TALKING THROUGH A METHOD

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We explore what it means to build and verify an arithmetic procedure. Specifically, we look closely at a conversation by five college students preparing to become elementary school (age range 5-12) teachers in an experimental mathematics course. These students work within a number system they have just constructed, exploring potential methods for written arithmetic. [1] We argue, based on this case study, that the mathematical potential of pre-service teachers may be much greater than has often been believed. In particular, we offer evidence to challenge the widespread belief that teachers of younger students are themselves not able to do mathematics thoughtfully – that they see mathematical activity as doing what a teacher tells them to do, in situations where correct solutions must be achieved by one given method.

The students that we follow here report important changes in perspective. They work against a background of widely held beliefs that they have come to question:

> Is it true that a procedure is correct, provided merely that it gives 'right answers' in a few test cases?
>
> Is it true that mathematics offers few significant alternatives for working through important problems?

Simple answers to such questions might deflect us from productive thinking about classrooms, where social forces play upon the learners, and where it might be helpful to address significant complexity. (In particular there may be more than one discourse in play, even in a given classroom (Speiser and Walter, 1997). A first student might check procedures by experiment, while the second could insist on reasoned explanations. To engage the second discourse, teachers would need to think conceptually about procedures, while in the first discourse, one might only need to follow rules and monitor procedure steps.)

To make sense of what they had, in effect, been forced to memorize as children (Speiser and Walter, 2000), the students focused systematically on how to build convincing justifications for conclusions they had drawn. In this connection, it is interesting to revisit Benny (Erlwanger, 1975; Speiser and Walter, 2004), the twelve-year-old student whose work and thinking helped to anchor a powerful critique, by researchers, of prevailing practice at that time. Much as our five research subjects must have done as children, Benny implemented rote procedures, but declined to follow a trajectory proposed by his curriculum's designers.

Our students are greatly concerned to build convincing explanations. Benny, too, had addressed questions of validity, but in a very different way. For him, school mathematics was nothing more than arbitrary rules and processes. Such rules and processes, taken individually, might (or might not) make any sense, or even, taken together, be consistent. Hence for Benny, no sense ultimately needed to be made. Benny did, however, try to make sense of the kind of knowledge he was building and perhaps, especially, the social practice that he had to navigate.

The mathematics curriculum that Benny experienced (and resented deeply) implemented a particularly well-entrenched behaviorist approach. Erlwanger's analysis of Benny's case helped to expose a fundamental weakness of the resulting instruction, and, by implication, of behaviorism: that it fails to engage learners as thinkers; specifically that it ignores the need for learner's mathematics to make sense, both internally and as a way to work with realistic situations. In what follows, we emphasize a further problem raised in part by Benny in remarks to Erlwanger about what he called "the rules": his tacit suggestion that one could in principle construct rules as one wished – in effect concoct the whole of mathematics as an arbitrary game divorced from sense or meaning (see page 33 this issue).

Suppose instead that the mathematics had to make consistent sense and also connect usefully to realistic situations – one's freedom for invention should, as a consequence, be limited. In this discussion we will probe, through close analysis of student work on the foundations of arithmetic, the potential scope for creativity, for personal invention, which even very basic mathematics, when conducted with due reference to sense and meaning, might support in realistic social contexts. There are three guiding questions:

- what kinds of choices do these students make?
- what functions might these choices serve?
- what changes can we find in how these students think and see themselves?

To explore how learners work with the variety of possible procedures for arithmetic, we will emphasize the development of *methods* that distinct procedures could be said to share. By *method* we will mean a well-defined approach that may be shared by several procedures. To discuss the functions of the choices that our students make, we trace what might constrain the choices that they make. Such constraints often reflect important needs felt by the learners: perhaps to make sense mathematically, or to come to grips with past experience. These students' past experiences seem to share important similarities with Benny's.

Analytical perspectives

Under behaviorist assumptions, we respond to stimuli in the environment based on internalized experience. Dewey

For the Learning of Mathematics **24**, 3 (November, 2004)
FLM Publishing Association, Edmonton, Alberta, Canada

(1896) deconstructed this conception in a celebrated paper on the reflex arc. For Dewey, the behaviorists had got the process backwards. Problem situations evoke exploration as a response (through a sense of lack, a sense of something missing). In other words, we *seek* a stimulus when we respond to a *perceived need*. Dewey stressed how stimuli emerge from (rather than trigger) actions:

> Neither mere sensation, nor mere movement, can ever be either stimulus or response; only an act can be that; the sensation as stimulus means the lack of and search for such an objective stimulus, or orderly placing of an act; just as mere movement as response means the lack of and search for the right act to complete a given co-ordination. (p. 106)

Dewey's work initiated a progression of important research in psychology, including classic studies of perception by the Gibsons (both J. and E.) and, more recently, Siegler's (1996) cognitive investigations of children's mathematical learning. Here J. Gibson discusses stimuli.

> The observer who is awake and alert does not wait passively for stimuli to impinge on his receptors; he seeks them. He explores the available fields of light, sound, odor, and contact, selecting what is relevant and extracting the information [...]. The passive detection of an impinging stimulus soon gives way to active perception [...] The classical concept of a sense organ is of a passive receiver, and it is called a receptor. But the eyes, ears, nose, mouth, and skin are in fact mobile, exploratory, orienting. (1966, pp. 32-33)

In this view, perception builds from active exploration in an environment in part shaped by the perceiver. Similar arguments hold also for cognition, where ideas and understandings build, through reflection, from responses to felt needs. Siegler (1996) concentrates especially on how to give convincing evidence for change. To make sense of change as it is taking place, he stresses that one needs to follow and make sense of learner's *choices*, in detail and over time:

> What difference does it make whether children know and use multiple cognitive approaches, rather than just one? The import can be illustrated in the context of problem-solving strategies. Strategies for solving problems differ in their accuracy, in how long they take to execute, in their demands on processing resources, and in the range of problems to which they apply. These varying advantages and disadvantages allow children who chose strategies wisely to adapt to the demands of changing circumstances. (p. 16)

In the student work illustrated in the following discussion, alternatives to select from might be strategies, tools for presentation, ways to write a process down symbolically, or perhaps ways to justify results obtained. For clarity, we will concentrate on one student, Marci, to help bring the collective work of the five collaborators into focus.

Background

In September 1995, we began to teach two parallel sections of the two-semester mathematics content course for future elementary school teachers at our university, as teaching experiments. [2] Each section met twice a week for two hours each time. The students worked in groups of five or six, maintained throughout the semester.

Our main interventions took the form of tasks. These were designed to raise fundamental mathematical issues for extended exploration, especially the construction and selection of alternatives. We wanted, through these tasks, to trigger conversations, by the students, that could make their thinking more available to us as well as to each other. We began the course with four weeks of exploration of standard arithmetic, using base-10 blocks (*units*, *longs*, *flats* and *solids*) and Cuisenaire rods. In the fifth week, we asked our students to move away from base-10 numbers so as to work more directly with quantities and operations, not just the symbols used to write and implement them. [3] We hoped that students would begin to think of written numbers and symbolic calculations as tools to accomplish useful purposes that they could design and redesign as needed. In the task we gave them, the design itself would be the problem:

> Use the blocks below [see Figure 1, the students chose *piggy* to denote the smallest block, instead of *unit* including "solids" if you wish, to come up with a number system that only uses the symbols O, A, B, C, and D. Use the blocks to help develop and explain your number system. Your number system should represent any number, no matter how large, and allow you to add, subtract, multiply and divide.

After intense discussion, the five students arrived at a base-5 place value number system [4] – a 5 × 5 array displays in columns, left to right, the numbers A through AOO, as written in this system (see Figure 2, depicting a physical construction of torn paper counters built by Marci to be sure that AOO, at lower right, completes a flat).

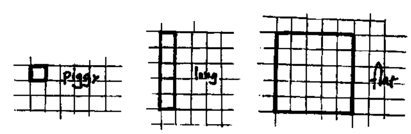

Figure 1: The blocks, as shown on the task sheet.

Thus counting begins with the sequence A, B, C, D, AO, AA, AB, and then continues as shown in Figure 2.

As Marci tore successive counters, she placed the (unmarked) counters on a table, saying the corresponding number names. This presentation completed the verification that the given number system permitted counting in a way that matched the structure of the given blocks.

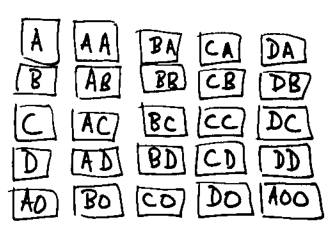

Figure 2: Marci's 5 × 5 array of counters (drawn by the authors based on the videotape).

Data and analysis

With their new number system, the group began immediately to build and test potential methods for arithmetic. We present several examples, key moments of a continuous, extended conversation of less than half an hour, by the group with no teacher interventions.

First example

Discussion centers on the computation of AA × AA, by Marci (see Figure 3). Marci had proposed this problem, which Holly called "a hard core multiplication," as the first to be tried. In effect, Marci takes the algorithmic bull directly by the horns, by mimicking a standard base-10 computation without explicit comment and placing her work directly on the table for discussion.

Holly: How in the heck are you going to look at that and say... oh yeah, I forgot... you know? [Holly points to Marci's work, written in Holly's notebook, where the second line of the multiplication has been shifted one place value to the left.]

Mandy: Because if you're used to just using your number system. But if they've only ever had these five symbols...

Holly: But they're never going to look at that and say, oh, yeah... [moves her hand to the left over Marci's second line to represent the shift in place value]... got to put this in... leave a space there.

Marci: Yeah.

Mandy: [quietly] Why can't they do that?

Marci: [to Holly] Why not?

Holly: I wouldn't be that smart!

These students clearly use the pronoun *they* for children they will later teach. Hence their discourse functions in at least two contexts:

1. We have the task at hand, seen as mathematics to be worked through.
2. These students take into account potential work and thinking by their future students.

Each context, in our view, can trigger and constrain the building and discussion of ideas. [5] The two contexts intermesh here, as they will (but less explicitly) in later calculations. Marci's computation in base 5 reproduces steps that she already knows (from prior data) would make sense in base 10. But what sense, *in base 5*, does this calculation make? This question, triggered in context 1, could just as well be posed in context 2, where learners' past experience, developmental needs and social practices may need to be considered – and this is how Holly, in particular, responds.

The following discussion emphasizes that the choice of how one should make sense here might best be clarified in

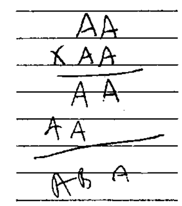

Figure 3: Marci's calculation of AA × AA, from her notes.

context 2, where the constraints come from the needs of working classrooms. Once Marci has thrust the need for careful grounding into the foreground, that need will motivate how the students formulate and then address the next few problems.

Second example

Marci opens the discussion of the next example, a computation by Holly. This discussion validates alternative procedures that can share a common grounding. The specific problem is to find AC + DB. In view of the complexity that emerged in Marci's first example, the group chose an addition problem next, most likely to isolate place value as a focus for concern.

In the symbolic calculation (see Figure 4), at the left, the second line AO has been shifted one place to the left, and the bottom line, AAO, records the sum of AO and the shifted AO.

Marci: [points to Holly's written calculation] Oh, that's a different way of doing it, instead of carrying it and bringing it over there.

Holly: That's what we had done in class.

Marci: She didn't carry. Instead she just added it down over there.

Holly: I just brought it down. I like that way better than carrying. It's just what we've done in class.

Katie: Just putting them in the right columns.

Holly: Right. That's a DB right? [She draws blocks for both AC and DB, as shown in Figure 4.] It'll take me forever. Okay. That's a DB, right?

Marci: Yep.

Holly: So [labeling her drawings of 'longs'] AO, BO, CO, DO, AOO.

Marci: No! A, AO... AAO, AAO.

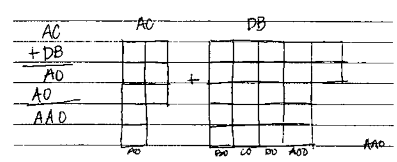

Figure 4: Holly's written solution for AC + DB from her notes.

Katie: No, it'd be AOO, and then with these one's [points to the 'piggies'] it'd be an AAO.

Marci: Oh, yeah.

Katie: So, it works.

Marci: So it did work. Cool.

The students use the drawing to help explain imagined work with blocks. Reading *longs* first, they find A + D = AO, giving the second, shifted AO in the written calculation. There are also C + B = AO *piggies*, hence the first (unshifted) AO. While the work with blocks looks standard, Holly's written calculation seems neither rote nor typical for adding. This example shows how openly these students welcome alternative procedures, given grounding that they see as appropriate. Here the explanation builds directly from the meaning of addition, symbolized as work with blocks. Considering our first guiding question, about procedures and their underlying methods, these students may be said to share a common method. Indeed, Holly's non-standard, alternative procedure, and the way she grounds it, strongly suggests the presence of a *method* that her written steps instantiate, not just a single algorithm.

Third example

In this example, proposed by Holly, the group returns to products. With the product BC × D as anchor, we will follow Marci first, then Katie, as they provide alternative solutions. Several group members had already sketched proposed solutions, but their results did not agree. At this point, Marci and Katie pulled back from the collective discourse to work through, each one on her own, more detailed calculations.

In Marci's calculation (see Figure 5) she records the repeated addition, BC times, of the quantity D. In particular (as shown in the videotape) Marci first enumerates C copies of D, and then evaluates B copies of D. Distinct spatial areas correspond to different stages of the calculation.

Marci begins writing at the upper left, to sketch a computation template for determining the given product: first the operator BC, then, below the digit C, the quantity D on which BC will operate, and finally a horizontal line. This completes the first stage of her template (shown in gray in Figure 6, which is an enlarged version of the top left of Figure 5).

Then Marci starts the computation. First she lets the multiplier C (for *piggies*) operate on D. Above and to the right of the template, she sketches three rows of four counters for the product C × D. This product is BB. Marci writes its first digit in small script in two places, then the second digit once, full size, on the bottom line. It appears that she has left a blank space in the *longs* place-value of the bottom line, and has written the smaller B above and to the left of this blank space.

Next, Marci turns attention to the partial product B × D, where the first factor B will operate as *longs* (see Figure 7).

Because B counts *longs*, the product AC = B × D will later be read as counting *longs*, but at the moment Marci evaluates the product B × D, just as she did for C × D, with a drawn array. To the right of her initial calculation template, Marci writes a second template, like the first except in the bottom line, where she omits the small B above the blank for *longs*. Using her new array, Marci finds B × D = AC. But AC (as noted above) counts *longs*. For the final steps (see Figure 8) Marci first writes AC (shown in black), and then, continuing the count of *longs*, she writes a B below AC, to represent the *longs* provided by her prior partial product (D × C = BB *piggies*) below AC, writes an addition sign, draws a horizontal line, and then records the sum BO.

Hence Marci finds BO new *longs* for the full product D × BC. Then she returns to her second template (gray). There (in black), she writes the latter product, BOB.

So far, Marci's notebook entry presents parts of her own thinking to herself, as she does not share it at this time. We take her presentation at face value, as a *notebook entry*, where Marci records findings, with schematic indications of supporting evidence, for potential later use. Viewed from this perspective, the entry's spatial organization can be seen as meaningfully horizontal, although it emerged over time in what might seem a quite haphazard way. There is a central row consisting of two calculation templates, surrounded by two rows, one above and one below, of annotation. We suspect that this horizontal organization has been designed to facilitate the possibility of later reference. Indeed, the rows of annotation collect evidence, in the form of images together with symbolic readings of those images, which should be helpful to check further or explain the given computation and its grounding. Because we see a conceptually meaningful spatial design, we can view this notebook entry as a way of organizing *not so much the steps by which calculation was performed*, but rather results found, their grounding, and potential future lines of reasoning.

Looking back at these examples, we see several levels of motivation for the actions and decisions taken. These students posed the problems to be solved. These problems, as the students' discussion shows, helped them to focus on key issues that they sought to clarify. In Dewey's terms, the problems were designed as *responses* to recognized needs, and so served as stimuli for further exploration. The explorations took the form of calculations. Key steps were justified by reference to the meaning of the numbers and the operations, either by means of images, or through the 'architecture' of written symbolic calculations.

Returning to the group, we cannot yet conclude, at least from the collective discourse, that BOB is a correct evaluation of BC × D, although Marci has given a strong case for

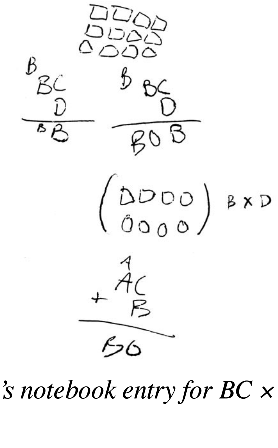

Figure 5: Marci's notebook entry for BC × D.

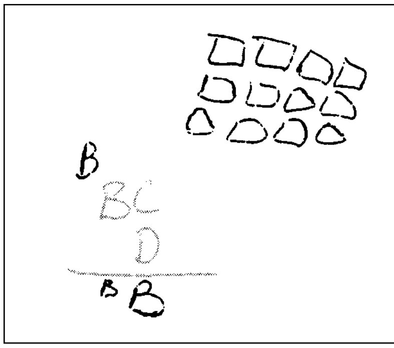

Figure 6: Marci's first template (gray), its first entries, and her first array.

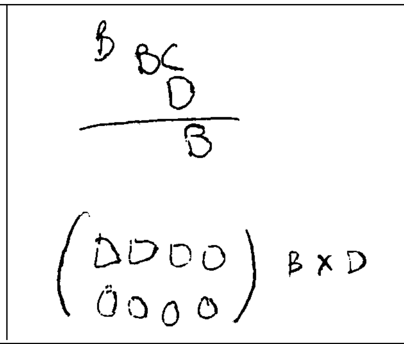

Figure 7: Marci's second template and array (see Figure 5).

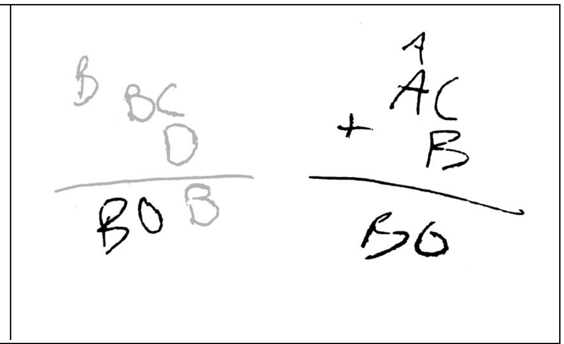

Figure 8: Marci's final steps (black), see Figure 5 for positionings in the full notebook entry.

it. She assembled her solution by combining the results of partial computations organized by place value. Nonetheless, it still may not be clear that *all the pieces needed* have been brought into the picture.

Fourth example

Perhaps Katie withdrew from the group's discussion to address the product, not by assembling it from pieces, but by dissecting it as a completed whole. In barely one minute she returns with the computation shown in Figure 9.

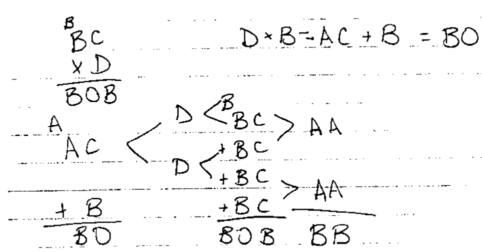

Figure 9: Katie's computation, from her notes.

Here, to emphasize the product as a whole, Katie has chosen the smaller factor D, rather than BC, to be the operator. We consider first the calculation at the center, where Katie adds D copies of BC. She finds this product by repeated doubling. Katie works outward from the central column, using pointed brackets to connect the numbers being doubled to their doubles. The repertoire of basic facts she needs is very small; simply A × A = B; B × B = D; C × C = AA; and D × D = AC. In her computation, she will use these facts, and then (later, just once) a fifth: B × C = AO. Viewed this way, indeed, the calculation can be followed easily, through its vertical and horizontal architecture. Note also that this architecture clearly shows the basic steps, *but without determining their order*. Hence, a reader might check Katie's steps in any of several alternative ways.

In this way, Katie focuses attention on how she breaks the product down to basic steps, to show she obtains the full product from the given basic steps. Unlike Marci, who concentrates on showing that a basic algorithmic pattern holds for the same reasons in base 5 and base 10, Katie uses special features of base five (especially the ease of doubling) to simplify the basic steps and hence focus on what she sees as essential.

We see significant creative initiative, not just at the strategic level (posing tasks to probe key issues), but also at the level of technique, especially in the way that Katie takes advantage of special features of base-5 arithmetic. Her creativity itself responds to the group's quest for certainty; indeed no mere replication of a prior method could, by definition, meet the need for *independent* checking. In effect, she has to build a different method.

Discussion

Consider first, in context 1, the mathematics of the three examples in relation to the tasks at hand. Marci opens with a standard algorithm, transposed to base 5, which triggers a discussion, in context 2, of the kinds of reasoning and explanation children might need in order to build understanding. In the next example, the addition problem, such reasoning and explanation are explored, using imagined blocks, pictures and written symbols. [6] When the group returns to multiplication, such grounding supports two impressive presentations:

- Marci's rethinking of a standard template using variations on her original array of counters
- Katie's independent checking strategy, which anchors directly to repeated adding as a meaning for the product.

The students had wide choices here, and the progression of their thinking seems to mark a path away from rote procedures, toward a carefully considered technical and expressive freedom that includes, indeed seems to require, the purposeful design of new procedures. Such procedures, for these students, must be grounded clearly in the meanings of the operations and explicit structures given to the quantities they work with by means of their number system.

Through these developments, we have followed Marci in her interactions with the other members of the group. She began by framing what turned out to be a helpful stimulus for deeper exploration by the group, and she continued, in her notebook, by working through, in detail, a conscious variation on important features of the algorithm that she first proposed. She builds arrays to check key calculation steps, and notes carefully her steps and their results, including a rethinking of the template with which she began. At this point, having checked her own work carefully, she can consider Katie's independent calculation, which confirms not just Marci's result, but also, to a great extent, her method. Working in this way, the 'official' school culture now seems far behind them both.

Almost three months later, completing the semester in December, Katie and Marci wrote parallel but independent journal entries that touch on the issues we have raised. Here, for example, Katie reflects about the changes that she saw in her own thinking:

> [...] I soon discovered that there was a lot about mathematics that I took for granted. I never asked the question of why things worked the way they did, I just always believed that they did and put my trust in the teacher to not teach me anything wrong. During this past semester I have found myself constantly asking why a concept works the way it does. Why, when dividing fractions do you multiply the top fraction by the reciprocal of the bottom fraction to get the answer. I always knew it was how it was done but I never understood why. Simple mathematics that I thought I knew like the back of my hand became new and exciting to me. I think that if all children were taught in this manner from the beginning of his or her education, the love of mathematics would grow in schools. Students wouldn't be so confused. They would understand why. (Speiser and Walter, 2000, p. 86)

Katie directly confronts the prevailing 'official' discourse of school mathematics that she had encountered as a growing child. She spells out what she has come to feel is wrong with it – no roots in experience, no reasoning, and hence, for her, no truth. She locates these conclusions firmly in context 2, by connecting to potential gains that *any* child can make.

While Katie emphasized how learners might build understanding, Marci (also writing in December) reflects on changes in her way of understanding:

> As written in my journal, I have never been a visual math leaner. I learned math by formulas and equations and never worried about why or how [...] It's amazing that in the course of one semester my thinking in math could change so drastically. And not only do I think of math differently, but it's interesting to find me thinking of life in a much more illustrated or visual way.
>
> [...] When we first worked with groups and had to find answers to problems, we all tried to think of what the problem had to do with our previous learning of math. Along the course of the semester, our group started to look at math in a different light. I knew that the problems we did had simple enough algorithms to figure out the answer, but it was more interesting to use the blocks or rods and really look into the heart of the problem. (Speiser and Walter, 2000, pp. 86-87)

In her first paragraph, Marci indicates that she is testing ways of thinking, gained in context 1, in the extended context of her *life*. Marci's emphasis on "visual or illustrated" ways of thinking sets imagination strongly in the foreground, in the sense of building and inspecting images, as we have seen already in her notebook entry (Figure 5). Next, Marci reflects (in her journal, also in December) on the variety of productive thinking that she sees around her:

> [...] Granted, I'm not a very visual (or should I say creative) thinker, in fact, I have never thrived on creativity, but I have learned to think more in this way. This class has helped me realize that people look at math in different ways. As a teacher, that is important for me to know. Had I not realized this, all of my students would have had to learn math in the same way I did, memorizing formulas and never asking why.

Like Katie, Marci concludes in context 2, when she affirms the need for variation, for productive contrast – in short for the kind of exploration and discovery through which she could reconsider her identity as learner and thinker. In a word, these students explicitly reject important aspects of the way they learned arithmetic under 'official' auspices [7]:

- For Katie, the main thrust might be to uncover and convey compelling reasons for important constructs – not simply why they might be right, but also how and why they may have been selected.

- For Marci, the focal issue might be how specific concrete images can be designed and used to anchor reasoning, facilitate discovery, and so make important choices possible.

In our introduction, we described a seeming conflict in how mathematics might be learned and understood, between mere replication of standard (and perhaps demonstrably correct) procedures and the building (through exploration) of more grounded understanding. For the five education students we have followed, however, we find no evidence of such a conflict *once these students have addressed their past experience as mathematics learners*. To address that past experience, as we have seen, they chose to work inventively within a range of possibilities, grounded in a careful, even abstract view of numbers and operations. Within that range, the standard algorithms may actually *gain* significance, because they can be seen to represent important general ideas. Such ideas can serve equally to underpin alternative procedures, or even (as with Katie's checking strategy) alternative methods.

Perhaps the key point is *perspective*. In this study, primarily of one extended case, we have taken a particular perspective, a way of choosing what to look at, what to analyze, and how to go about the process. We feel that this particular analysis suggests strongly that the potential of preservice elementary teachers to build deep, productive understanding of important mathematics – and to value such understanding highly – is much greater than has been widely believed. Further, what we have learned from these five students, especially about how they see themselves as teachers in the making, suggests equally strongly that the potential for what can happen in school classrooms may be similarly greater. After all, each teacher's understanding, and indeed how much he or she loves what he or she will teach, should contribute strongly to conditions that will help or hinder children's growth.

[Notes and references can be found on page 32 (ed.)]