SEEDS OF ALGEBRAIC THINKING: TOWARDS A RESEARCH AGENDA

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A three-year old child is playing doctor with her stuffed animals. She has a number of possible 'patients' from which to choose and she also demonstrates a sense of what can be a patient; a pencil probably would not be a good candidate, but her stuffed panda would—as would her stuffed pig or cat (Figure 1). The role of 'patient' is a placeholder that can be filled by some toys but not others. We can see the child displaying a precursor to the idea of *variable* in formal algebra. Though probably not thinking about variables *per se,* she is using a cognitive resource that we might call 'replacement.' The idea that there is a position (or role, or 'slot') to be filled in by a limited number of potential candidates is a resource elicited and refined in thinking about variables in later algebraic activity. And this matters, because variables are essential in formalizing structure, pattern, and generalization in algebra, and thus a key focus of algebra instruction (Kaput, 2008; Kieran, 2004).

Children have a wealth of early experiences that have the potential to develop into resources for thinking algebraically. Gattegno (1988) argues these experiences begin well before a child reaches age 3; even in utero, a fetus learns to respond to predictable patterns. In infancy and beyond, children gain relevant experiences as they babble, play with blocks, stuffed animals, or toys.

How do these insights about early child development interact with mathematics education research on the development of algebraic thinking? In the past decades, the field has developed a convincing body of literature about the capacity of elementary school students to reason about sophisticated algebraic concepts and representations (Cai & Knuth, 2001; see Stephens, Ellis, Blanton & Brizuela, 2017 for a recent review). In developing this line of work, however, researchers have tended to first identify important benchmarks of school algebraic competencies, such as viewing the equal sign as a relational symbol rather than an indicator to compute, and then track children's thinking toward those competencies. This approach could be described as mapping back from particular algebraic competencies to identify potential earlier forms of these *same* algebraic competencies. For example, steps toward holding a relational view of the equal sign might move from holding an operational perspective (seeing the equal sign as a symbol to do a computation, reading left to right) to shifting to a more symmetric perspective, attending to both sides in tandem. We believe such an approach is useful for supporting and monitoring children's formal algebraic development, however, it is necessary to distinguish this kind of research-supported design work from building a theoretical perspective on the development of

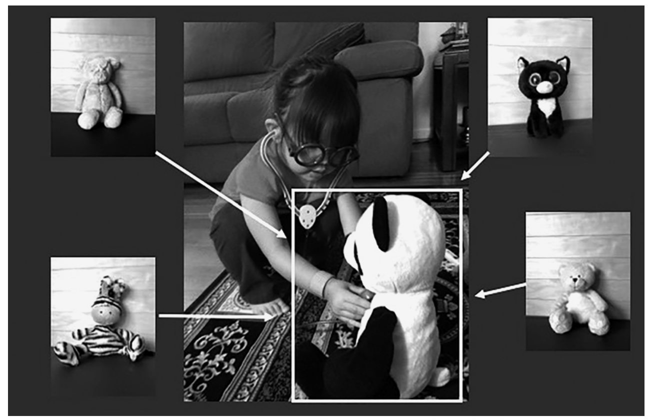

Figure 1. Playing doctor.

algebraic thinking that involves attending to potential pre-instructional sources of the conceptions. Without bringing a developmental perspective to bear on instructional design, children's pre-instructional cognitive resources will remain an untapped resource in supporting their formal learning of algebra. As Mason (2008) describes, children are "enculturated into a climate of non-use of many of their own powers" (p. 68). The question that remains is "how [...] can children's natural powers be harnessed to enrich the emergence of algebraic thinking?" (p. 68).

In this article, we seek to develop a perspective on algebraic thinking that foregrounds the algebraic potential of patterns of thinking that emerge through children's embodied experience and their early pre-instructional experiences, such as through play. In their review of the literature on learning mathematics through play, Wager and Parks (2014) point out that, "almost no work has been done examining the role of patterning and algebraic thinking in children's play" (p. 219). While there is general educational interest in the mathematical potential of children's out of school activity, including the support provided by strategic games, for young children, the mathematics identified in such experiences is more commonly framed as being about either counting or measurement as opposed to algebra. This lack of focus on patterning and algebraic thinking in out of school and play activities is, we believe at least partly, a remnant from earlier perspectives on the development of algebraic thinking that were informed by a view that children must learn arithmetic and counting before algebra.

Mason argues that the field of research on algebraic thinking does not currently have a systematic way of identifying and connecting the resources from children's early experience to algebraic thinking (Mason, 2017). We hypothesize that one thing that has been holding the field back from recognizing the algebraic potential of young children's early experiences is an emphasis on learning trajectories that involve specific definitions of what counts as algebra-relevant knowledge. The systems perspective that we develop in the next section offers a way to recognize a broader range of knowledge, including knowledge derived from early experiences, as algebra-relevant.

A new lens – seeds of algebraic thinking

We argue that a theoretical perspective is needed that can help us recognize the kinds of potentially relevant resources we might find in fine-grained observations of children's out of school experiences, including play. In our approach, we call the pre-instructional cognitive resources we seek to identify 'seeds of algebraic thinking.' We propose that seeds of algebraic reasoning are constructed early on in a child's life and develop as they interact with their world. Like *image schemata* (Lakoff & Johnson, 1980), we view seeds of algebraic thinking as experientially derived sub-conceptual knowledge structures that may be rooted in repeated sensorimotor experience as abstract patterns that help individuals make sense of future experiences. In this way, seeds become resources for a number of algebraic structures, ideas, and concepts.

In particular, we propose that seeds have the following characteristics:

1. *Seeds have long histories.* Seeds are not ideas that are taught in formal instruction, but rather are abstracted over many experiences that take place before school. The experiences related to the development of a seed may come from a child's very early experiences, such as playing with stuffed animals.

2. *Seeds are of a sub-conceptual grain-size.* Seeds get invoked in many contexts and seeds that get invoked in algebraic thinking processes are not necessarily themselves intrinsically 'algebraic' in character (nor even necessarily overtly mathematical in character)—such as 'replacement.' What is often labeled as 'algebraic thinking' or 'proto-algebraic thinking' can more productively be thought of as an ensemble of more fine-grained cognitive resources.

3. *Seeds are contextually activated and therefore not inherently correct or incorrect.* Because seeds emerge from recognizing patterns in experience, seeds play a role in a child's thinking in many contexts, not just algebra. Related to this, seeds can be productively or unproductively applied in a given context and thus are neither correct nor incorrect in and of themselves.

For example, the idea of *balance* has been discussed widely in literature that spans mathematical and scientific thinking. At its smallest grain-size, *balance* is an example of a seed of algebraic thinking. It is a sub-conceptual knowledge structure, acquired at a young age, that undergirds larger conceptual chains of reasoning. Children first encounter the idea when they are very young as they learn to roll over, sit and eventually stand. They encounter the ideas again, as they play with toys or objects around them. They reencounter the idea later in life in varied contexts, such as on the playground or building with blocks. The idea of balance itself is neither correct nor incorrect, but can be used productively or unproductively when applied to a particular context. For example, children's conception of the equal sign is discussed in literature as either an *operational* (seen as a symbol to compute) or *relational* (seen as a symbol of equality). The early acquired seed of balance is instrumental in viewing the equal sign as a relational symbol (as opposed to a signifier to compute an operation)—a key to formal algebraic thinking. While the balance seed must be activated when thinking about the larger, more coarse-grained algebraic concept of equality, it is not inherently tied to that idea. It could also be activated in other contexts, such as on the playground or while playing with blocks.

The broader view of algebraic thinking that we describe here is consistent with a "Knowledge in Pieces" epistemological perspective (diSessa, Sherin & Levin, 2016). Knowledge in Pieces (KiP) models knowledge as a complex system of many local abstractions of experience. In this view, learning is a process of 'tuning' knowledge systems towards more productive configurations through ongoing feedback regarding the appropriate use of particular elements and subsystems. diSessa (1996) used the knowledge systems perspective (and his construct of phenomenological primitives, 'p-prims', a particular class of knowledge elements related to individuals' sense of physical mechanism) to show how physics subjects' explanations of physical phenomena could be modeled as being comprised of multiple productive primitive elements, each rooted in physical experience. This approach to understanding students' explanations as generated by multiple finer-grained knowledge elements, has practical implications. For example, it suggests that the core building blocks of physics novices' explanations are sensible and grounded in experience thus the instructional approach should be aimed at supporting more appropriate coordination and use of knowledge elements (Smith, diSessa & Roschelle, 1993). In contrast, the instructional approach implied by viewing the entire explanation as a unitary misconception (in this case, the particular explanation aligned with what others have labeled the 'impetus theory' misconception, McCloskey, 1983) is replacement or displacement of the entire misconceived explanation.

Our conjecture related to the development of algebraic thinking, guided by Knowledge in Pieces, is that algebra-relevant knowledge is a large knowledge system. Thus, we conceptualize algebraic knowledge, not from the perspective of topics or processes, as defined by standards documents or textbooks, but instead as a complex system comprised of algebra-relevant cognitive resources gained through experience. Similar to the case of decomposing the 'impetus theory' into finer-grained and potentially productive knowledge structures as described above, we conjecture that it may be productive to view chains of thinking that we could characterize as 'algebraic' as also being comprised of a collection of more fine-grained knowledge elements.

Developing algebraic thinking as the coordination of multiple seeds—an example

In this section, we present an example of student thinking that can be seen as bridging 'arithmetic' and 'algebraic' problem solving approaches. Our goal in sharing this example is to highlight how analyzing student thinking in terms of smaller-grained knowledge resources that have their roots in pre-instructional activities can shape the way that we interpret students' developing algebraic reasoning processes in school contexts.

The example concerns a seventh-grade pre-algebra student, Liam, who has invented a novel approach for solving algebra word problems. Initially, Liam used systematic and purposeful guessing and checking, invoking seeds of algebraic thinking such as *direct co-variation* ('If the input variable increases, then the output variable will increase') and *inbetweenness* ('If a guess for an input is too low and another is too high, then the true value should lie in between these two values.') to solve algebra word problems with an underlying linear structure. For example, when Liam chose a value that resulted in an output that was too high, he would note this and then purposefully choose an input value that was lower for his next guess. Once a guess that was too high and then another guess that was too low were obtained, Liam coordinated these inferences to make the deduction that the input value that would satisfy the constraints of the problem context should lie in between the two inputs he had chosen. He would continue with this process of purposefully choosing trial values that would result in him getting closer and closer to the solution to the problem.

During the sessions with a tutor/researcher, Liam subsequently refined his approach based on purposeful guessing and checking into an approach that involved a sophisticated and essentially algebraic algorithm that can be recognized as linear interpolation. That is, while his later approach still relied upon on purposefully selecting multiple trial input values, instead of a general strategy of getting closer and closer, he extracted more quantified information from the data he was generating about his trial values and their results. In particular, he computed the 'unit worth' of one guess, considering the amount the output would change corresponding to a change of one in the input. Liam then took the output corresponding to a particular trial input as a reference and figured out how far that output was away from the target output. Finally, Liam used the unit worth of one guess to figure out how much he should change the input by, in order to produce the change in output he had just computed that he needed.

In Levin (2018) the specific conditions under which Liam refined his strategy are explored, and in the interest of space and bringing attention to the main point here—coordination of knowledge resources that play a role in the development of algebraic reasoning—we present here only the final form of his algorithm. Liam was working on the following problem:

> Anne is twice as old as Paul. Bill is five less than Anne's age. Together, Anne's and Bill's ages sum to 147. How old are Paul, Bill, and Anne?

He had organized and recorded his work in the chart shown in Figure 2.

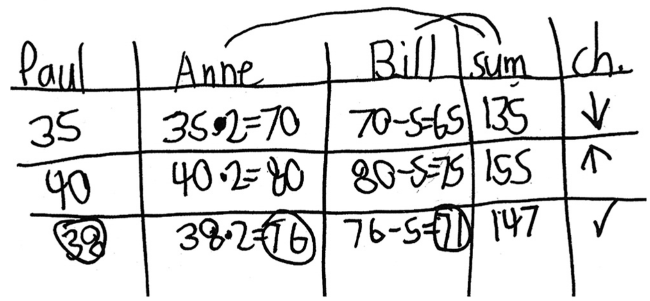

Figure 2. Liam's chart.

> OK. So I'll guess Paul is *[pause, 8 seconds]* 35. Okay, so that was a bit too low *[recording the resulting output of 135 in the chart]*. I will guess *[pause, 6 seconds]*. I'll just guess 40. It's probably less time consuming than trying to find out the relation from this *[Paul]* to that *[Sum]* in some sort of equation. *[pause]* Okay so 40 times 2 equals 80. 80 minus 5 equals 75. So that's 155. Or is that 155? 80 plus 75 equals, hold on. So it *[the input corresponding to 155]* is too much. *[pause, 8 seconds]*. Okay, so the difference between these two *[between 135 and 155]* is 20. The difference between these *[35 and 40]* is 5; 20 divided by 5 is 4. So that means that every time I change this by one *[left hand touching Paul's column]* this *[right hand descends and indicates Sum column]* changes four. So I should lower this *[40 in Paul's column]*. *[Liam writes 38 in the input column, which then solves the problem.]*

In terms of the particular seeds of algebraic thinking that can be identified in this episode, we focus in particular on knowledge resources associated with *co-variation* and *inbetweenness*, describing each in terms of our criteria for seeds.

Knowledge resources related to co-variation have early roots, helping individuals to predict the effect of controlling a cause (an independent variable). For example, consider the situation of a child waiting for her baby pool to be filled up enough for her to play in it, on a hot summer day. The quantity she cares about, the height of the water varies with the time she waits. Assuming the hose runs at a constant rate, this is a linear function. As time increases, the height fo the water increases. The two quantities vary together. Note that co-variation schemes are, in and of themselves, neither correct nor incorrect. What makes them appropriate or not in reasoning about a given context is the variation that underlies the situation. For example, if the situation the student was reasoning about involved depreciation or non-linear variation, reasoning on the basis of direct proportional variation would be problematic.

The second example of a seed that we highlight in the episode of reasoning above is what we call 'in-betweeness.' Examples of inbetweenness in early childhood abound. A discrete version of 'inbetweenness' is invoked in the tale of Goldilocks and the Three Bears when Goldilocks finds the daddy bear's porridge too hot, the mommy bear's porridge too cold and baby bear's porridge to be 'just right.' For a continuous example of inbetweenness, consider something many parents and children do together: chart children's growth regularly on a wall in their home. Let's say that Benjamin's parents measure his height as 52 inches in March, but they forget to measure him again until August. They find that he's 54 inches by August. Thus, they know that sometime between March and August he was 53 inches. With even younger children, we can imagine a child thinking about inbetweenness as they play with blocks or arrange toys. Finally, the idea of inbetweenness itself is neither correct nor incorrect as its productivity lies within the context in which it is used. Eventually, the idea of inbetweeness can be formally expressed as the Intermediate Value Theorem.

While Liam's strategy as described in the example above does not involve formal symbolic manipulations, there are reasons to describe aspects of it as 'algebraic' whereas his earlier approach was 'arithmetic' in nature. For example, the second strategy involves an algorithm that will work for entire classes of problems to *determine* an unknown solution. In contrast, his earlier strategy involved converging to the answer through a sequence of purposeful, albeit, specific numerical calculations. This shift is consequential because guessing and checking, even with purposeful selection of next guesses, is often cast as an arithmetic approach that is undesirable and incompatible with later forms of algebraic thinking. Instead, the case here shows the continuity between an approach that at the macro level expresses arithmetic qualities (guessing and checking) and an approach that at the macro level expresses algebraic qualities (linear interpolation) via the coordination of seeds of algebraic thinking such as co-variation schemes and inbetweeness [2].

Discussion

Many researchers have recognized that children have 'powers' gained from their experiences from infancy on, and that these can be valuable resources to leverage in formal instruction. However, our current ways of conceptualizing algebraic thinking in the K-12 field largely exist at the level of concepts and/or competencies and do not provide us a way to see and leverage (and often end up obscuring) children's sub-conceptual, pre-instructional resources. This is unfortunate because many children struggle in algebra classes and yet the rich resources that could support their learning of algebra go untapped. If we could support teachers in noticing and leveraging children's pre-instructional algebraic resources in the formal algebra classroom, algebra instruction could be transformed.

In this article, we have proposed 'seeds of algebraic thinking,' a lens we believe can help us attend to the wealth of children's pre-instructional algebraic resources. The nature of the elements we consider differs from other work in algebraic thinking in that seeds are not directly related to the algebraic concept at hand, but rather they are pre-instructional resources that a child acquires over the course of his or her life through a range of experiences. Our examples of balance, replacement, and inbetweenness, demonstrate that seeds can be invoked in informal as well as formal algebraic contexts and are not limited to algebraic reasoning.

This work has implications for current lines of research on the teaching and learning of algebra. The way a seed is invoked and combined with other resources can give us insight into a new level of complexity in understanding chains of thinking. For instance, current literature discusses a relational view of the equal sign as 'replacing' the operational view as children's thinking becomes *more* algebraic. In our examples we see that seeds could be cued and brought together in a range of ways that would challenge a binary categorization of the thinking around concepts such as the equal sign. Younger (4–5 year old) children do not yet have substantial experience with the equal sign, however they do have an idea of 'balance' that is a productive resource in later developing a relational perspective. Both an operational and relational view of the equal sign are highly coordinated for students in mathematical contexts and seeds like balance can be invoked in ways that might not fall squarely in one category or the other. This approach to exploring the development of algebraic thinking can complement the current models that exist and inform school algebra instruction, by introducing a perspective on the development of algebraic thinking that considers the sources of children's ideas, and can ultimately help us learn to harness their natural powers in formal algebra instruction.

Reconceptualizing the nature and form of algebraic thinking has implications for classroom teachers in two important ways. First is an epistemological stance that acknowledges that children's intuitive ways of thinking can be useful while learning formal mathematics. This supports much prior work that identifies the ways infants and very young children think mathematically as well as a long history of work calling for formal instruction to leverage children's early mathematical activity. This framing allows teachers to be primed to notice, not only formal algebraic chains of reasoning, but the pre-instructional resources children bring when thinking algebraically. The second is part of the longer research agenda of identifying seeds of algebraic thinking and presenting them in ways that help teachers become aware of their existence in order to leverage those resources to support formal algebra instruction.

Mason (2017) argues that an effective way to elicit children's informal resources in formal instruction is for teachers to provide opportunities in the classroom "for noticing what might previously have passed by un-noticed" (p. 2) and for teachers to be able to recognize and leverage these resources. When provided these opportunities, teachers need to be able to recognize fleeting moments in which children bring resources that can support their formal learning of the subject. Our hope is that by identifying seeds of algebraic thinking and articulating their nature we can ultimately support teachers in noticing these elements of children's thinking that may have otherwise been overlooked. If teachers are able to acknowledge ways pre-instructional experiences can serve as algebraic resources for children, and recognize them as expressed in the classroom, they can work to build on this important set of resources in ways that support formal algebraic reasoning.

Notes

[1] Both authors contributed equally to this manuscript.

[2] Interestingly, from a historical lens, one can recognize Liam's invented approach as the method of double false position (Berlinghoff & Gouvea, 2004). The method of false position was known in antiquity, found in documents such as the Rhind papyrus (2000 BCE), the Hindu Bakhshali manuscript (600 CE) and explained in documents of the Arab mathemati-

cian Al-Khwarizmi as well as Leonardo of Pisa (Fibonacci, c. 1200 CE). In the eighteenth century the African American mathematician, Benjamin Banneker, was well-known for inventing and solving mathematical puzzles employing the method of single and double false position.

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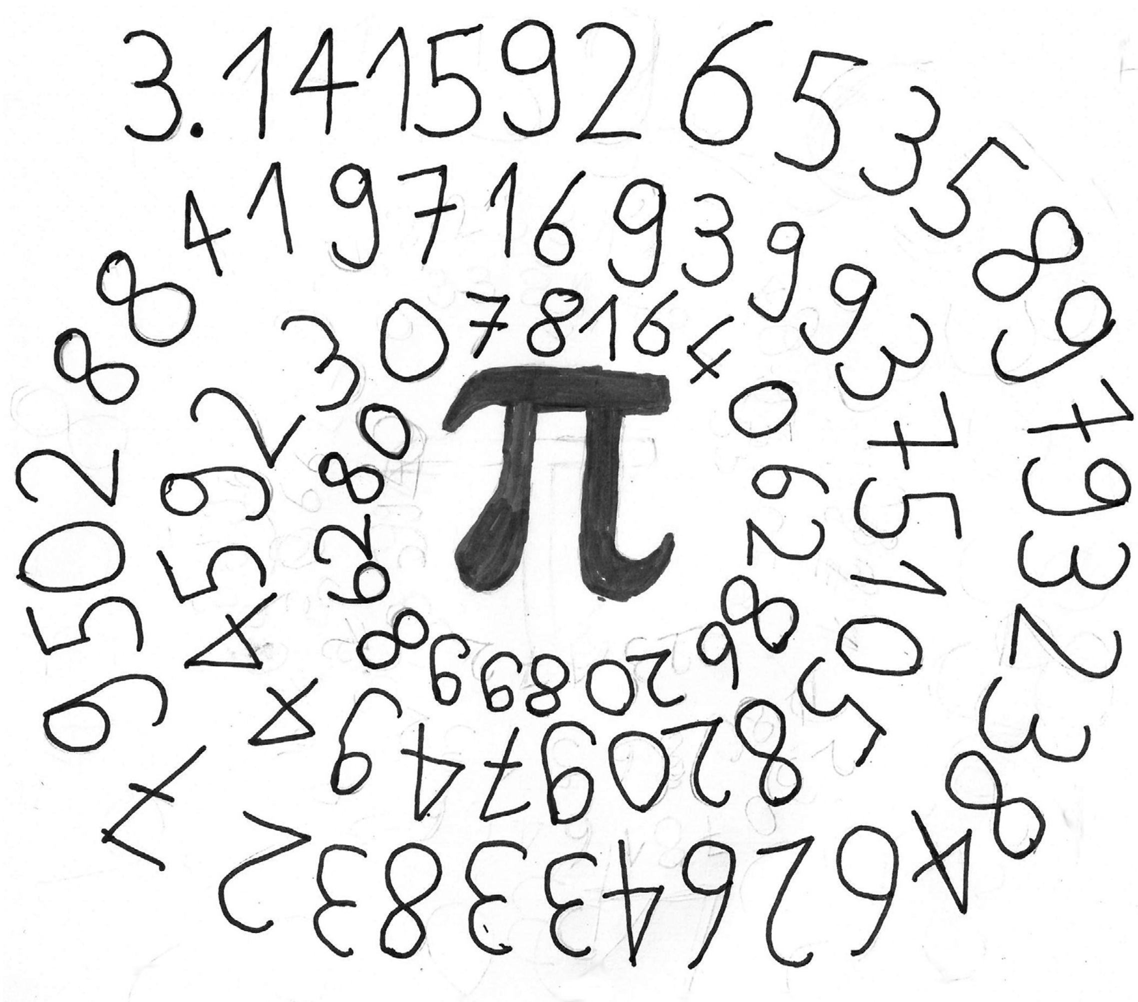

Drawn by Sophia, age 11, on being asked to "Draw something mathematical"