept of innate knowledge conflicts with the basic assumptions of constructivism, but need not imply acceptance of Plato's ideal world Indeed, Chomsky views innate knowledge in the less exotic framework of evolutionary biology (Chomsky, 1988), and a different elaboration on the possible source of innate knowledge may be found in Barnes (1997) Incidentally, a favourite aphorism among intuitionists is the much more theological view (of Kronecker) regarding the *a priori* quality of the number faculty

[10] In fact, Chomsky (personal communication, 1999) has informed me that the quotations cited here are not entirely accurate

[11] This original model has since been replaced by more sophisticated models (Chomsky, personal communication, 1999)

[12] As an early exception, Skemp (1987) noticed with enthusiasm what he saw as a neuro-psychological confirmation of his analytically based theory of visual-geometrical thinking as opposed to verbal-algebraic thinking; Skemp's theory was published in 1971, and the relevant work in neuro-psychology (due to Shannon) in 1980

[13] In a certain physiological sense, they are clearly not more complicated, as they may be described as chemical processes I trust the reader will not find this a relevant objection.

[14] This is in a way also the original setting, as described in section 1.

[15] In Chomsky (1957, n 3), it is also explicitly noted that:

> any formalized system of mathematics or logic will fail to constitute a finite state language

[16] The definition of the concept of *register* goes back to Halliday, McIntosh and Strevens (1964 p. 87), although formulated in different wording and with a narrower definition of language than given here The use of the concept of register in connection with mathematics seems to originate in Strevens (1974), and is extensively discussed in Pimm (1987), Winsløw (1998) and Morgan (1998)

[17] It may not be clear to the reader what should be perceived as mathematical communication. In this case, this proposition could alternatively serve as a (very loose) definition

[18] Symbolic language does not necessarily use visual media, such as writing, but most people find it hard to handle more complex symbol strings otherwise, and so at least at a post-elementary level the act of symbol writing is important also in the creative aspects of symbolic language The important point is that this is a matter of practice, not of principle

[19] In more technical terms: the *phrase structure* is the same, but in addition to usual lexical insertion, some elements of a phrase may be filled by symbol strings One may think of the latter as 'replacing' a natural language string by the symbol string, although no such replacement actually takes place For instance, the (simplified) phrase structure 'name-verb-predicate' could yield 'John looks happy', or '*f* is continuous'

[20] To be precise, the analysis here concerns the present-day, mainstream mathematical register as built from the English version of natural language. Even if most of what is said here remains true in all Germanic and Romance languages the author knows of, it is known that there are significant differences (arising from fundamental differences in syntax) between the forms of interplay of very different natural languages (e g English and Japanese) with mathematics, cf e g Hosoi (1983)

[21] By 'article', I refer to the grammatical *function*, not to morphology, where it is customary (in English) to classify only the words 'a', 'an', and 'the' as articles.

[22] The *kernel* of a phrase is obtained by deleting as many elements as possible while keeping the phrase grammatically correct. In most cases (the most common exception being phrases containing conjunctions), the kernel is uniquely determined, and in most cases (except exclamatory phrases and the like), the kernel contains a verb in finite form

[23] A monumental collection of such studies may be found in Cocking and Mestre (1988).

[24] That is, one has identified even purely *intrinsic* effects, in the sense of Saxe (1988).

[25] As mentioned in [20], it also depends on the participating natural language

[26] The term 'text' is used here in the extended sense of communication theory, corresponding to our extended notion of language, to mean any manifestation of language use

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