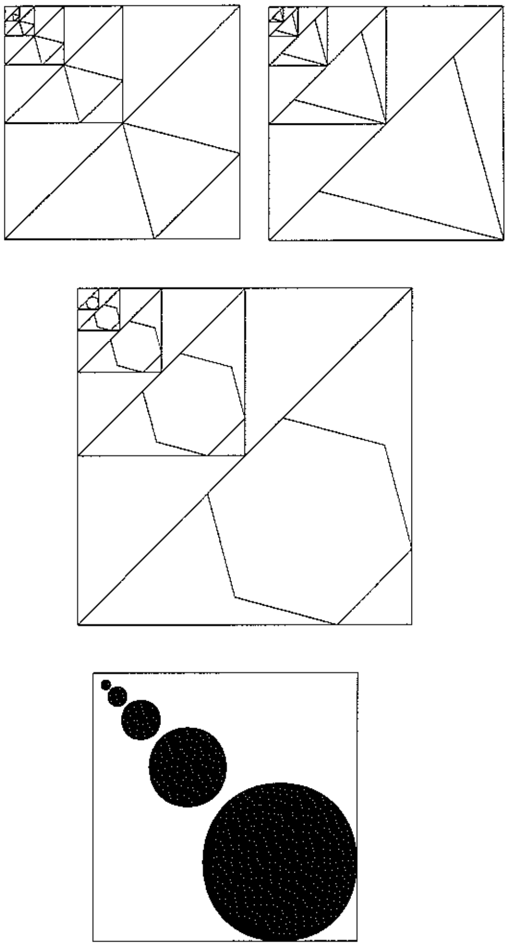


Figure 5:

Figure 5 is an extension for other polygons in van Doesburg's square, plus a set of circles which are the incircles of the triangles See Sharp (in press) for more detail.

Reference

Sharp, J (in press) 'Geometry inspires art – art inspires geometry', *Mathematics in School*.

Re-constructing a Painting with Geometric Eyes

NATHALIE SINCLAIR

Upon receiving issue **21**(2) of *For the Learning of Mathematics*, I turned right to the 'cover story', which featured Marion Walter's (2001) vivid mathematisation of Theo van Doesburg's painting *Arithmetic Composition 1*. I had just begun working with a small group of grade eight students who were working independently on a geometry course I was to visit every week with some tasks using *The Geometer's Sketchpad*. The first chapter of their school textbook focuses on inductive reasoning. As I was preparing a suitable task, I remembered the many interesting mathematical patterns Marion had mined, including the inductive geometric pattern suggested by the painting's self-similarity

By asking the students to construct the painting with *Sketchpad*, I hoped to probe some of their inductive reasoning skills and, also, to introduce a few new *Sketchpad* commands – this was only our third class together and we had not yet constructed a square! I was also interested in seeing how my students would interpret the painting, not so much as art critics, but rather in terms of the mathematical properties and relationships they would notice. The image on the canvas may be singular and static, but the process of construction, which van Doesburg emphasises in his work (his painting is not just an *arrangement* of squares), seems more multiple and dynamic. Might the painting pose a problem that could be 'solved' in many different ways?

First steps

To start, we briefly discussed the painting. Not surprisingly, the students noticed the diagonal symmetry, as well as the leftward 'leaning' of the painting. They seemed quickly drawn to the tilted black squares, but then noticed a host of other squares Using the language of inductive reasoning, we talked about what the 'next step' of the painting would be, as well as the 'previous step' I was hoping to elicit some of Marion's mathematical observations, such as the relationship between successive horizontal squares, but the students were anxious to begin

Everyone first attempted to construct the largest horizontal square: after all, it is the container, frame and boundary of the entire painting *Constructing* a square in *Sketchpad* is not a trivial matter; one must first know what defines a square and then know how to use the appropriate tools. I have noticed that most students start by using the *segment* tool to draw four equal sides (the salient property of the square) and then attempt, when the time comes, to arrange the segments at right angles (the more tacit property) I let the students I was working with draw squares using only the *segment* tool and then showed them how to use the *circle* tool to construct equal segments. Since they had already learned to construct both perpendicular and parallel lines, they were then able to construct their container squares Except Aleah She was stuck on her horizontal segment, insisting on 'turning it' up to a vertical position – not wanting perhaps to bother with circles and perpendicular lines. I showed her how to turn her segment using the *rotate* command. Once she had completed her square, she proudly showed the technique to Sara

I asked Sara which technique she preferred, Aleah's rotation method or her own 'compass and straightedge' one? Sara thought the rotation method much easier and much quicker to perform (given the grammar of *Sketchpad's* tools at least, where rotation is a one-step action) But she described the compass and straightedge method as "more perfect and more mathematical" I am not sure whether Sara has a classical aesthetic or whether she had been enculturated into believing that things which are more technical, more complicated, are (in turn) more mathematical. Whatever the reason, she managed to convince Becca and Zhavain, but not Aleah

Since van Doesburg's painting features many squares, I showed the students how to create a custom tool that would allow subsequent squares to be created effortlessly *Sketchpad's* custom tools are accompanied by scripts, which provide a symbolic representation of the steps involved in the construction associated with the tool, as well as the 'givens' that must be selected prior to using that tool Aleah thought that the brevity of her script would help convince her classmates of the rotation method's superiority (revealing another familiar mathematical aesthetic of economy), but alas, they were stubbornly committed.

Equipped with their square tools, the students were ready to return to the problem. They varied widely in their approaches

A horizontal, iterative approach

Zhavain focused first on the series of horizontal squares. He constructed the mid-point M of segment AB and used vertices A and M as givens to construct another square AMON using his custom tool (see Figure 1)

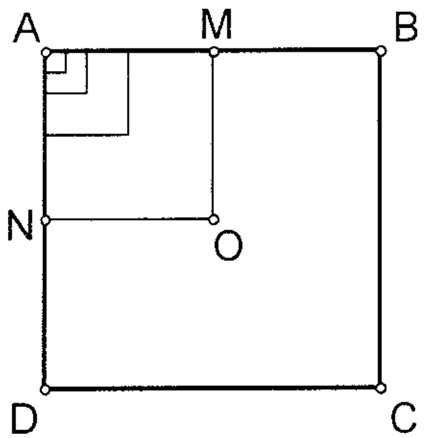


Figure 1:

This strategy he repeated by finding the mid-point of AM and constructing a second square At this point, he asked me whether there was a way of "just doing it over and over" He only had one more square to construct, but perhaps he was already thinking of extending van Doesburg's progression When students say 'over and over', they are ready to be introduced to iteration, a tool I find helpful in clarifying induction steps. (In order to effect an iterated series in *Sketchpad*, students must specify the map between successive terms - forcing a crucial change of attention from the objects students create to the relationships between them.) Zhavain explained to me that in order to obtain his second square, he had simply found the mid-point of the first square's top side and then used it, with A, to create the third square. We asked *Sketchpad* to apply this relation between squares iteratively, generating from the second square a third, from the third a fourth, and so on

Zhavain was quite taken with his iterated squares, increasing the number of iterations until the smallest square (the last of the series) became indiscernible. He then noticed the gnomons and decided to investigate their areas. Looking at the measure of areas, as reported by the software, he found that they were "dividing by four each time", thus, with Marion, noting the geometric - as opposed to arithmetic - nature of the composition

A tilted-square approach

Instead of focusing on the horizontal squares, Sara found her attention drawn to the biggest tilted black square. She used her square tool to construct a square with one vertex V on BC and another vertex W on CD (see Figure 2)

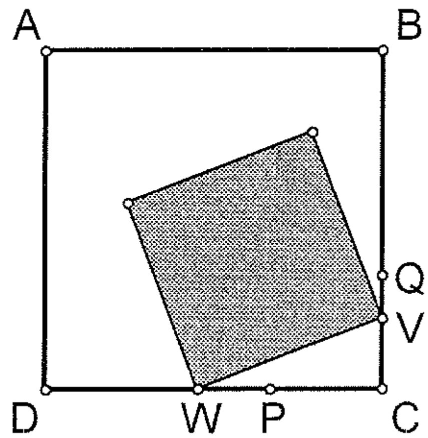


Figure 2:

She then slid the two vertices V and W along their segments until the tilted square "looked right". Choosing these locations heuristically - rather than by analytic derivation - at least allowed Sara to get started on the problem But she grew unhappy upon further reflection. If V and W could be dragged to the right locations, could they not also be dragged to terribly wrong locations? Having conjectured that W should be two-thirds of the way from D to C, she wanted to know whether there was any way to make a W that would "stick exactly there". (In wanting to change the potential movement of the vertices to a secured position, Sara's situation nicely reifies the construction versus drawing dilemma of dynamic geometry.) As it happens, *Sketchpad* allows fixing a ratio, which I showed her how to do This allowed her to construct point P, as well as Q, and then to construct the tilted square on PQ using her custom tool

Sara was happier with her new, fixed, tilted square and turned her attention to the bracket (which corresponds to the next horizontal square) that would hold the next one. She constructed MO and ON (in Figure 1) in relation to AB and AD, respectively. She then repeated her process, by first finding the two-thirds point on NO, and so on. At which point, Zhavain - peering over - told Sara that she could use "a trick" to avoid going through all the steps again. I left them working together

A diagonal approach

Becca had noticed the diagonal symmetry in our initial discussion and her approach used the (invisible) diagonal as a scaffold for the tilted square. She first constructed the segment AC (thus adding a new object to the configuration - often a sophisticated move in geometry). Using her square tool, she constructed a square on the right of the diagonal

She dragged the square up and down the diagonal somewhat, then deleted her square and constructed the mid-point O of the diagonal. Using O, and an arbitrary point L on AC, she re-constructed a square on the diagonal. She then constructed the symmetric square on the left of the diagonal, using points L and O. Finally, by bisecting each square by a segment parallel to the diagonal, she creating a tilted square centred on AC (shown shaded here).

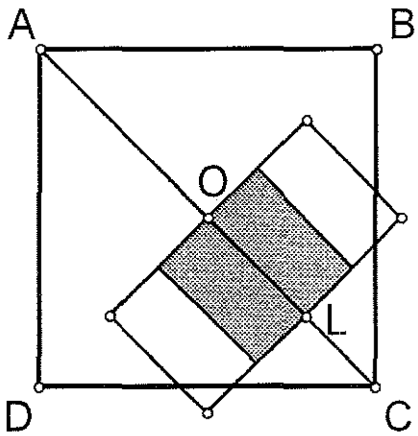
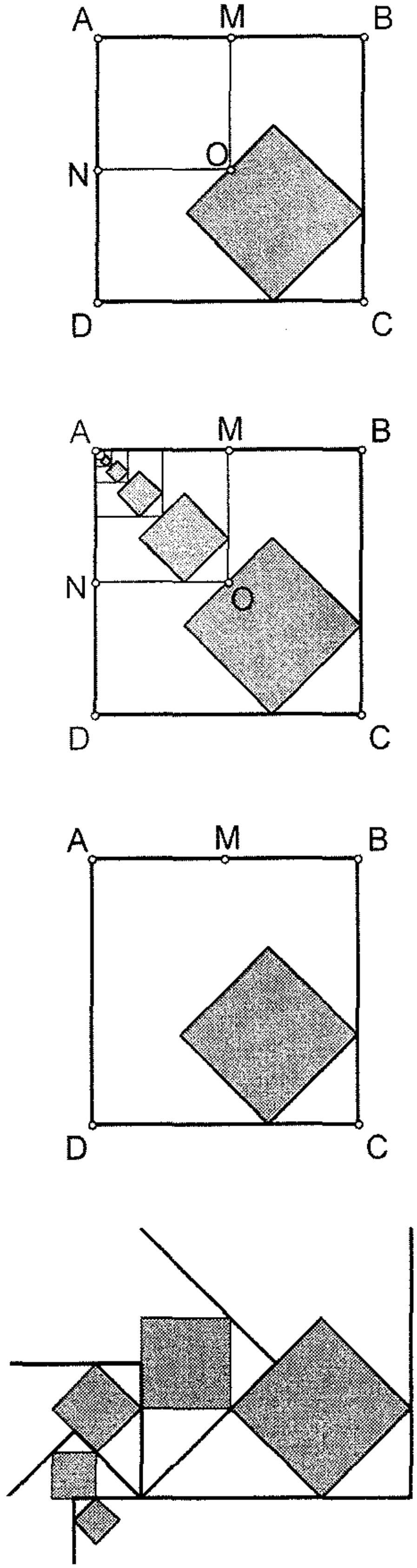


Figure 3:

This allowed her to drag point L until the vertices of the shaded square fell on the sides DC and BC of the contained square. As with Sara, Becca's approach let her get her feet wet quickly. However, she too was unsatisfied with the instability of her dragged point So she began a process of locating the more exact position for L by repeated bisection: the construction of successive mid-points is a process with an oscillating convergence. The mid-point K of OC was not right, nor was the mid-point of KC, but she was convinced that by 'mid-pointing' enough times she would finally find her target. In fact, the correct L is five-sixths of the way from A to C (thus unattainable through repeated halving): but Becca found that she could converge quite quickly to her target and seemed content to use the sixth point she obtained (Visually, her 107/128 is indistinguishable from 5/6, being less than a pixel off.) Becca might not have realised she was approximating, but felt more satisfied with her method of moving deterministically toward the target than with her dragging method. (Though never reaching 5/6, Becca's method quite efficiently converges superlinearly.)

The students reached the end of their constructions, having settled into their different approaches By this time, Zhavain and Sara had figured out that by iterating the "starting construction" (see Figure 4(a)), they could generate the whole painting (and more!). With four iterations, they produced the van Doesburg Plus (see Figure 4(b)). Aleah showed them that the starting construction could be even simpler by removing segments MO and NO (see Figure 4(c)). While trying to find an even simpler starting construction, Aleah accidentally created a variation on van Doesburg (Figure 4(d)), a composition she deemed much more artistic!



(a) Starting construction

(b) van Doesburg Plus

(c) Simpler starting construction

(d) Aleah's variation on van Doesburg

Figures 4a-4d:

Some concluding comments

I would like to close with three brief observations

First, each student saw a very different van Doesburg painting, if we are to believe the evidence of the constructions. An occasional concern aired against dynamic geometry is to observe how likely it is – given the varying forms of constructed determinacy within even a simple diagram – that individual students exploring such diagrams will confront a bewildering array of local behaviours, only some of which are elevated, mathematically, to properties of the figure. Here, we see the same effect in a static diagram, a single image being 'read' very differently by different students: subtle variations in initial perceptions lead students to strikingly distinct emergent mathematisations

Second, neither the painting itself, nor its re-construction with *Sketchpad*, seems to suggest or require the tools of dynamic geometry. However, the power of dragging, here in the aspect of approximation, revealed itself during both Sara's and Becca's constructions They separately relied on dragging as a heuristic in their problem solving I find it curious that both ultimately abandoned their approximations, which seemed at least functionally useful Their insistence on moving from dragged approximations to more stable configurations reminds me of some ancient Greek geometers' aversion to *neusis* constructions [1]. While classroom geometry – unlike analysis or calculus – has all but banned approximate solutions to problems, geometers, as well as educators, have perhaps under-acknowledged the value of approximation techniques in the problem-solving process In fact, geometry might easily rival the conventional numerical setting used to exercise students' estimation and approximation skills

Third, I am struck by the frequency with which the students judged their work by the aesthetic dimension of those judgements Zhavain was intrigued by the clever 'trick' that would by-pass all those intermediate steps Aleah showed her penchant for brevity and simplicity, while Sara and Becca both strove for exactness and certainty. Their desire to find the most basic 'starting construction' also reveals an inclination commensurate with Euclid's desire for identifying the elements of geometry But what made the students feel it was appropriate to make and share their judgements? Perhaps the artistic dimension of the task's source domain provoked them Or perhaps it was the nature of the task itself The students knew exactly what they had to reproduce and could always test their progress against the finished product – without having to rely on my evaluation.

A transition from the empirically visual to the analytic is discernible in both Becca's and Sara's constructions, a transition largely motivated by their quest for exactness. That aesthetic predilection draws each of them away from the contingency of the screen to the determinacy of mathematics. We often hear mathematics defined as being the study of numbers and shapes, or the description of nature. Such definitions – focused as they are on the objects of the discipline – ignore the animating purposes of mathematical activity. Among other things, mathematics perhaps best satisfies the human desire to experience exactness, that fitting moment when, in the words of the poet Paul Valéry, 'a certain position of the bolt [ ] positively closes the lock'

Notes

[1] *Neusis*, or 'verging , is an operation that involves marking a certain length on a ruler, which is then slid into place in a diagram Archimedes uses *neusis* constructions in his work *On Spirals* – it is also possible to trisect an angle using this means Of course, *neusis* constructions are not permitted in the straightedge and compass restriction to which many ancient Greek geometers such as Euclid adhered It remains unclear to historians at what point in time, and for what reasons, *neusis* constructions were deemed to be unacceptable (Artmann, 1999).

References

Artmann, B (1999) *Euclid: the Creation of Mathematics* New York, NY, Springer-Verlag

Walter, M (2001) 'Looking at a painting with a mathematical eye', *For the Learning of Mathematics* **21**(2), 26-30