# Arguments from Physics in Mathematical Proofs: an Educational Perspective

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The first premise of this article is that proof must be part of any mathematics curriculum that aims, as it should, to reflect mathematics itself and the important role of proof within it. The second is that the most significant potential contribution of proof in the classroom is in the promotion of mathematical understanding, a role it plays in mathematical practice as well (Thurston, 1994). Many educators, proceeding from these premises, have given consideration to how proof is best used in the classroom. There has been a significant reorientation among them, especially in the past twenty years, towards the use of intuition in the teaching of proof (Dörfler and Fischer, 1979). Wittmann and Müller (1988), for example, speak of 'intuitive proof' *(inhaltlichanschaulicher Beweis)*, while both Hanna (1990) and Dreyfus and Hadas (1996) draw on the distinction between explanatory and non-explanatory proofs.

These educators have concentrated on the internal aspect of proof, however; in other words, they have focused in the main on its function within mathematics and have discussed issues such as the relative value of intuition and the study of formal deductions in teaching proof (Hanna and Jahnke, 1993; 1996). However, Jahnke (1978) and Winter (1983) have earlier argued that the usual opposition between 'intuitive' and 'deductive' is unacceptable, and that mathematical proof should not be seen as a turning away from observation and measurement, but rather as it should be seen as a guide to the intelligent exploration of phenomena.

This article seeks to redress this imbalance somewhat by investigating proof primarily from the viewpoint of the relationship between physics and mathematical proof: that is, by using ideas from physics to model a mathematical problem. The specific question this article poses is two-fold: what is the possible role of arguments from physics within mathematical proof and how should this role be reflected in the classroom?

## Previous scholarly work

The first part of this question has to do with mathematics itself. The close co-operation between mathematicians and theoretical physicists has led to a heightened awareness of the many benefits that mathematics derives from physics (Jaffe and Quinn, 1993). Jaffe (1997) points out that physics has traditionally been a source of important problems for mathematics, contributing in this way to its progress, and that in turn mathematical results have helped solve difficult problems in physics.

In an article on the phenomenology of proof, Rota (1997) maintains that the benefits of this close association are to be seen in mathematical proof in particular. Even when a proof has established without a doubt that a theorem is true, mathematicians often remain dissatisfied with it because it has provided little or no insight as to *why* the theorem is true. In such a case, physical concepts and models can often make an important contribution to understanding and can even help mathematicians devise purely mathematical proofs of a more explanatory nature. In addition, however, it can also be useful for mathematicians to incorporate an argument from physics into a mathematical proof as an integral part.

The second part of our question relates to mathematics education. In approaching this issue, we have taken our cue in large part from two publications that deal directly with the role of arguments from physics in the classroom: Winter (1978) and Polya (1954). References to physical laws do appear in other educational publications, but only as remarks in passing. Castelnuovo (1971), for example, introduces projections and shadows when treating the notion of similarity. Struve (1990) discusses geometry as an empirical science in contrast to geometry as a theoretical system. In Bender and Schreiber (1985), one finds a different conception of the relation of empirical and theoretical geometry, based on the ideas of H. Dingler. The following paragraphs discuss work published on closely related issues.

Some recent publications describe various approaches to making proof meaningful in the classroom with the help of empirical arguments. Dreyfus and Hadas (1996) show that teaching geometry using dynamic software can bring students to realize the importance of proof, because they see that proof is required to explain empirical results that are unexpected or counter-intuitive. De Villiers (1995), Mariotti (1995) and Mason (1991) discuss several dynamic geometry constructions, illustrate problem-solving methods not possible with pencil and paper and advocate the use of dynamic software for fostering new insights into traditional geometry theorems. Greer (1996), as well, describes the use of empirical arguments for proving.

There is also a movement among mathematics educators to base the teaching of mathematics in general upon its various applications. One facet of this movement is the suggestion of a closer relationship between mathematics and the other sciences (OECD, 1991). Other facets include Realistic Mathematics Education (RME), the name that has been given to a theoretical framework which advocates using reality as a source for mathematisation (Freudenthal, 1983; Streefland, 1991), as well as a number of other projects that seek to strengthen the role of applications in mathematics teaching in various ways. For the higher grades of school teaching, one must also take into consideration the publications of the ISTRON group (Blum, 1993) and the International Conference on the Teaching of Mathematical Modelling and Applications (ICTMA - see Niss *et al.*, 1991)



*For the Learning of Mathematics* 22, 3 (November, 2002)
FLM Publishing Association, Kingston, Ontario, Canada

None of these proposals, however, deals explicitly with the teaching of proof

## What is meant by 'arguments from physics within mathematical proofs'?

To explain better the idea behind the proposed investigation, we would like to draw a clear distinction between using arguments from physics within mathematical proofs and merely using physical representations or illustrations of mathematical concepts or theorems. An example of the latter is the representation of the commutative law for multiplying natural numbers by a rectangular configuration of pebbles (Figure 1).

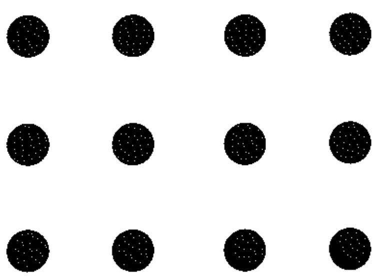

*Figure 1: The commutative law of multiplication*

The idea is that, given a pair of natural numbers, say 3 and 4, it is possible to build a rectangular array of pebbles of length 4 and width 3 representing the product $3 \cdot 4$. Rotation of this array by 90° will lead to an array representing the product $4 \cdot 3$ without changing the number of pebbles. Thus, the identity $3 \cdot 4 = 4 \cdot 3$ is proved. The generalization to every pair of natural numbers (i.e. the proof that $m \cdot n = n \cdot m$) relies on the argument that whatever can be done with the specific numbers 3 and 4 can be done with every other pair of specific numbers

A decisive feature of this well-known intuitive proof of the commutative law for natural numbers is that every operation involved is immediately accessible to our senses. A physical model is used which directly represents the essential properties of the mathematical notion in question. Only the statement that we can perform these operations with every pair of natural numbers would require that we think of all possible rectangular configurations of pebbles and thus transcend the visible world

In contrast to such a mere physical representation, let us look at a typical example of what we mean by an argument from physics in a mathematical proof. It is a well-known theorem of elementary geometry that, given an arbitrary quadrilateral ABCD, the mid-points of its sides R, S, U, V form a parallelogram (see Figure 2). A purely geometrical proof of this surprising result might divide the quadrilateral into two triangles and apply a similarity argument

An argument from mechanics, on the other hand, would consider points A, B, C, D as being loaded with equal masses (each of measure 1) and connected by rigid but weightless rods: it would be based on the postulate stating that "any system of masses has only one centre of gravity".

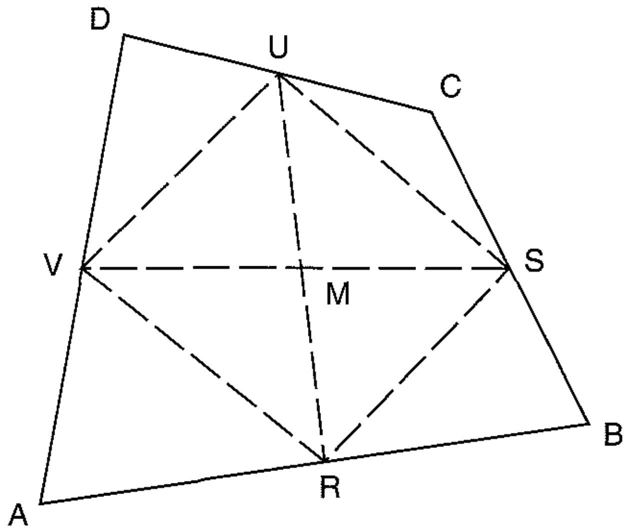


*Figure 2: The midpoints R, S, U, V form a parallelogram*

Of course, the whole system of weight 4 has a centre of gravity, which we are going to determine The two sub-systems AB and CD each have weight 2 and their respective centres of gravity are their midpoints R and U. From static considerations, we may replace AB and CD by R and U, each loaded with mass 2. But AB and CD make up the whole system ABCD Its centre of gravity is therefore the mid-point M of RU. In the same way, we can consider ABCD as made up of BC and DA. Therefore, the centre of gravity of ABCD is the mid-point of SV. Since the centre of gravity is unique, this mid-point must be M, which means that M cuts both RU and SV into equal parts Thus, RSUV, whose diagonals are RU and SV, is a parallelogram.

In this proof, the whole argument depends on the fact that every physical body with a mass has a centre of gravity. Of course, this centre of gravity is something we may point at after we have found or constructed it, and thus it does exist in our visible world: in this sense, it is comparable to the pebbles of the previous proof. But there is a fundamental difference between the centre of gravity and the pebbles, because the centre of gravity owes its meaning to a complex physical law governing the mechanical behaviour of physical bodies.

Roughly speaking, the centre of gravity has both a static and a dynamic meaning. The static meaning is that a body is in equilibrium if it is supported at its centre of gravity. The dynamic meaning is that in many situations a moving body may be replaced by its centre of gravity. Though this law of mechanics governs our visible world, the law itself is not visible. It is something theoretical, a construction of our mind. In terms of its potential function in a logical argument, it may be viewed as just another mathematical theorem which we have at our disposal. Thus, invoking an argument from physics in a mathematical proof is very different from simply using a physical representation: it means relying on a physical principle as if it were a theorem of mathematics.

We can also learn from this example that an argument from physics may not only broaden our basis of argumentation by providing us with an additional principle, but may also add to our intuitive understanding of the mathematics involved. To an untutored mind, it may seem rather surprising that every quadrilateral, however irregular, has a property of such high regularity. When one considers the quadrilateral as a four-point system, however, it is immediately clear that its centre of gravity must also divide into equal parts the two levers that connect the mid-points of opposite sides

There are historical as well as educational examples of the use of arguments from physics in mathematical proofs When a purely mathematical proof of a theorem looks elusive or awkward, mathematicians have often found that the introduction of concepts and arguments from physics yields a straightforward proof. A famous example is Archimedes' use of the law of the lever for determining volumes and areas Another equally famous example, from the calculus of variations, is the so-called Dirichlet principle, which asserts the existence of particular minimal surfaces as solutions of certain boundary value problems In the nineteenth century, Dirichlet and Riemann took this principle as obvious for physical reasons. Weierstraß later criticized its use, however, forcing mathematicians to look for a purely mathematical proof of the principle. This was quite hard to achieve, but in the end the effort led to considerable progress in the calculus of variations (Monna, 1975).

A less well-known example is provided by Giovanni Ceva (1648–1734) In 1678, he published a small booklet of about 80 pages entitled *De lineis rectis se invicem secantibus statica constructio* ("Statical construction of straight lines cutting each other") In the preface, Ceva explained that instead of applying geometrical constructions by ruler and compass, as geometers usually do, his idea was to 'replace lines by weights'. By attacking problems of geometry purely through the use of statics, he was able to disentangle difficulties which up to then had proven beyond reach. Using his method, Ceva found the theorem which still bears his name and demonstrates the very general conditions under which the three lines joining a vertex of a triangle with a point on the opposite side intersect at a single point (see below). Pierre Varignon (1654–1722) was also working from the new foundation of statics when he found the theorem discussed above dealing with a parallelogram inscribed within an arbitrary quadrilateral Considerations of statics can be very powerful in elementary geometry as well, as we will see in the next section.

Let us turn now from the history of mathematics to mathematics education. Examples of the application of laws of physics to mathematical proofs in an educational context can be found in the chapter entitled 'Physical mathematics' of Polya's (1954) book *Mathematics and Plausible Reasoning*. Polya uses principles from optics and mechanics to solve a series of optimisation problems in a very illuminating way. An especially striking, elegant and famous example (which, by the way, can be found in earlier publications by other mathematicians) is the construction of the Fermat point of a triangle, where the triangle is modelled by a mechanical system consisting of a perforated plate and weighted ropes (Polya, 1954; see below).

A collection of theorems of elementary geometry that can be proved most easily by applying the law of the lever or the notion of centre of gravity can be found in Winter (1978) In the next section, we will discuss some examples. Other relevant publications are two books from Russia, Uspenskii (1961) and Kogan (1974), and a nice article by Tokieda (1998). An example from the differential calculus is provided by the mean value theorem If we interpret the derivative of a function as the velocity at a given instant, then the mean value theorem follows directly from the observation that a car going from A to B must have had, at least at one point, the mean velocity as its actual velocity.

Such applications of physics do much more than *illustrate* a theorem. By introducing productive concepts, they make possible a more satisfactory proof of the theorem, and one that, on the basis of an isomorphism between the mathematical and the physical constructs, is arguably no less rigorous. In this, they differ from much of what has come to be called 'experimental mathematics', which in its essence consists of generalizations from instances.

There are cases, of course, in which the use of concepts and arguments from physics leads the mathematician to the realization that there is a logical necessity that has yet to be proven. Often, however, arguments from physics are primarily a way for the mathematician to produce a more elegant proof. Frequently, such a proof may also be illuminating, in different ways. It may reveal the essential features of a complex mathematical structure or point out more clearly the relevance of a theorem to other areas of mathematics or to other scientific disciplines Using an argument from physics may also help create a 'holistic' version of a proof, one that can be grasped in its entirety, as opposed to an elaborate and barely surveyable mathematical argument

These broader benefits are invaluable even to the practising mathematician, so they clearly have great potential for promoting understanding among students. Unfortunately, this potential is not being exploited, because concepts and arguments from physics have not been integrated into the classroom teaching of proof to any great extent and certainly not in any organized way. This is not surprising, since there is no body of research work on this topic that might provide guidance and tools for teachers and curriculum developers.

## Examples

We begin with some examples showing the fruitfulness of centre of gravity arguments. One has to assume (and to make plausible to the students) these three principles, treated as postulates:

- the law of the lever;
- every geometrical configuration (a line, a plane figure or a solid) has a centre of gravity;
- the centre of gravity of a configuration can be determined by determining the centres of gravity of its sub-systems and then composing the whole from its parts

**Example 1**

By a rather simple physical argument, we can demonstrate the theorem of geometry that the medians of a triangle meet at a single point. From the three principles above, we can immediately find the centre of gravity of the triangle by first considering the triangle as loaded at its vertices with equal masses of weight 1 (Figure 3) (The vertices are considered to be connected by rigid and weightless rods.)

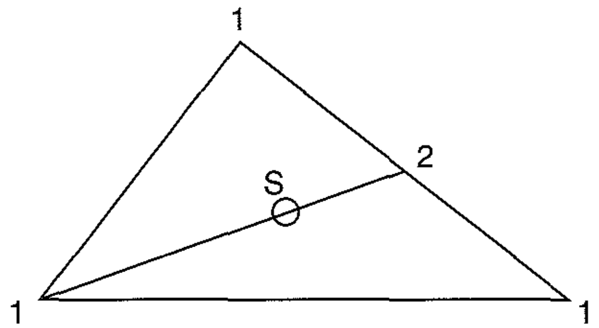


*Figure 3: Triangle with equal masses at its vertices*

Then, the mid-point of a side is its centre of gravity, loaded with weight 2. If we connect this mid-point to the third vertex to form a median, the centre of gravity of the whole triangle must lie on this median, and, by the law of the lever, must divide it in the ratio 2:1 Since this construction can be repeated using the other two sides, the three medians must meet in one and the same point, the centre of gravity.

Example 1 has actually been the subject of a teaching experiment that sought to determine the extent to which arguments from physics might help students understand and prove that the three medians of a triangle meet at one point, which is the centre of gravity of the triangle (Hanna, Jahnke, DeBruyn and Lomas, 2001) The results of this experiment were somewhat encouraging From the students' comments and the assessment of their work by the teacher, it was quite clear that several students showed an understanding of the role of principles from physics in proving a mathematical theorem However, some of the students were misled into viewing the proof as only a generalization from an empirical observation.

Examples 2, 3 and 4 are additional examples that we have not yet tried out in the classroom.

**Example 2**

The considerable power of arguments from statics is impressively demonstrated by the fact that the preceding argument can be generalised immediately to three dimensions, to show that the four lines connecting the vertices of an arbitrary tetrahedron to the intersection points of the three medians of the opposite triangles meet in one and the same point To prove this complex geometrical theorem, we consider the tetrahedron as loaded at its four vertices with equal masses of weight 1 (Figure 4).

Again, the vertices are connected by rigid and weightless rods. Then, we consider the centre of gravity of one part-triangle. It will be the point of intersection of the three medians of this triangle and will be loaded with weight 3. Therefore, the centre of gravity of the whole tetrahedron

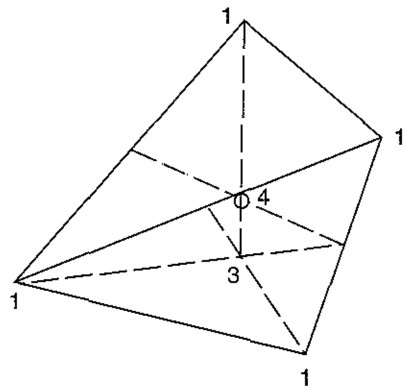


*Figure 4: A tetrahedron with equal masses at its verticies*

must lie on the line connecting this point and the remaining vertex and, by the law of the lever, it must divide this line in the ratio 3:1. Since this argument may be applied to every part-triangle and its opposite vertex, it is clear that all these lines meet in one and the same point, the tetrahedron's centre of gravity It is easy to invent further theorems of this type.

**Example 3**

Ceva's famous theorem gives a general condition under which three internal lines of a triangle (lines joining a vertex to a point on the opposite side) will intersect at a single point From this general condition, one can derive special theorems for angle bisectors, medians, heights and perpendicular bisectors. (In the following we will refer to internal lines as 'transversals'. Transversals which satisfy Ceva's condition are known as 'Cevians' - see Figure 5.)

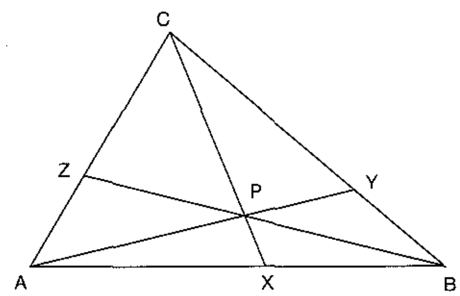


*Figure 5: Cevians in a triangle*

The theorem states (see Figure 5) that three transversals AY, BZ and CX of a triangle ABC intersect at a single point P (that is, they are concurrent) if and only if the two products formed by 'alternating segments' are equal - that is:

$$\overline{AX} \cdot \overline{BY} \cdot \overline{CZ} = \overline{XB} \cdot \overline{YC} \cdot \overline{ZA}$$

or

$$\frac{\overline{AX} \cdot \overline{BY} \cdot \overline{CZ}}{\overline{XB} \cdot \overline{YC} \cdot \overline{ZA}} = 1$$

To prove this theorem, we again consider the vertices of the triangle as loaded with masses, but this time of different weights. The general idea is to define the weights in such a way that the intersection of the three Cevians becomes the centre of gravity of the triangle. Specifically, the weights are defined so that:

$$\text{mass in } A : \text{mass in } B = \overline{XB} : \overline{AX}$$

and

$$\text{mass in } B : \text{mass in } C = \overline{YC} : \overline{BY}$$

Then X and Y must be the centres of gravity of AB and BC respectively. Now, we suppose that the three transversals meet at P. Since X is centre of gravity of AB, the centre of gravity of triangle ABC must lie on CX. The same argument applied to Y and BC shows that centre of gravity of triangle ABC must lie on AY. Therefore, P is centre of gravity of triangle ABC. From this, it follows that Z is the centre of gravity of AC, hence:

$$\text{mass in } C : \text{mass in } A = \overline{AZ} : \overline{ZC}$$

Therefore:

$$1 = (\text{mass in } A : \text{mass in } B) \cdot (\text{mass in } B : \text{mass in } C) \cdot (\text{mass in } C : \text{mass in } A)$$

$$= \frac{\overline{XB} \cdot \overline{YC} \cdot \overline{AZ}}{\overline{AX} \cdot \overline{BY} \cdot \overline{ZC}}$$

To show the converse, we suppose that the alternating products are equal, i.e. that their quotient is equal to 1. Again, the weights in A, B, C are defined as above, such that X and Y are the centres of gravity of AB and BC respectively. Then it follows, from the condition for the alternating products which we now suppose, that:

$$\text{mass in } C : \text{mass in } A =$$
$$(\text{mass in } C : \text{mass in } B) \cdot (\text{mass in } B : \text{mass in } A) =$$
$$\frac{\overline{BY} \cdot \overline{AX}}{\overline{YC} \cdot \overline{XB}} = \frac{\overline{AZ}}{\overline{ZC}}$$

Therefore, Z is centre of gravity of AC. This means that the three transversals, since each connects one vertex with the centre of gravity of the opposite side, must go through the centre of gravity of the whole triangle and thus meet in a single point.

**Example 4**

Different mechanical principles feature in this example, the determination of the Fermat point of a triangle. These are, first, the addition of forces by the parallelogram rule and, second, the fact that a system of connected bodies in our gravitational field tends to a state where its common centre of gravity is in its lowest possible position (the potential energy is minimised).

The Fermat point of a triangle ABC is the point X which yields the minimal sum of distances from X to the three vertices (Figure 6). An example would be provided by the optimal location of a power plant supplying three cities.

Solutions of the problem using methods from elementary geometry or calculus are neither simple nor obvious. The

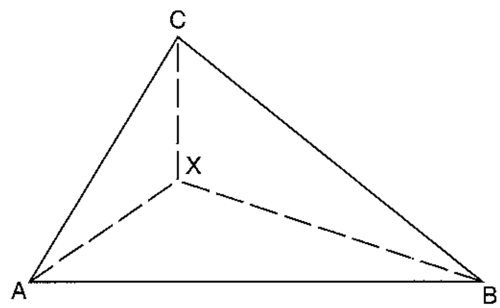

*Figure 6: Fermat's point*

following mechanical device, however, will lead to an immediate solution. Let us consider three pulleys A, B, C free to turn on nails fixed in a vertical wall (Figure 7).

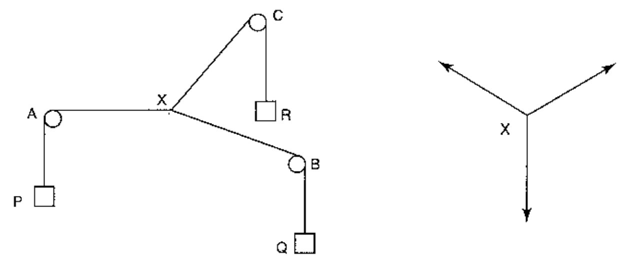

*Figure 7: A mechanical device to find Fermat's point*

The three strings XAP, XBQ, XCR pass over the pulleys. At their common end-point, they are attached to each other, and at the other ends of the strings hang the equal weights P, Q, R.

To which position will X move? Of course, this must be a position where the three forces acting on X due to the weights balance each other. On the other hand, X will move to the point at which the three weights P, Q, R, taken together, hang as low as possible. This is the state of minimum potential energy. This means that AP + BQ + CR takes on a maximum.

Since the sum of the lengths of the three strings is constant, it follows that AX + BX + CX takes on a minimum. Hence, X will move to the Fermat point of triangle ABC. Since in the position of equilibrium the three forces balance each other, and since the weights are equal, then, for reasons of symmetry, the directions of the forces must be equally inclined to each other.

Therefore:

$$\angle AXB = \angle CXA = \angle BXC = 120°$$

A point obeying this condition can easily be constructed.

Looking back on this clever idea, one has the impression of a real insight. From the outset, one has the feeling that the Fermat point is characterised by a sort of symmetry and the physical analogy makes the nature of this symmetry immediately clear. The price one has to pay for this intuitive insight is no more than the application of a holistic, very general and basically non-mathematical principle.

## Broader educational aims

There are broader educational reasons for studying the use of arguments from physics within mathematical proofs. We will sum up these in the following statements.

- As mentioned earlier, there is a trend in all Western countries away from using proof in the classroom. In our view, this development threatens to undercut the educational value of mathematics teaching and should be countered by fresh approaches to the teaching of proof.

- Given the emerging interest in restoring proof to the mathematics curriculum, this research is highly topical, in particular because it proposes ideas that can help overcome previous difficulties with the teaching of proof.

- In our view, the present trend to the systematization of experimental mathematics, which concerns itself with the production of data that are 'completely' reliable and the effective communication of insights (Borwein, Borwein, Girgensohn and Parnes, 1996) should be reflected by an increased emphasis on experimental mathematics in schools. But experimental mathematics in schools should not be restricted to 'mathematics with computers'. From an educational point of view, this would be a dangerous development. Rather, experimental mathematics should include a strong component of the classical applications of mathematics to the physical world. In doing experimental mathematics, students and their teachers should be guided by the question of how mathematics helps to explore and understand the world around us.

- As discussed above, experimental mathematics does not stand in opposition to proof. On the contrary, it is closely aligned with it. If experimental work is aimed at real observation, it has to be intelligent. Therefore, it requires the building of models, the invention of arguments to the question 'why', the study of consequences from assumptions, etc.

- In Western countries, physics - the discipline nearest to mathematics - has become less and less a required subject. To maintain meaningful and interdisciplinary mathematics teaching, it may therefore become necessary to include some elementary physics in the mathematics curriculum. Of course, this will have to be done carefully, bearing in mind the value of the manifold applications from the social sciences which have entered the curriculum in the last few decades. Nevertheless, we think that some adjustments to the curriculum will be necessary if we are to convey to students a more valid and balanced view of mathematics.

- The holistic aspect which many arguments from physics can bring to mathematical proofs, as mentioned above, is an important part of mathematical competence that is frequently under- estimated. Instead, there is a predominance of step-by-step procedures. Finding good examples of instances where arguments from physics are useful in mathematical proofs will help develop a way of teaching and learning mathematics which is more balanced in this regard.

## The contribution of arguments from physics to the educational evolution of proof

The educational aspect of our question actually comprises three tightly linked issues.

- How can the actual role of arguments from physics in mathematical practice best be reflected in the curriculum?

- How can such arguments best be used in the classroom to promote the understanding of specific mathematics topics?

- How can arguments from physics contribute to the development of an adequate understanding of proof on the part of students?

To address the first issue, one would have to examine the epistemology of mathematics implied by much of present classroom practice and compare it with the accounts of the nature of mathematics implied by the practice of mathematics itself or espoused by mathematicians and philosophers of mathematics.

Implicit differences of epistemology are important. For example, students are often taught that the angle sum theorem for triangles is true in general just because it has been proven mathematically. Ignoring the fact that measurements have shown this relationship to hold true for real triangles as well, this practice implies a very specific and limited view of the nature of mathematics and its relationship to the outside world (for a full elaboration of this argument, see Hanna and Jahnke, 1996, pp. 892-899). Students do not share this view, however, bringing to the classroom the belief that geometry has something to say about the triangles they find around them. In this, they may unwittingly be closer than the curriculum to the broader view of the nature of mathematics held by most practising mathematicians. For this reason, it should come as no surprise to educators when students are taken aback, misinterpreting the assertion that mathematical proof is sufficient in geometry to mean that empirical truth can be arrived at by pure deduction.

It would seem that educators themselves need to come to the classroom with a more satisfactory understanding of the nature of mathematics, one that encompasses its relationship with the empirical sciences and everyday human experience. Of course, the curriculum itself should be informed by the same understanding.

The second issue is the use of proof for the promotion of understanding. Students being introduced to mathematical proof come to the classroom with preconceived notions and complex epistemological uncertainty. Educators need to understand both much better than they do today. When confronted with the proof of a theorem, for example, students quite often say that they have understood the proof, but still ask for additional empirical testing. From a purely mathematical viewpoint such a request seems quite unreasonable and teachers usually take it as an indication that the students did not really understand what a mathematical proof is. From the viewpoint of a theoretical physicist, however, the same request would seem quite natural; no physicist would accept a fact as true simply on the basis of a theoretical deduction. Thus, a consideration of the role of mathematical proof in theoretical physics may well shed light on the way in which students view proof.

Keeping in mind the viewpoint of the theoretical physicist is useful when analysing how students approach proof when using dynamic geometry software such as *Cabri-Géomètre* or *Geometer's Sketchpad,* which allow exploratory work similar to experimental physics. Comparing students with theoretical physicists also promises to be of help in understanding how teachers might best cope with the questions that may be created in students' minds by the use of concepts and arguments from physics in mathematical proofs.

Introducing concepts and arguments from physics into the teaching of geometry could have another healthy effect on the evolution of students' understanding of proof. One of the most difficult problems faced by educators when they start doing proofs with their students is the systematic nature of Euclidean geometry. Today, of course, nobody would teach Euclidean geometry in an axiomatic way. And yet a closer analysis of geometry textbooks and the practice of teaching would show that Euclid's system is always present. It determines to a large extent the sequence of theorems and the arguments students are allowed to use in a proof. Teachers have a mental picture of a 'grand theory', of which they are bringing only a small part to the attention of their students.

As a consequence of this practice, geometry is bound to appear arbitrary and dogmatic to the students. Why are they asked to prove the angle sum theorem, but are allowed to use facts about angles formed by parallel lines intersected by a third line, rather than vice versa? Years ago, Freudenthal and other educators proposed the idea of *local organisation* in geometry to overcome this arbitrariness:

> in introductory geometry the student can be led to learn to organize shapes and phenomena in space by means of geometrical concepts and their properties. At a higher level he should organize these concepts and their properties by means of logical relations. Above this level, this relational system can become a subject of investigation (Freudenthal, 1973, p. 458)

What Freudenthal had in mind is shown by his example of the theorem on the perpendicular bisectors of a triangle. It is not necessary, in his view, to give a complete proof that calls upon the entire (implicit) background of the Euclidean system, starting with the equidistance property of perpendicular bisectors and then progressing to the fact that they meet in one and the same point. Rather, one can concentrate on certain aspects which, for one reason or another, are of interest in the specific teaching situation, while taking other aspects for granted. Thus, local organization aims at the exploration of a certain configuration and not at establishing a purely deductive truth within a large system.

In line with Freudenthal's idea of 'local organisation', we would propose a distinction between 'large' and 'small' theories. Instead of building up a large theory (called, for example, Euclidean geometry) in the course of the curriculum, it seems to be more appropriate to work in several small theories around fundamental and stimulating applications. The physical mechanisms described and analysed in elementary statics could provide fruitful examples of such small theories. If such examples were included in the curriculum, geometry might lose a lot of its image as pure theory, detached from applications. In such a learning environment, arguments from physics would find their natural place. Proof would lose much of its dogmatic and ritualistic flavour and regain its original meaning as a way to search for answers to the question 'why'.

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