Communicating Mathematics is Also a Human Activity

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That was the title I had committed myself to for a lecture to the 1981 Easter Course of the Association of Teachers of Mathematics, held in Lancaster, England. Really it was only a subtitle Six years earlier David Wheeler had given his lecture, choosing the title *Humanising Mathematical Education,* and in the subsequent write-up [1] he issued the following challenge

> *Perhaps it is a title that should be allocated to one lecture at each of the next ten conferences and a series of speakers invited to work on it. In this way, after ten years, we might find that ATM had made a contribution to the task of humanising mathematical education, the value of which no-one could question.*

Each course since has usually had one speaker dealing with some aspect of this theme. (See, e g [2,3]) My own modest contribution seemed to be appropriate for someone who was editor of the Association's journal. After all, communication is also an aspect of mathematical education All the same, I felt, with David Wheeler:

> *Having originally chosen a title which defined a large enough space for me to feel sure that I would be able to find some place in it from which to speak when the time came, I now feel that I chose not too wisely but too well I see it now as a powerful and evocative title that covers far more than I dreamed, and I know that I have only a tenuous grip on a few aspects of it.*

There are indeed many aspects of communication in this context, and broadly I would like to consider three of them.

(1) *Teachers' communication of ideas about mathematics to children.*
(2) *Children's communication of ideas about mathematics to teachers or to each other.* (If you want to oversimplify you can call (1) "questions" and (2) "answers"!)
(3) *Teachers' communication of ideas about (1) and (2) to each other.*

If I can jump to (3) for a moment, I have some comments on the lecture as a means of communication The write-up of David Wheeler's lecture was headed "Reconstructed from notes for a lecture.. " In a way my lecture itself was reconstructed from the notes for it. There seems to be a great difference between written and spoken words, and I find that in reconstructing this article from both my notes for the lecture and my memories of what I actually said, I am having to rearrange things considerably and express things quite differently. I can never understand how whole conferences on mathematical education — or on anything else — can be based on the curious phenomenon of people reading to other people what they have already written, and even in some cases first circulated to the listeners Either words are intended to be read, or they are intended to be heard. My worst experience, and the only time I have read a lecture, was delivering a painfully constructed paper in corrected Danish to an audience of Danes in a painful accent, wondering at times if either I or they understood what I was saying!

I have in fact had occasion several times to speak in *English* to foreign audiences, and I find that the secret is not just to speak slowly and clearly, avoiding colloquialisms and obscure constructions, but also to leave gaps in between the sentences, so that the listeners can translate one sentence before they hear the next Such pauses are, in a smaller and different way, built into our everyday speech, which contains a mass of legitimate redundant language as well as ums and ers and "you know" and "I mean". These give us, both speakers and listeners, time to understand the essential words And these space-fillers are necessary If not, we would all speak in telegrams

Now, there is something about mathematics in particular that lends itself to a conciseness which makes it difficult to understand if taken at speed. This can make the mathematics lecturer who reads his notes completely incomprehensible, and students are presumably then meant merely to copy for later study But reading mathematics is a skill that students need to be trained in, so that they read it as a series of statements, each to be considered carefully, rather than as a poem or a novel where the sense either takes second place to the music of the words, or is sufficiently clear because of the built-in redundancies As a case in point, I began reading Hofstadter's deceptively entertaining book [4] as though it were a novel, and was brought up sharply on page 34!

Children also need time to think in between the things they read or hear, and also in between the things they say. Sometimes they provide their own space-fillers in the form of words, and a teacher needs to tolerate the muddle while they sort it out. Witness Cheryl, aged 10, in the following extract from a lesson.

I was wanting to know what number, when squared, gave 15129, and I had offered to square any number for the children on my calculator, but they all had to agree on which number to try. We already had

	71	5041
and		
	233	54389,

and there were suggestions of 139, 232, 137 and 217. Cheryl now said:

If it's over 232 it can't ... that's too much. if it's less it might be able to . but if it's only just about — like — 232, that won't be it because that's too much too; 217 might give us a low number . It might give us a low number but it might give us a clue to what number it is, though. If it gives us a smaller number — 217 — if it gives us a smaller number in the answer it means it's smaller than that, and if it's less than that — or more — wait a minute — if it's more *than that . . . it's less than that, I think . . . If the answer comes out a big number, bigger than that number, it means it's going to be less than 217.*

So, she got there, all on her own, and with no interruption from some overhelpful teacher.

At the other extreme, Theresa seemed to need encouragement to talk in order to get any mathematics done at all.

I watched her take a sheet of paper and write on it her name, the date, "p. 31", and a "1" in the margin. Then she looked around at the rest of the class who were busily organising themselves.

"What are you doing?" I asked. She showed me the cover of her topic book. Then she stared round the room again.

"You're on that page, are you?" She nodded. She read question 1 to herself. It said: "359 × 8547 = 3036064. How can you tell from the unit figure that the answer is wrong?"

Theresa looked round the room. She turned over to p. 32 to see what was there and turned back again. She asked the girl next to her what she was doing, and went to fetch a book for her.

She wrote "2" in the margin. Question 2 said "Some numbers are both triangle and rectangle numbers. Name those below 20." Theresa wrote "1, 3,"

"What's next?" I asked.

"Five."

"Is it a triangle?"

"Yes."

"Show me."

I gave Theresa another piece of paper and she drew five circles in a row then completed a triangle.

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        0
      0   0
    0   0   0
  0   0   0   0
0   0   0   0   0
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"That's not five, is it?"

Theresa made no reply.

"How many is it?"

She counted, and said "Fifteen". She added fifteen to her list, and then with sudden rapidity put 21 and 28. A pause.

"What have you got to find?"

Theresa pointed to the question.

"Have you got all the triangle numbers there?" I asked.

No reply. A long pause.

We continued in this way for almost 40 minutes. Out of nearly 50 responses to my questions (Theresa never volunteered information or asked questions herself) 3 were gestures, 5 were drawings, 11 were numbers, 24 were "Yes," "No" or "Don't know", 2 were statements of fact and only 3 gave reasons for anything.

But the teacher can overdo the encouragement to talk, especially when he becomes blinkered by his own idea about the situation.

John: A prime number is . . . a number which only one and the number itself goes into it.
DF: Aha. So, is 52 prime?
—: No.
DF: Why not?
Jonathan: 2 goes into it.
Cheryl: There's another number as well.
—: 4. 5. 42.
DF: If 2 goes into it, what else do you know goes into it?

(I have now developed a blockage about finding the factor to pair with 2 by calculating 52 ÷ 2, but the children do not know this!)

—: 1.
DF: Cheryl said something about another number as well. Does the calculator help?
Cheryl: Yes.
DF: How?
Cheryl: You can use it to times all different numbers to see if you get 52.
DF: If you *know* that 2 goes into 52, *how* do you know that 2 goes into 52?
Cheryl: 'Cos it ends in a 2.
—: Or 4. Or 6. Or 8. Or 10.

We discussed even numbers.

DF: So, if 2 goes into 52, what else will go into it?
Edward: 26.
DF: How do you know?
Edward: Because if you add 26 and 26 you get 52.
Cheryl: 2 × 5 is 10 . . . 3 × 5 is . . . 10 × 2 is 20 . . . and I've got it in my head but I can't say it out.

(Cheryl's desperation is catching. I become desperate too . . .)

DF: Well, if 2 goes into 52 exactly, how many times does it go in?
Steven: 26.
DF: How did you know that?
Steven: 'Cos 2 25s are 50, and another one makes 52, and if you share out the other 2 you get 26.
DF: Yes. Is there any other way you could work out how many 2s in 52?
Cheryl: Use a calculator.
DF: Yes. How?
Jonathan: You would times the thing you thought by 2.
DF: Is that what you did?
Jonathan: Yes.
DF: What did you start off with?
Jonathan: 26.

This does not help my problem either, and in my anxiety to pursue my one idea I am ignoring all the fruitful ideas the children are presenting. So neither class nor teacher is communicating effectively, but it is my fault.

I repeat, children need time and space within which to talk. A corollary is that most teachers talk too much. The 13-year-old daughter of my colleague, George Robertson, came home from school one day and said, "You know, Daddy, she's the sort of teacher who tells you something so many times you forget it." The sentiment is echoed by Piet Hein [5] in a poem called *If you Know What I Mean*:

A poet should be of the
old-fashioned meaningless brand:
obscure, esoteric, symbolic, —
the critics demand it;
so if there's a poem of mine
that you do understand
I'll gladly explain what it means
till you don't understand it.

On one occasion I watched a student teacher discovering that John understood something and Peter did not. She asked John to explain it to Peter. I suggested afterwards that she could have asked Peter to explain it to John. It is those who do not understand who need to talk.

But it is not always so easy. And writing is more difficult for children than talking.

"I don't know what to write," said the 11-year-old.

"Tell me what you did," I said.

She told me.

"Now write that down," I said.

But it is literally easier said than done.

Children at best begin with a purely factual account. Sally, aged 11, does write what she did:

> *At first I had to get a piece of paper. Then I had to put two circles at the top of the paper. And see how many lines joined the two circles together. The answer was 4. Then I had to put two more circles at the top but with the edges touching. The answer was 3. Then I had to put two more circles on the paper with the circles overlapping. The answer was 2. Then I put two more circles on the paper and put one circle just inside the big circle but touching. The answer was one. I had to do two more circles which were wright in each other and the answer was 0. After that I had to see if I could get three circles and see I could get up to twelve lines touching all three or just two at once.*

Some of my 13-year-olds began to be more descriptive, discursive, decorative:

> *These groups of matrices will be applied to certain geometrical shapes, to see what transformations take place.*
> *The first shape to play guinea pig to all three matrices will be the ordinary square. We will use axes to use as guides placing the original shape in the top right hand corner.*

In other words, they wrote essays in mathematics, as they would in most other subjects, but perhaps with more dialectic. The trouble was that, four or five years later, the task was to stop them writing in this very *human* way — about mathematics as they did it — and, for the sake of public examinations, produce the expected precision and concision that Charles, for instance, was able to produce:

Following are axioms

① $m(S) \geq 0$ ② $m(\emptyset) = 0$

③ $m(A \cup B) = m(A) + m(B) - m(A \cap B)$

1 $m(\mathcal{E}) = m(A) + m(A')$

$m(\mathcal{E}) = m(A \cup A)$ [since $\mathcal{E} = A \cup A'$]

$= m(A) + m(A) - m(A \cap A')$ [axiom ③]

$= m(A) + m(A) - m(\emptyset)$ [since $A \cap A = \emptyset$]

$= m(A) + m(A)$ [axiom ②]

In a way, this is *dehumanised*. But maybe it is not, because by this stage this form of language is shared by the students and it is therefore a perfectly valid and human means of communication.

What we need to know is how to achieve this, that is, how to make such concise symbols a shared language, an idea included in the submission by the Leapfrogs Group to the Committee of Enquiry into the Teaching of Mathematics in Schools:

> *In addition to being a human activity, mathematics is also a social activity. It follows that mathematics has to be communicated — theorems are enunciated to inform others, proofs are written or spoken to convince or to be tested by others, and symbolisms need to be explained and usable by others.*

The following extract is from a lesson in which pupils aged 12 were investigating transformations, represented by capital letters, and pairs of successive transformations, represented by juxtaposition. The discussion between a pupil (P) and me (T) is about whether to use an 'equals' sign between two pairs of transformations, and precisely what it means.

P: Sir, I think I've got one that clashes.
T: You've got one that clashes?
P: Yes. BC and AD.
T: BC and AD are what? (Writes "BC AD" on blackboard.)
P: I think they both come out in the same way.
T: What can you say about these two, then? (Points to BC and AD on blackboard.)
P: They're either both AD or both BC.
T: Well, what can we say about BC and AD, then?
P: Well, when you do them from the identity, they both come out to the same place.
T: So it doesn't matter whether you do that one (BC) or that one (AD)? They get to the same place.
Are these the same, then?
P: They're not the same, but they come out to the same

place

T: Then what can I say about these two?

P: They come out the same, but they're not the same operation

T: So they're the same in a way?

P: They're the same in one sense, and they're not the same in another. It depends what they're doing.

T: So, can I write that they are equal? (Writes "=" between them) O.K ?

P: It all depends what way you're looking at it If you're looking at it — do they come out to the same place? — then in that case they're equal, but — are they the same operation? — they're not

T: Is that equals sign all right then?

P: It depends which way you look at it

T: Well, if you know what's been going on, and I write an equals sign, then do you know what I mean by the equals sign?

P: Yes.

T: So that's all right, then

P: They both come out to the same thing.

T: Are you happy about that?

P: Yes.

The pupil has caught the essence of an equivalence relation, but this happens because he is telling me, rather than me telling him. All I am doing is asking questions, but it is worth looking back carefully to see what questions I ask

Perhaps the art of teaching mathematics is the art of asking the right questions, but it was always my aim in the classroom gradually to enable or to encourage the children to ask the questions, rather than the teacher. This subtly involves a change from asking questions or giving instructions to presenting situations in which children can ask their own questions, determine the rules, and in general make their own mathematics One ingredient of this approach is the way in which the situation is communicated, a point made very well by *Language and mathematics* [6], in which extracts are given from written work by two children about the "same" topic

> *But one was the result of a written workcard while the other was a record, by a child, of what went on in a class lesson that was mainly a discussion The different initial presentations, one written and the other verbal, produce essentially different results. Whereas talking can hold ambiguity, temporariness and uncertainty, written instructions seem to imply a fixed reality, a unique meaning.*

It is interesting to look at mathematics textbooks over the last fifteen years and observe the varying literary styles in which they are written. It is still possible to find "sums", or tersely given instructions, but very often the conciseness of questions and instructions is interspersed with descriptive prose with more human interest For example, the "chat":

> *This figure is a Magic Square, and is one of the oldest pieces of arithmetic we know There is a Chinese legend which tells how, one day about 4000 years ago, as the Emperor Yu was standing on the bank of the Yellow River, a divine tortoise come out of the river. On its back were two mysterious signs, one of them being this magic square.*
> *The rather strange patterns stand for numbers If we count the dots, we can rewrite the magic squares using figures.*
> *Can you see why it was called a magic square? Try adding the sets of numbers which form lines across, or down, or diagonally from corner to corner.*
> *Magic squares were used as charms in Europe a few hundred years ago, and still are used in some parts of the world. Since the first magic square was devised, many people have found pleasure in discovering ways of making magic squares and other similar figures*

is followed by:

> *Ex 1 Copy these magic squares and put in the missing figures.*

Take the 'chat' too far, however, and the books do not sell! One such series began thus:

> *Patterns are always fun to make. Have you ever thought how very dull the world would be without pattern? The pattern of the bare branches of a tree against a winter sky, of light and shade on a range of hills, of pebbles on a beach or the seeds in a sunflower head — Nature is full of pattern. Man has noticed this and he has often copied Nature and decorated the things he has built and made. The result is that he has produced some very beautiful things. It is only when he forgets pattern that he ends up with things which are dull and ugly.*

and continued in this way throughout, with many stimulating suggestions for mathematical activity, but no numbered questions!

On the whole, I always found it so much easier to communicate mathematics orally, when among other things all the ambiguities could be discussed, as illustrated by the previous discussion about the "equals" sign. Writing questions for children is fraught with difficulties, as panels of teachers found in the early days of the Certificate of Secondary Education when setting examinations for average 16-year-olds; they had to choose between making the questions mathematically correct, and enabling the candidates to understand them!

As an extreme case of non-communication from teacher to child I offer the following, mainly for its entertainment value! It was a home-produced worksheet, and the first question read:

> The set of letters in ESTATE
> E = {E, S, T, A}.
> Write out the set of letters in the word COMMITTEE

I watched the 11-year-old write:

C = {C, O, M, I, T, E}

The next question was:

> Describe this set in words.
> X = {1, 2, 3, 4, 5, 6}
>
> The pupil wrote:
> X = {one, tw, hr, fu, iv, sx}

I need to conclude with some remarks about teachers communicating with each other This is an editorial interest, because this is what articles in journals are about. I also heve a professional interest in it, because my full-time job is concerned with in-service education. And, after all, this is wholly what the course/conference was for, at which I gave the lecture.

Recently I received an article, submitted as a draft. (Would that all writers were so modest!) It began:

> *The pupils were given a piece of graph paper and told to draw a right-angled triangle (isosceles) of side one.*
> *They were then instructed to draw the squares on the three sides and to find their areas. They then observed that the square on the hypotenuse was 2 units in area and the other two squares gave the same area when added.*
> *The teacher asked them to test this result on a 2 unit right-angled (isosceles) triangle*
> *All went smoothly*

There followed an account of more of the lesson, then:

> *The idea emerged that if we boxed in the square A this would be sufficient to ensure we could draw A having already drawn B and C.*
> *It was then a simple step to use some basic algebra to verify the theorem in any right-angle of sides* X and Y

Now what seemed so disappointing was that the bulk of it was at the level of 11- or 13-year olds, a purely factual description of what happened. Yet the exceptions only *hinted* at what else was happening: "All went smoothly" — what on earth does that mean? "The idea emerged that . . ." — How? "It was then a simple step . . ." — Whose step? Simple for whom?

It is not so easy. We really need to know more about how children communicate with themselves, i e. each with himself, not with each other. But we do not even understand our own insights into mathematics. How many times have I sat down with, say, an elementary teacher working at some mathematics I have presented, who has called me across with, "I don't get it" We talk for a few minutes, then suddenly she says, "I get it!" I always try to get them to tell me what happened in between the two states that caused the transition, but no one can ever catch it When we can, perhaps we will have solved the greatest problem in mathematical education.

All I can do is to encourage teachers to write about what happens with children. This has a sort of Hawthorne effect, in that (i) in the telling of it one can heighten one's awareness of what happened, and (ii) there is an effect on the lesson itself of the consciousness of a subsequent report, enhanced perhaps by some means of recording the activity as it happens My own lessons were always enhanced by the presence of visitors when they were there, not least because I had a heightened awareness which was due to seeing things though their eyes. This is also connected with the purposes of giving observation lessons at courses or conferences — this is a most effective way of communicating with teachers about communicating with children It presents an activity to be commented on, and the question of successs or failure is only incidental In fact, we can probably learn more from the "failures", where so much more happens than "all went smoothly". Eric Love, for instance, recognised this when he wrote [7]:

> *To enable children to use mathematics in unfamiliar situations, they must be encouraged to invent mathematical problems themselves and to devise their own methods*
> *Here is an account of a lesson that attempted to do this. The lesson was a failure, but, perhaps, an instructive one.*

Like any lesson, my lecture/this article has merely presented something that happened, an event to be commented on, in the hope that my thoughts will stir the listener or reader into producing his or her own thoughts, which will be much more relevant. Perhaps communication is only an isomorphism at best

If it is too much of a muddle for that, then let Piet Hein [5] have the last word in setting up a smoke screen for me.

LEST FOOLS SHOULD FAIL
True wisdom knows
it must comprise
some nonsense
as a compromise,
lest fools should fail
to find it wise.

References

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