On Mathematical Problem Posing*

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In mathematics classes at all levels of schooling in all countries of the world, students can be observed solving problems. The quality and authenticity of these mathematics problems has been the subject of many discussions and debates in recent years. Much of this attention has resulted in a rich, more diverse collection of problems being incorporated into school mathematics curricula. Although the problems themselves have received much scrutiny, less attention has been paid to diversifying the sources for the problems that students are asked to consider in school. Students are almost always asked to solve only the problems that have been presented by a teacher or a textbook. Students are rarely, if ever, given opportunities to pose in some public way their own mathematics problems. Traditional transmission/reception models of mathematics instruction and learning, which emphasized students passively receiving knowledge as a result of transmission teaching, were compatible with a pedagogy that placed the responsibility for problem posing exclusively in the hands of teachers and textbook authors On the other hand, contemporary constructivist theories of teaching and learning require that we acknowledge the importance of student-generated problem posing as a component of instructional activity.

Problem posing has been identified by some distinguished leaders in mathematics and mathematics education as an important aspect of mathematics education [e.g., Freudenthal, 1973; Polya, 1954]. And problem posing has recently begun to receive increased attention in the literature on curricular and pedagogical innovation in mathematics education. In the United States, for example, recent reports, such as the *Curriculum and evaluation standards for school mathematics* [NCTM, 1989] and the *Professional standards for teaching mathematics* [NCTM, 1991], have called for an increase in the use of problem-posing activities in the mathematics classroom. Both reports have suggested the inclusion of activities emphasizing *student-generated* problems in addition to having students solve pre-formulated problems, as is clearly illustrated in the following excerpt from the *Professional standards for teaching mathematics*:

> Teaching mathematics from a problem-solving perspective entails more than solving nonroutine but often isolated problems or typical textbook types of problems. It involves the notion that the very essence of studying mathematics is itself an exercise in exploring, conjecturing, examining, and testing—all aspects of problem solving. Tasks should be created and presented that are accessible to students and extend their knowledge of mathematics and problem solving. Students should be given opportunities to formulate problems from given situations and create new problems by modifying the conditions of a given problem [NCTM, 1991, p. 95]

Despite this interest, however, there is no coherent, comprehensive account of problem posing as a part of mathematics curriculum and instruction nor has there been systematic research on mathematical problem posing [Kilpatrick, 1987]. For the past several years, I have been working with colleagues and students on a number of investigations into various aspects of problem posing. [1] Our experiences in studying mathematical problem posing, and our reading of the work of others interested in this area have formed the basis for this paper.

This paper begins with a brief introduction to the types of activities and cognitive processes that have been referred to as problem posing, and then identifies and discusses various perspectives from which one can view the role and place of problem posing in the school mathematics curriculum. Kilpatrick has argued that "problem formulating should be viewed not only as a goal of instruction but also as means of instruction" [1987, p. 123], and both views of problem posing will be evident in this paper The nature and findings of some research related to mathematical problem posing is also discussed in order to characterize some of the available research evidence associated with each of the perspectives discussed and to suggest some important issues in need of further investigation. To illustrate the international interest in mathematical problem posing, examples of research and opinion from around the world are discussed.

What is mathematical problem posing?

Problem posing refers to both the generation of new problems and the re-formulation, of given problems. Thus, posing can occur before, during, or after the solution of a problem.

One kind of problem posing, usually referred to as problem formulation or re-formulation, occurs within the process of problem solving When solving a nontrivial problem a solver engages in this form of problem posing by recreating a given problem in some ways to make it more accessible for solution. Problem formulation represents a kind of problem posing process because the solver transforms a given statement of a problem into a new version that becomes the focus of solving. Problem formulation is related to planning, since it may involve posing problems

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that represent subgoals for the larger problem. Polya's heuristic advice, "Think of a related, more accessible problem," suggests another way in which problem formulation involves problem posing. If the source of the original problem is outside the solver, the problem posing occurs as the given problem is reformulated and "personalized" through the process of re-formulation. The operative question that stimulates this form of posing is: how can I formulate this problem so that it can be solved?

At least since Duncker's [1945] observation that problem solving consists of successive re-formulations of an initial problem, problem formulation has been extensively studied by psychologists interested in understanding complex problem solving. According to contemporary information-processing models of complex problem solving, a problem is solved by establishing a series of successively more refined problem representation which incorporate relationships between the given information and the desired goal, and into which new information is added as subgoals are satisfied. One of the major findings of an extensive body of research on the differences between experts and novices in a variety of complex task domains is that experts tend to spend considerable time engaging in problem formulation and re-formulation, usually engaging in qualitative rather than quantitative analysis, in contrast to novices who spend relatively little time in formulation and re-formulation [Silver & Marshall, 1989]. For relatively simple problems, problem formulation may occur primarily in the early stages of problem solving, but in extended mathematical investigations, "problem formulation and problem solution go hand in hand, each eliciting the other as the investigation progresses" [Davis, 1985, p. 23].

Not all problem posing occurs within the process of solving a complex problem. Problem posing can also occur at times when the goal is not the *solution* of a given problem but the *creation* of a new problem from a situation or experience. Such problem posing can occur *prior* to any problem solving, as would be the case if problems were generated from a given contrived or naturalistic situation. This type of problem generation is also sometimes referred to as problem formulation, but the process being described here is different from that described above. [2] Problem posing can also occur *after* having solved a particular problem, when one might examine the conditions of the problem to generate alternative related problems. This kind of problem posing is associated with the "Looking back" phase of problem solving discussed by Polya [1957]. Brown and Walter [1983] have written extensively about a version of this type of problem posing, in which problem conditions and constraints are examined and freely changed through a process they refer to as "What-if?" and "What-if-not?". The operative question that drives these kinds of problem posing is: What new problems are suggested by this situation, problem or experience?

Perspectives on mathematical problem posing

Having discussed briefly the nature of mathematical problem posing, we now turn our attention to several perspectives from which to view the importance and role of mathematical problem posing as an object of pedagogical and research attention. The purpose here is not to focus on sharp distinctions among these perspectives, since they are not mutually exclusive, but rather to use these perspectives as lenses through which to view various research studies and instructional interventions that have been undertaken.

Problem posing as a feature of creative activity or exceptional mathematical ability

Problem posing has long been viewed as a characteristic of creative activity or exceptional talent. For example, Hadamard [1945] identified the ability to find key research questions as an indicator of exceptional mathematical talent. Related observations have been made about professionals in various science fields [e.g., Mansfield & Busse, 1981]. Similarly, Getzels and Csikszentmihalyi [1976] studied artistic creativity and characterized problem finding as being central to the creative artistic experience.

The apparent link between posing and creativity is clear from the fact that posing tasks have been included in tests designed to identify creative individuals. For example, Getzels and Jackson [1962] developed a battery of tests to measure creativity, of which one task asked subjects to pose mathematical problems that could be answered using information provided in a set of stories about real-world situations. Getzels and Jackson scored the subjects' problems according to the complexity of the procedures that would be used to obtain a solution (i.e., the number and type of arithmetic operations used), and they used the results as a measure of creativity. Balka [1974] also asked subjects to pose mathematical problems that could be answered on the basis of information provided in a set of stories about real-world situations. Analysis of the responses attended to three aspects: fluency, flexibility, and originality. Fluency refers to the number of problems posed or questions generated, flexibility to the number of different categories of problems generated, and originality to how rare the response is in the set of all responses. This analytic scheme closely parallels that used in many approaches to measuring creativity. [Torrance, 1966]

The relationship of problem posing to exceptional mathematical ability has also been explored. For example, Krutetskii [1976] and Ellerton [1986] each contrasted the problem posing of subjects with different ability levels in mathematics. In his study of mathematical "giftedness", Krutetskii [1976] used as one of his measures of exceptional talent a problem-posing task in which students were presented with problems in which there was an unstated question (e.g., "A pupil bought $2x$ notebooks in one store, and in another bought 1.5 times as many "), for which the student was required to pose and then answer a question on the basis of the given information. Krutetskii argued that there was a problem that "naturally followed" from the given information, and he found that high ability subjects were able to "see" this problem and pose it directly; whereas, students of lesser ability either required hints or were unable to pose the question. In Ellerton's [1986] study, students were asked to pose a mathematics problem that would be difficult for a friend to solve. She found that the "more able" students posed problems of greater computational difficulty (i.e., more complex numbers and requiring more operations for solution) that did their "less able" peers.

Since problem posing has been embedded in the assessment of creativity or mathematical talent, it is reasonable to assume that there is some link between posing and creativity. In fact, creativity has been associated both with novel productions [Newell, Shaw & Simon, 1962] and with ill-structured problem solving [Voss & Means, 1989], so a relationship to problem posing seems clear. On the other hand, the general relationship between creativity and problem posing is unclear. Haylock [1987] reviewed a set of studies that examined the relation between creativity and various aspects of mathematics, and he found an incomplete basis for asserting a relationship. More recently, Leung [1993] studied the relationship between the problems posed by a group of preservice elementary school teachers and their performance on tests of creativity and mathematical knowledge. After rating the posed problems along several dimensions of cognitive and mathematical complexity, she found essentially no relationship with their scores on the test of creativity. On the other hand, Leung did report a strong relationship between the mathematical knowledge of the subjects and the quality of the problems they posed.

Because of the association of problem posing with the identification of persons with exceptional creativity or talent, one might infer that instruction related to problem posing would be appropriate only for "gifted" students. However, Leung's findings suggest that mathematical problem posing as an activity need not be reserved for use only with students identified as exceptionally talented or creative. In fact, problem posing is a salient feature of broad-based, inquiry-oriented approaches to education that are discussed next.

Problem posing as a feature of inquiry-oriented instruction

In classrooms where children are encouraged to be autonomous learners, problem posing would be a natural and frequent occurrence. According to Ernest [1991], unlike inquiry approaches that emphasize discovery or problem solving, an investigatory approach to inquiry-oriented mathematics teaching is characterized by having responsibility for problem formulation and solution rest as much with the students as with the teacher. In the *Curriculum and evaluation standards,* one finds expressions of support for this view of problem posing in the mathematics classroom, as providing situations in which "mathematical ideas have originated with the children rather than the teacher" [NCTM, 1989, p. 24]. This sentiment is further illustrated in the following excerpt from the same document:

> Experiences designed to foster continued intellectual curiosity and increasing independence should encourage students to become self-directed learners who routinely engage in constructing, symbolizing, applying, and generalizing mathematical ideas." [NCTM, 1989, p. 128]

Collins [1986] has identified three different general goals for inquiry-oriented teaching: (a) to help students construct general rules, theories or principles that are already known and match an expert's understanding of a domain, (b) to help students construct genuinely novel theories or principles that emerge from their inquiry, and (c) to teach students how to solve problems through the use of self-questioning and self-regulatory techniques and metacognitive skills. Collins identified Berberman's *discovery* teaching as an example of mathematics teaching directed at the first goal, and he cited Schoenfeld's [1985] problem-solving instruction as an example of instruction directed toward the third goal. Although Collins did not give an example of mathematics instruction directed toward the second goal, he would have been justified in citing the work of Brown and Walter.

Brown and Walter [1983] have written extensively about their incorporation of problem posing in the teaching of mathematics at the college and precollege levels. Their instructional approach emphasizes the generation of new problems from previously solved problems by varying the conditions or goals of the original problem. The essence of Brown and Walter's "What-if-not?" process is the systematic variation of problem conditions or goals. A rationale for this type of posing-oriented instruction, which is closely aligned with the general inquiry-oriented philosophy discussed above, is presented by Brown [1984].

Some form of inquiry-oriented instruction has long been offered to students from social and economic elite groups, but it has generally been denied to those who come from less privileged backgrounds. Despite this historical pattern, inquiry-oriented instruction can be seen to have close connections to arguments for emancipatory education for all students [e.g., Freire, 1970; Gerdes, 1985]. Ernest [1991] provides a fairly complete summary of this view by showing how an inquiry-oriented pedagogy, with an emphasis on problem solving and problem posing, can be used to challenge the rigid hierarchies associated with conventional conceptions of mathematics, mathematics curriculum and mathematical ability. Through such pedagogy, Ernest argues, mathematics can be empowering for all learners and not just for those who are privileged by the current social, political and economic arrangements. One version of this approach is being implemented in the United States through the QUASAR project, which provides mathematics instructional programs aimed at high-level thinking, reasoning and inquiry to students (grades 6-8) from economically disadvantaged communities [Silver, Smith & Nelson, in press]. Authors writing from a feminist perspective [e.g., Noddings, 1984] have also shown that inquiry-oriented instruction can be used in ways that honor alternative ways of knowing and solving problems.

Problem posing has figured prominently in some inquiry-oriented instruction that has freed students and teachers from the textbook as the main source of wisdom and problems in a school mathematics course. Several authors have written about instructional experiments in which students have written a mathematics textbook for themselves or for children who will be in the class at a later time. Van den Brink [1987] reported such an experiment with children in first grade in the Netherlands. They were each asked to write and illustrate a page of arithmetic sums for children who would be entering first grade during the following year. Streefland [1987, 1991] has also employed similar authorship experiences for students as part of his "realistic mathematics education" instruction in the Netherlands. In the United States, Healy [1993] has used a similar approach with secondary school students studying geometry. In Healy's "Build-a-book" approach students do not use a commercial textbook but create their own book of important findings based on their geometric investigations.

Another example is drawn from Australia, where a primary grade teacher has written of her experiences in using problem posing as a central feature of her mathematics instruction involving a group of children over two and one-half years, spanning grades K-2 [Skinner, 1991]. In her teaching, Skinner had her students engage in an extensive amount of problem posing. They shared their posed problems with each other, and these formed the basis for much of the problem-solving activity in the class. Skinner also incorporated larger investigation-oriented work that proved ill-structured problems which engaged the students for relatively long periods of time.

Winograd [1991] has provided another example of mathematics instruction emphasizing problem posing. He provided fifth-grade students with a year-long experience in which they wrote, shared with classmates, and solved original story problems. Winograd did not have a comparison group, but he reported a generally positive impact of the problem authorship experience on students' achievement, and especially on their disposition toward mathematics. Similarly, problem posing has been a prominent feature of geometry instruction based on the use of the *Geometric supposer* [Yerushalmy, Chazan & Gordon, 1993] and also of the video-based, inquiry-oriented instruction developed by Bransford and his colleagues [Bransford, Hasselbring, Barron, Kulewicz, Littlefield & Goin, 1988], and positive claims about student outcomes have been made in that work.

In general, inquiry-oriented instructional activity has not been subjected to serious scrutiny, either with respect to the role of the problem posing in the instruction or to the long-term impact of the instruction on the students. The authors have provided some description of the instruction and of the students' responses or work, but there has been little or no systematic analysis of the nature of the problem posing and inquiry that occurred or of the impact that these experiences had on students' mathematical performance. Inquiry-oriented instruction can be closely tied to mathematics or it can be based more on a general framework. In the next section, we consider approaches that would be closely tied to mathematical activity.

Problem posing as a prominent feature of mathematical activity

One argument for focusing curricular or research attention on the generative process of problem posing is that it is central to the discipline of mathematics and the nature of mathematical thinking. When mathematicians engage in the intellectual work of the discipline, it can be argued that self directed problem posing is an important characteristic [Polya, 1954]. Mathematicians may solve some problems that have been posed for them by others or may work on problems that have been identified as important problems in the literature, but it is more common for them to formulate their own problems, based on their personal experience and interests [Poincaré, 1948]. Rather than being presented for solution by an outside source, mathematical problems often arise out of attempts to generalize a known result, or they represent tentative conjectures for working hypotheses, or they appear as subproblems embedded in the search for the solution to some larger problem. Thus, it has been argued that professional mathematicians, whether working in pure or applied mathematics, frequently encounter ill-structured problems and situations which require problem posing and conjecturing, and their intellectual goal is often the generation of novel conjectures or results [Pollak, 1987].

The relation of this view to the inclusion of problem posing in the curriculum is evident in the following excerpt from the *Curriculum and evaluation standards for school mathematics*: "Students in grades 9-12 should also have some experience recognizing and formulating this own problems, an activity that is at the heart of doing mathematics" [NCTM, 1989, p. 138]. Such views are compatible with the emerging view that to understand what mathematics is, one needs to understand the activities or practice of persons who are makers of mathematics. A view of mathematical knowing as a practice (in the sense of professional practice) comes from analyses of the history and philosophy of mathematics [e.g., Lakatos, 1976; Kitcher, 1984], which highlight important social aspects of mathematics that remained hidden from view in classical logical analyses. Those who view the purpose of mathematics education as providing students with authentic experiences like those that characterize the activity of professional mathematicians would identify problem posing as an important component because of its apparently central role in the creation of mathematics.

It has been argued that ill-structured problem situations are often encountered by those who create or apply mathematics in professional activity and that such situations serve as a major source of problem posing done by professionals in the field of mathematics. Moreover, Hadamard [1945] identified the ability to find key research questions as an important characteristic of talented mathematicians. Nevertheless, beyond anecdotal accounts, little direct evidence of problem posing by mathematicians has been produced. Ill-structured problem solving of the sort done by mathematicians has not been systematically investigated, but it has been studied in some other professional domains. For example, Reitman [1965] examined the processes utilized by artists and composers in large-scale ill-structured problem settings, like musical composition. He argued that the observation of persons solving ill-structured problems exposed many more differences in the memory structures of respective solvers than became exposed when they solved well-structured problems. Simon [1973] extended Reitman's analysis of ill-structured task domains and suggested that although there was little difference in the processes required to solve well-structured or ill-structured problems, ill-structured problems required a wider range of processes in formulating and solving the problem and in recognizing the solution when it was obtained, and that much of the cognitive activity in such problem solving is directed at structuring the task. Thus, ill-structured problem provide a rich arena in which to study complex cognitive activity, such as problem posing.

Some research has considered the application of mathematics to ill-structured problems. For example, Lesh and colleagues [Lesh, 1981; Lesh, Landau, & Hamilton, 1983] characterized the processes used by young adolescents as they solved applied mathematical problems embedded in real world, meaningful contexts. The findings of this research suggested that the processes used in solving applied problems were somewhat different from the processes observed when the same students solved well-structured school mathematics problems. In particular, in applied problem solving more cognitive attention was devoted to the processes of formulation and re-formulation during problem solving. Thus, significant mathematical problem posing activity occurs not only in the creation of mathematics by professional mathematicians but also in the thoughtful application of mathematics by students. Therefore, problem posing would also be a salient feature of instruction designed by those who view the purpose of mathematics education as being less about introducing students to the culture of professional mathematics and more about assisting students to learn the ways of thinking and reasoning employed by those who apply mathematics and quantitative reasoning effectively to solve real-world problems. Since most students will not become professional mathematicians, an education that prepares them to be intelligent users of mathematics in order to solve problems of importance or interest to them may be better suited for them than one which is based on the activity of professional mathematics, and extensive experience in problem posing would be an important component of instruction aimed at such a goal [Blum & Niss, 1991].

The research discussed above provides a foundation on which to build, but further study of the posing and solving processes involved in the solution of ill-structured, applied problems is needed. The general connection between problem posing and many forms of problem solving is further discussed in the next section.

Problem posing as a means to improving students' problem solving

Probably the most frequently cited motivation for curricular and instructional interest in problem posing is its perceived potential value in assisting students to become better problem solvers. In fact, a perceived connection between mathematical problem posing and curricular goals related to problem solving permeates the NCTM *Professional teaching standards*. Interestingly, although problem posing has not been a common feature of mathematics instruction, advocacy for problem posing as a means of improving students' problem-solving performance is not a new idea. For example, Connor and Hawkins [1936] argued that having students generate their own problems improved their ability to apply arithmetic concepts and skills in solving problems. Twenty years later, Koenker [1958] included problem posing as one of 20 ways to help students improve their problem solving.

Problem posing has been incorporated as a feature of some Japanese experimental teaching which employs problem posing as a means of assisting students to analyze problems more completely, thereby enhancing students' problem-solving competence. Several authors [Shimada, 1977; Hashimoto & Sawada, 1984; Nohda, 1986] have described various versions of a style of teaching, known as "open approach teaching" or teaching with "open-end or open-ended problems." Their descriptions, and those of others, suggest various ways in which problem posing is embedded in the instruction. For example, Hashimoto [19877], has described and provided a transcript of a lesson in which students pose mathematical problems on the basis of one solved the previous day.

Another interesting analysis of problem posing has been done by Sweller and his colleagues in Australia [e.g., Sweller, Mawer, & Ward, 1983; Owen & Sweller, 1985]. Some of Sweller's studies have involved ill-structured mathematics problems from the domain of geometry and trigonometry. In general, these studies have demonstrated that subjects are far more likely to use means-ends analysis on goal-specific problems (given an angle in a figure, find the value of a particular other angle in the figure) than on non-goal-specific problems (given an angle in a figure, find the measure of as many other angles as you can). Moreover, Sweller's results show that, although means-ends analysis is a powerful problem-solving strategy, the unavailability of means-ends analysis in non-goal-specific problems may lead subjects not only to use more expert-like, forward-directed problem-solving behavior but also to develop powerful problem-solving schemas, thereby positively affecting students' learning. This work suggests that students' engagement with problem posing and conjecture formulation activities, in the context of solving ill-structured mathematics problems, can have a positive effect on their subsequent knowledge and problem solving.

A few experimental or quasi-experimental studies have been conducted in the United States, in which students receiving a form of mathematics instruction in which problem posing has been embedded are contrasted with students who have comparable instruction without the posing experience. Keil [1965] found that sixth-grade students who had experience writing and then solving their own mathematics problems in response to a situation did better on tests of mathematics achievement than students who simply solved textbook story problems. Perez [1985] found similar results with college students studying remedial mathematics, and he also reported that the experimental treatment, which involved some writing and some rewriting of story problems, had a positive effect on students' attitudes toward mathematics. Unfortunately, these studies did not examine the direct impact of the instructional experience on students' problem generation itself.

Despite the interest in problem posing because of its potential to improve problem solving, no clear, simple link has been established between competence in posing and solving. Silver & Cai [1993] examined the responses of middle school students (grades 6 and 7) to a task asking them to generate three problems on the basis of a brief story (Jerome, Elliott, and Arturo took turns driving home from a trip Arturo drove 80 miles more than Elliott. Elliott drove twice as many miles as Jerome. Jerome drove 50 miles.). The student-generated problems were classified according to mathematical complexity (number of operations required for solution), and this measure of problem posing was compared to students' performance in solving eight open-ended mathematical problems. Silver and Cai found a strong positive relationship between posing and solving performance. On the other hand, Silver & Mamona [1989] found no overt link between the problem posing of middle school mathematics teachers and their problem solving. In that study, Silver and Mamona asked the teachers to pose problems in the context of a task environment (or microworld) called Billiard Ball Mathematics, consisting of an idealized rectangular billiard ball table with pockets only at the corners and on which a single ball is hit from the lower left corner at an angle of 45° to the sides. Problem posing occurred prior to and immediately after the teachers solved a specific problem concerning the relationship between the dimensions of the table and the final destination of the ball. Except for the fact that subjects' post-solution posing was influenced by their problem-solving experience (i.e., in post-solution posing, they posed more problems like the one they solved than they had in the pre-solution posing), there was no other relationship between posing and solving that could be detected. Clearly, there is a need for further research that examines the complex relationship between problem posing and problem solving In addition, there is also interest in exploring the relation of posing to other aspects of mathematical knowing and mathematical performance.

Problem posing as a window into students' mathematical understanding

Interest in problem posing as a means of helping students become sensitive to facts and relations embedded in situations has been evident for a long time. For example, Brueckner [1932] advocated the use of student-generated problems as a means of helping students to develop a sense of number relations and to generalize number concepts. In generating problems based on the mathematical ideas and relations embedded in situations, students engage in "mathematizing" those situations. Such experience may assist them to overcome the well-documented tendency of students to fail to connect mathematics sensibly to situations when they are asked to solve preformulated problems [Silver 7 Shapiro, 1992]. The following excerpt from the *Professional teaching standards* illustrates this point of view: "Writing stories to go with division sentences may help students to focus on the meaning of the procedure" [NCTM, 1991, p. 29].

In England, Hart [1981] used problem posing as one research technique to examine students" understanding of important mathematical concepts. By providing answers or equations and asking students to generate problem situations that would correspond to the given answer or equation, Hart showed that one could open a window through which to view children's thinking. More recently, Greer and McCaan [1991] used Hart's approach by providing multiplication and division calculations to students (ages 9-15) in Northern Ireland and asking them to generate story problems that matched a given calculation. A similar approach has been used by Simon [1993] and by Silver and Burkett [1993] in studies with preservice elementary school teachers' understanding of division. A variation on this approach was used by Ellerton [1986], who, without presenting any additional context or stimulus, simply asked Australian students (ages 11-13) to create a problem that would be difficult for a friend to solve. Based on the children's choice of numbers in the problems (e.g., fractions that did or did not permit cancellation), Ellerton made inferences about some aspects of the children's mathematical knowledge. Another technique, discussed above in reference to Krutetskii's [1976] study of mathematically talented students, is the use of problems with an unstated question. As these brief descriptions suggest, some researchers have found problem posing to have potential as a means of exploring the nature of students' understanding of mathematical ideas.

Regrettably, the research cited above has generally found a fairly weak connection between real life situations and mathematical ideas or symbols. For example, Greer and McCann found that some students used a fraction to represent a number of people in a posed problem; this finding is similar to the finding that many students will solve problems by providing answers that have weak connection to the real world setting described in the problem [Silver & Shapiro, 1992]. Unfortunately, a lack of concern about sensible connection to real world settings has been reported in studies of posing by preservice elementary school teachers [Silver & Burkett, 1993; Simon, 1993]. Moreover, Ellerton found that students' conceptions of difficulty seemed to be linked almost entirely to computational complexity rather than to situational or semantic complexity. These findings are consistent with conventional mathematics instruction, which tends not to relate mathematics to real world settings in any systematic manner, and they appear to be closely related to van den Brink's observation concerning the arithmetic books constructed by first-grade children in his study: "A striking aspect of the books was that arithmetic as applicable knowledge only appeared in the class book when it had been learned that way" [1987, p 47] Thus, it appears that problem posing provides not only a window through which to view students' understandings of mathematics but also a mirror which reflects the content and character of their school mathematics experience. Opening the problem posing window also affords an opportunity to view aspects of students' attitudes and dispositions toward mathematics

Problem posing as a means of improving student disposition toward mathematics

There are several different aspects of problem posing that are thought to have important relationships to student disposition toward mathematics. For example, posing offers a means of connecting mathematics to students' interests. As the *Curriculum and evaluation standards* suggests: "Students should have opportunities to formulate problems and questions that stem from their own interests" [NCTM, 1989, p. 67]. Nevertheless, personal interest is not the sole motivation for posing problems Within a classroom community, students could be encouraged to pose problems that others in the class might find interesting or novel. In a study of one such instructional experiment, Winograd [1991] reported that the fifth-grade students in his study appeared to be highly motivated to pose problems that their classmates would find interesting or difficult. He also noted that students' personal interest was sustained in his study through a process of sharing problems with others.

There is also a reciprocal expectation regarding problem posing, since engagement with problem generation is also thought to stimulate student interest in mathematics. Students who have difficulty with mathematics are sometimes characterized by a syndrome of fear and avoidance known as mathematics anxiety. Some have claimed that mathematics anxiety can be reduced through problem posing [Moses, Bjork & Goldenberg, 1990], since student participation in problem posing makes mathematics seem less "intimidating" [Brown & Walter, 1983]. In fact, Perez [1985] taught college-age students studying remedial mathematics, and therefore likely to have mathematics anxiety and poor attitudes toward mathematics, using a problem-posing approach. He reported improvement in the students' attitudes toward mathematics, as well as in their achievement.

In general, reports of problem-posing instruction do not discuss instances in which students have rejected or reacted negatively to this instructional approach Nevertheless, it seems plausible that some students, perhaps especially those who have been successful for a long period of time in school settings characterized by didactic, teacher-directed instruction, would react negatively to a style of teaching that was less directive and placed on them more responsibility for learning. For these students, there may be little desire or motivation to alter the existing power relations in the classroom, or to alter the hierarchical assumptions underlying current conceptions of mathematical performance. There is evidence from other sources that students can sometimes resist changes in classroom instruction that require them to deal with higher levels of uncertainty about expectations or higher levels of responsibility for their own learning [e g., Davis & McKnight, 1976; Doyle & Carter, 1982].

Some evidence of the plausibility that some students might reject or resist mathematics instruction based on posing comes from two recent non-instructional studies. Comments made by a few of the middle school teachers who were part of the study by Silver and Mamona [1989] indicated hostility toward the task requirement to pose their own mathematical problems (e g., "This is stupid!", "Why are we being asked to do this?") Similarly, Silver and Cai [1993] found that some students in grades 6-7, when asked to pose three problems on the basis of a story situation, expressed profound dismay at being asked to do this (e.g., "This is unfair", "My teacher didn't teach us how to do this."). Understanding low students, especially those who have been successful in less inquiry-oriented classrooms, do or do not make a transition to participation in problem posing and acceptance of posing-oriented instruction is an important research topic.

Healy [1993] provides an example that illustrates how an emphasis on student-generated problem posing can humanize and personalize mathematics learning and instruction in profound ways. For many of his students, mathematics became something other than a neutral body of knowledge filled with abstract ideas and symbolism that others had created and which was accessible only through imitation and memorization. Instead, many students became passionately concerned about mathematical issues that they were investigating because of personal interest and commitment. Clearly, the affective dimension of such an instructional experience is significant, as is exemplified in the following quote from one of Healy's students after three months of the course: "In this class we make enemies out of friends arguing over things we couldn't have cared less about last summer" [1993, p. x]. Even in a less competitive setting than the one implied by this student's comment, one would expect the passionate, personal engagement of students with mathematical ideas to produce learning situations in which affective and cognitive issues would both have great import.

Conclusion

In their historical account of the treatment of problem solving in the mathematics curriculum, Stanic and Kilpatrick [1988] argued that problem solving could be viewed as a means to teach desired curricular material or it could itself be viewed as an educational end or goal. Similarly, in this paper, problem posing has been discussed in ways that correspond to it being viewed as a means to achieve other curricular or instructional ends or as an educational goal itself. Many purposes for which problem posing might be included as a feature of school mathematics have been considered, as has research evidence associated with these purposes. From the fairly unsystematic collection of findings that characterizes the research literature on mathematical problem posing, many studies were cited and some of their findings and approaches organized for presentation in this paper.

From the perspective of research, three major conclusions seem warranted from this review. First, it is clear that problem-posing tasks can provide researchers with both a window through which to view students' mathematical thinking and a mirror in which to see a reflection of students' mathematical experiences. Second, problem-posing experiences provide a potentially rich arena in which to explore the interplay between the cognitive and affective dimensions of students' mathematical learning. Finally, much more systematic research is needed on the impact of problem-posing experiences on students' problem posing, problem solving, mathematical understanding and disposition toward mathematics.

Coda

Problem-posing experiences can afford students opportunities to develop personal relationships with mathematics. The process of personalizing and humanizing mathematics for students through the use of open-ended problem-posing tasks invites them to express their lived experiences, and this can have important consequences for teachers and for researchers. For example, if allowed to do so, students may pose problems different from the ones that the teacher or researcher had in mind. In all of the problem-posing studies I have conducted, at least some of the subjects have given responses or engaged in behavior that was entirely unexpected. For example, in studies involving the Billiard Ball Mathematics task [e.g., Silver & Mamona, 1989], many subjects have not only imported aspects of their experience in playing billiards to pose problems but also generated problems that are not the "standard" problems that have neat mathematical solutions. When one poses a problem, one may not know whether or not the problem will have a simple solution, or any solution at all.

Another consequence of personalizing and humanizing mathematics through problem posing is that students can and will respond in ways that reflect their personal commitments and values. In some cases, this personal attachment can have positive consequences, as in the work reported by van den Brink [1987], in which he reported that the young children in his study made almost no mistakes in their self-constructed arithmetic books. This stands in sharp contract to the commonly observed carelessness of students and their tolerance of mistakes in mathematics classrooms in which they feel no personal ownership of mathematics. In one recent study of problem posing [Silver & Cai, 1993], students posed problems on the basis of a story about three persons driving in a car, and their responses suggested that issues of morality, justice and human relationships may have been as important to some students as issues of formal mathematics. For example, students revealed an apparent concern about an equitable distribution of driving responsibilities when they posed the following kinds of questions: "If they each drive an equal amount, how many miles would each person drive?", "Why does Arturo drive so long?" and "Why did Elliott drive twice as far as Jerome?". From the perspective of the research being conducted in that study, which focused on the semantic and syntactic complexity of the problems generated by the students, some of these responses were treated as being of marginal interest. Viewed from a broader perspective, however, these responses suggest not only the power of problem posing as an experience in which people express themselves with respect to mathematical situations or ideas but also the complexity of the educational and research challenges connected with understanding what the posed problems themselves represent as products of human activity. For the reasons discussed here, such problem-posing experiences are likely to be both especially important and especially problematic in teaching or research settings involving culturally diverse groups of students.

It would be easy for instructional developers and psychological researchers to overemphasize the role of problem posing as a means to accomplish other aims, such as improving the learning or study of problem solving. Our orientation toward problem solving in mathematics is so strong that we could miss the value of problem posing for its own sake. As we proceed with an agenda of instruction and research related to mathematical problem posing, let us be mindful not only of the potential that posing may offer for accomplishing other goals but also that the unsolved questions themselves offer great promise to us and to our students. We would be wise to heed the advice of the poet, Rainer Maria Rilke:

> Be patient toward all that is unsolved
> in your heart
> Try to love the questions themselves

Notes

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[2] It is worth noting that teachers and textbook authors engage in this kind of posing when they pose questions for students to solve. Posing problems so as to be evocative of good mathematical thinking has received some attention in the literature [e.g., Butts, 1980]. In this paper, however, the primary focus is on problems that are generated by students

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Mars suddenly yielded up a gift, when with startling ease he refuted Copernicus on oscillation, showing by means of Tycho's data that the planet's orbit intersects the sun at a fixed angle to the orbit of the earth. There were other, smaller victories. At every advance, however, he found himself confronted again by the puzzle of the apparent variation in orbital velocity. He turned to the past for guidance. Ptolemy had saved the principle of uniform speed by means of the *punctum equans*, a point on the diameter of the orbit from which the velocity will appear invariable to an imaginary observer (whom it amused Kepler to imagine, a crusty old fellow, with his brass triquetrum and watering eye and smug, deluded certainty). Copernicus, shocked by Ptolemy's sleight of hand, had rejected the equant point as blasphemously inelegant, but yet had found nothing to put in its place except a clumsy combination of five uniform epicyclic motions superimposed upon one another. These were, all the same, clever and sophisticated manoeuvres, and saved the phenomena admirably. But had his great predecessor taken them, Kepler wondered, to represent the real state of things? The question troubled him. Was there an innate nobility, lacking in him, which set one above the merely empirical? Was his pursuit of the forms of physical reality irredeemably vulgar?

John Banville