FLASHES OF CREATIVITY

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From a letter from Sir W. R. Hamilton to his son, we know that the following conversation occurred between them each morning in the first half of October 1843.

"Well, Papa, can you *multiply* triplets?"

"No, I can only *add* and *subtract* them." (Graves, 1885, p. 434)

This conversation also occurred on the morning of 16 October 1843, several hours before the moment when quaternions were actually invented. This invention, which, according to Kleiner (2012), redirected an entire domain of algebra, concluded at least ten years of unsuccessful attempts to construct an algebra of triplets, comparable to the algebra of complex numbers (for more details see Whittaker, 1944). The story of the invention of quaternions can be seen as a great success of a prominent mathematician who somehow managed to get out of a long-term intellectual dead end by inventing a novel and unexpected idea. Hamilton managed to rise above his 'usual' (*i.e.,* demonstrated over a long time) level of mathematical creativity and ingenuity.

We chose to begin this article with this historical anecdote for two reasons. First, we focus here on the phenomenon of a one-off manifestation of mathematical creativity against the background of non-particularly-creative behavior among mathematics learners. We term this phenomenon a *flash of creativity.* To us, the moment of Hamilton's invention is an example of such a flash. Our interest in *flashes of creativity* stems from our and our colleagues' encounters with such manifestations in our research and pedagogical practice. Second, the story prepares appropriate ground for the analytical apparatus that we are going to employ for analyzing flashes of creativity. Though the history of mathematics knows quite a lot of flashes of creativity among mathematicians, we find the story of the Hamilton particularly illustrative because it includes not only a well-documented account of the invention, but also has traces of Hamilton's personality, life circumstances, shifts of social, emotional and cognitive nature.

So, let us come back to the story of Hamilton. Consider an excerpt from the aforementioned letter: "the *desire to discover* the laws of the multiplication referred to *regained with me a certain strength and earnestness,* which had *for years been dormant,* but was then on *the point* of being gratified" (Graves, 1885, p. 434, italics added). This excerpt encapsulates shifts in Hamilton's emotional state and intellectual motivation. We should add that, according to van Weerden and Wepster (2018), Hamilton was involved in interactions of different types during the year preceding the quaternions invention. Hamilton conducted intensive correspondence with the Board of Trinity which had expressed dissatisfaction with Hamilton's work in an observatory, leading him to consider giving up pure mathematics. More positively, Hamilton had many meetings with friends and colleagues who supported him and helped him to regain strength and energy.

When the Hamilton story is projected onto the realm of mathematics education, and in particular onto the realm of the student mathematical creativity, the following question arises: Which particular characteristics of a regular student's personality and environment and which emotional and cognitive triggers may facilitate the student to reach, even for a short time, an unusual (for her) level of creativity?

Of course, we do not equate a seminal invention by a prominent mathematician and a local invention of a middle-school student who solved a school-level problem in a surprising way. Our approach, however, is stimulated by Hadamard's (1945) vision of the problem-solving work by a student and by a research mathematician as the same in nature but different in level. Accordingly, our study lies within a relativist strain in creativity research (*e.g.,* Liljedahl & Sriraman, 2006; Kaufman & Beghetto, 2009).

Specifically, the goal of this article is to elaborate a particular configuration of conditions having the potential to facilitate flashes of creativity in school students. We illustrate our claims by a case of a high-school student who progressed from her not-particularly-creative behavior to an unforeseen by us flash of creativity, expressed in a productive solution to a challenging geometry problem. In order to handle the complexity and multi-dimensionality of the case, we adhere to a so-called 4P (creative Product, Person, Process and Press) perspective on creativity which was first suggested by Rhodes (1961) and is still among the widely used theoretical lenses on creativity.

Creative Product, Person, Process and Press

Pitta-Pantazi, Kattou and Christou (2018) interpreted the four interacting strands of 4P model in mathematics as follows. "(1) Product: the communication of a unique, novel and useful idea or concept; (2) Person: cognitive abilities, personality traits and biographical experiences; (3) Process: the methodology that produces a creative product; (4) Press: the environment where creative ideas are produced." (p. 29). Of note is that Pitta-Pantazi, Kattou and Christou applied a 4P structure to review past research on mathematical creativity and not as a research tool for characterizing creative ability or creative individuals. These scholars suggested that mathematics education research should focus on the interrelation of the elements of the 4P model, with special attention to characteristics of individuals and of learning environments having the potential to facilitate creative processes leading to creative products. This is exactly our intention when using the 4P model as an analytical tool. Namely, when we stumble upon a product or process which is unusually creative (for a particular individual), we attempt

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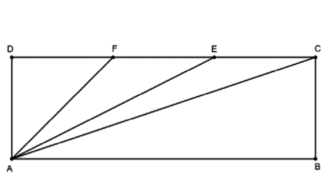
to understand an interrelation between the person who produced it and the environment in which the person acted. The following description of a student's problem-solving effort characterizes conditions for a flash of creativity and illustrates our analytical approach.

A flash of creativity: the case of Nika

It was difficult not to notice Nika in a tenth grade mathematics classroom. She participated in mathematics conversations with other students and with the teacher (Alik) frequently and openly. In many cases, Nika's contributions to the discussions were mathematically naïve, as when she proposed a way to find a derivative of a function at a point by just calculating a ratio $f(x) : x$. Nevertheless, some of her ideas later evolved into mathematically correct representations and solution methods, usually with the help of the teacher. Nika was not among the most successful (in terms of grades) students of her class, but she never failed a mathematics test. In other school subjects, her grades were excellent. Nika also loved dancing and performing.

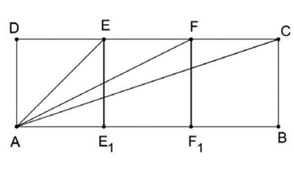
Nika's class, which studied mathematics in accordance with the Israeli advanced-level curriculum, took part in a research project exploring the affordances of combining classroom and online problem solving (Koichu, 2018; Koichu & Keller, 2019). During the school year, challenging mathematical problems were posted and collaboratively approached on the class online forum. On the one hand, these problems did not go beyond the school curriculum. On the other hand, they required a non-standard approach and encouraged students to stay on task for days rather than for minutes, as is frequently the case in regular mathematics lessons. One of the first problems presented to the students in the project was the Three Angles problem [1] (see Figure 1).

The problem was presented before the students had begun studying trigonometry, as it could be solved by several non-trigonometric approaches (the reader is invited to do so and then proceed reading). The problem was assigned as long-term homework, but no student solved it in the first two weeks. In the meantime, the teacher introduced trigonometric tools and told the students that the Three Angles problem can now be solved easily, but there are still several purely geometric solutions. After a mid-term exam (consisting of problems of calculus, trigonometry and geometry), in which Nika and some other students performed poorly, the teacher announced that a bonus would be given for solving the Three Angles problem. Some students responded and solved the problem using trigonometry. Their solutions, such as the one presented in Figure 2, were discussed in the classroom.



In the rectangle ABCD, AB=3AD. Points F and E divide DC into three equal parts. Find the sum of the angles: ∠FAB, ∠EAB, and ∠CAB.

Figure 1. The Three Angles problem.



$$\angle EAB + \angle FAB + \angle CAB =$$

$$\arctan\left(\frac{1}{1}\right) + \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{3}\right) = \frac{\pi}{2}$$

(using a calculator).

Figure 2. A trigonometric solution of the Three Angles problem.

The class moved on, and it was very surprising for us when after about three additional weeks Nika presented to the teacher a purely geometric solution of the problem (Figures 3 and 4).

Even a brief glance at Nika's solution (Figure 3) was enough to give an impression that Nika had done something distinctive. During two years when she was a student in Alik's class, she never submitted a solution organized in such a way. Namely, she presented not only the final solution but also the preceding attempts. Nika's sketches show that she introduced multiple auxiliary elements, which eventually

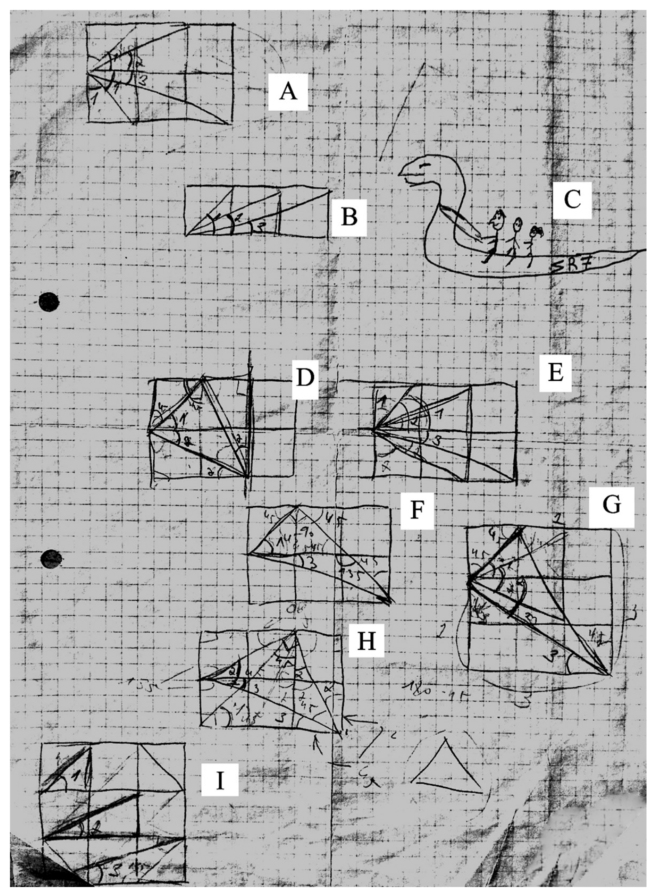


Figure 3. Nika's attempts. The successful attempt relates to sketch H.

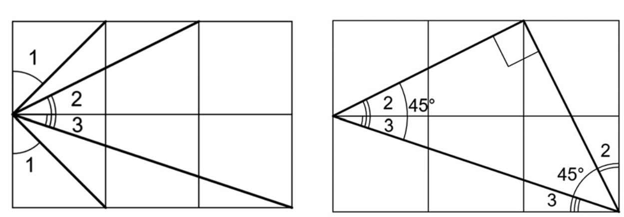


Figure 4. Our copy of Nika's sketches A and H.

helped her to apply principles of symmetry and unite previously unrelated components of the original diagram (Figure 4).

Nika was apparently very proud of herself, because she solved a difficult task applying a method that none of her classmates had used. Of course, she received the endorsement as well as the bonus points that helped her to improve her grade. After the described episode, Nika continued to be active in the mathematics lessons, but she produced no particularly creative solution to challenging problems, though we continued to provide the students with opportunities to do so.

Nika's one-off solution and persistence surprised us. In a conversation on "what happened?", Nika's initial explanation was as follows:

> I got 25 in the last exam and my parents were disappointed. I myself was not used to such grades. I began to study more seriously. My mom promised to buy me a new phone for improving my math grade. I felt uncomfortable that I did not solve this problem, and this drove me crazy. Do you know how much time I sat on it? I tried many types of symmetry.

This response intrigued us even more, as it contained multiple indications of shifts in her motivation, either external or internal, and engagement with the problem. To gain insight into Nika's flash of creativity, we attended to the data corpus of the above mentioned project: videotapes of several lessons in Nika's class and Nika's answers to the reflective questionnaire about the solution of the problem. We also conducted two interviews with Nika. The first interview focused on her solution, and the second on her mathematical history and habits of study. To recap, we attempt to contrast Nika's usual mathematical behavior with her actions during a flash of creativity. In the following section we present the contrasting characteristics in relation to the 4P model strands: Person, Press, Process and Product.

Same traits, new triggers

To recall, Nika was moderately successful in math tests until she failed a mid-term exam, but she had various external motivational stimuli to become more successful in mathematics than she had been long before the described episode. She mentioned in the interview the wish to be "number one in the class", the wish to get a new cellphone promised by her parents for good grades by the end of the school year and her interest in acquiring good mathematical knowledge that would be advantageous for her "future prosperity". In her words, "Yes, I had plenty of *[sources of motivation]*." On that basis, the introduction of the Three Angles problem created an additional motivational source, intellectual challenge: "What motivated me was knowing that there was another solution. It was interesting. I knew *[from a trigonometry solution which was discussed in the lesson]* that the answer is 90°. It was interesting to get there without trigo." In a way, Nika's ambitious personality responded to the challenge as a trigger liberating her from the experiences of dealing with 'ordinary' problems. In her words:

> I could do what I wanted in the shape, to insert the angles as I wanted, without boundaries and restrictions. It was nice. No constraints like alpha. One of these shapes would work. *[I tried about]* 30 sketches. I drew the shape and next one near to it. I started from simple *[ideas]* and then tried more complicated *[ideas]*. There were no constraints. It gave me freedom to solve *[the problem]* as I wanted.

In Nika's classroom, students compared their grades and strove for attention and appreciation from the teacher and peers. Apparently, Nika was a competitive person, too. During the interview, she compared herself to the strongest student in the class:

> He is a very good friend of all the 'ordinary' problems. In some respects, we are the same. As he does with all 'ordinary' problems, he can just see the problem and tell what the solution is. I can do the same *[pauses, apparently, tries to find a better description]*, but the point is that I don't solve them *[ordinary problems]*. He, if he gets the problem, he can solve it because *[…]* I don't know what he relies on. He is a genius, he is playing by the rules, let's put it that way.

Even before a flash of creativity, Nika was not afraid to show her original yet mostly erroneous attempts, as the regular classroom atmosphere of tolerance to alternative interpretations encouraged this. Failing a mid-term exam was an emotional shock to Nika. Her parents applied an enormous amount of pressure on her, and she suddenly saw herself as falling behind most of her classmates. She could improve the grade just by studying trigonometry, as some of her classmates had done, but she took another, non-trivial, path to get out from the emotional pressure and decided to pursue a genuine intellectual challenge. The Three Angles problem provided an opportunity for her not only to level up with the peers, but to outperform them. In Nika's words: "Then I decided: I can solve it by trigo, but everybody can solve it by trigo."

A combination of emotional pressure and intellectual challenge led Nika to enter a new for her course of actions. According to Nika, she allocated much more time to solving the problem than she did in her habitual mode of study. She testified: "I spent a lot of time struggling with the problem. I thought about it even in the middle of the lessons. It took me several days." Her search for knowledge supporting her self-determined quest was also very uncommon. Nika turned to an internet online forum, "a forum of mathematicians with a lot of gibberish", where she "learned a lot of not practical *[in relation to this particular problem]* stuff." In the forum, Nika was exposed to some essentially new mathematical practices, not directly connected to the problem. In her words from the questionnaire:

> I spent a whole Saturday learning *[on the Internet forum]* about geometric constructions inside of squares and rectangles and how to use them *[...]* I tried to apply what I learned to this problem. I tried many *[auxiliary]* constructions, which turned out to be inappropriate and wrong until I got the one that apparently looked right. I would call this construction 'beautiful'.

Notice that while engaging with the problem, Nika realized that there was a vast community interested in solutions of challenging mathematical problems. This was very different from what she described as her usual classroom behavior, when she solved problems "automatically, everybody solves, so do I. *[...]* I'd like to be as everybody and I solve. Except when we learn some new topic, then I need it *[to solve problems]*. I am not going to invest time from my life in drilling."

Earlier we presented Nika's work sheet containing her solution, which we considered remarkable based on how it looked. Our impression was confirmed: Nika's solution consisted of moves that were novel for her (as well as for the rest of the class). Also, we learned that Nika's approach was systematic:

> This is the method. You take a shape that you know you'd be able to construct again with the same givens, and you construct it within the new shape under new conditions. *[In the Three Angles problem],* I started from triangles, then I moved to the parts of the shape. I saw that this did not work and started looking for rectangles to construct. *[This is in order]* to put the shapes I knew something about into the given shape. To inscribe this into the shape *[rectangles and squares]*, whose angles I knew *[90°]*. To create the conditions that I could use. To create the frame, actually.

Having in mind this method, Nika was very persistent:

> I drew a lot of sketches. It did not help me, so I threw them in the trash bin. These *[points to a page with sketches]* are attempts towards a goal *[emphasis in intonation]*. Here *[points to sketch D in Figure 3]* the angle is a sum of 1+2, and here *[sketch F]* 1+3 and here *[sketch H]* 2+3.

She further explained that she began from reformulating the problem: "The angle 1 was 45°, there was no complication with it. I knew it at the level of 'statement—justification'. What was left for me was a question, if *[angles]* 2 and 3 give together 45°." To tackle this question, she drew an auxiliary square grid as a frame of reference and kept using it in all the new sketches (Figure 3, A-I). Then Nika decomposed the given sketch into separate elements (Figure 3, sketches B, I) and combined them (sketches D, F, G and eventually H). Finally, she expanded the original diagram by producing symmetrical reflections and noticing new equilateral triangles that she had created:

> I stretched my wings and considered different directions. I tried to reassemble the sketch as you taught us *[see the denoted points on sketch I]. [...]* I saw the congruence, this is one and this is one. This is two and this is two, 90 degrees here and there *[points on the right triangles having length 1 and 2 in sketch D].* I worked with blocks *[apparently, the square grid].* I completed the proof for myself here *[points to sketch H].*

So, how can the described flash of creativity be accounted for in terms of the 4Ps? First, one can see in Nika's explanations that she essentially remained the same person when she experienced the flash of creativity: she valued originality and enjoyed expressing herself by communication with others. Indeed, personal traits and learning environments cannot be changed overnight. In her class, the mathematics lessons continued to be of the same nature as before, during and after the described episode. All the lessons included opportunities to solve problems, discuss ideas, fail and receive help from the teacher (*cf.* Davies *et al.,* 2013). However, as we have seen, the described circumstances plus the influence of external and internal motivational stimuli plus an interesting problem that served as an intellectual trigger led Nika to an unusually creative process and product. In other words, it so happened that the Three Angles problem resonated well with the student personality and circumstances.

In sum, Nika spent several weeks thinking about the problem and undertook a very intensive effort. The solution relied on individual knowledge and resources plus resources gained through the specialized online forum. In order to solve the problem, the student voluntarily learned, among other things, a new (for her) heuristic principle of decomposition and recomposition of geometric shapes. The final attack consisted of systematic use of the chosen method, which lasted for hours. This course of action was not observed subsequently.

Discussion

We started this article by re-telling the story of Hamilton's invention of quaternions, and then presented a case of a high school student solving a challenging (for her) mathematical problem. Hamilton's invention was a result of his personal quest in which he produced a new mathematical idea. The idea was new also to Hamilton's colleagues and the mathematics community at large. In order to solve a challenging high-school-level geometry problem, Nika invented a mathematical idea which was new to her as well for her classmates. Thus, in line with Csikszentmihalyi and Sawyer's (2014) view on creativity as social phenomenon, creative products were evaluated by the members of a particular community, comparing a product by merit of its originality and usefulness against the other products within the same community. Hamilton was a talented and devoted mathematician functioning in a professional environment that awaits new results and has clear mechanisms for communicating and scrutinizing them. Nika was a student who valued originality and non-conformism and who studied mathematics in a learning environment tolerant of original ideas. Building on these parallels, and capitalizing on the work of Hadamard (1945), we suggest that Hamilton's and Nika's flashes of creativity were very different in level, but not so different in nature.

Let us proceed with the parallels. In both cases, productive ideas appeared after long not-particularly-productive periods. Moreover, intellectual and emotional triggers in the personal worlds of our heroes appeared just before inventions. Hamilton was uncomfortable in his social and professional environment (he almost lost his position at the Dansink Observatory!), but was dedicated to one mathematical problem of special interest to him. Nika was under social pressure and self-esteem threats when she chose to pursue an opportunity to solve a problem that became very special for her.

Thus, some of the conditions for both flashes of mathematical creativity were related to the personal traits and circumstances which may be seen as relatively stable. The additional conditions seem to be related to emotional and intellectual triggers: a stressful situation to get out of and a mathematical problem that does not let go and demands special attention for a relatively long period.

What might be some pedagogical implications of the above observations and suggestions? As argued and illustrated, a flash of creativity may occur when two strands of the 4P model, Person and Press, are already in favorable combination (see Figure 5). The learning environment should (at least) tolerate original ideas and solutions, and a student should attain some measure of proficiency and some inclination towards the originality in mathematical contexts. What seems to make a difference and induce a flash of creativity is a timely introduction of a challenge that can serve as an intellectual and emotional trigger for the student. In Nika's case, it was an opportunity to solve a problem which was intellectually and aesthetically appealing to her plus an expectation of getting out of a stressful situation and obtaining valuable rewards if and when succeeding in solving it.

Practically, even in educational environments that are favorable for mathematical creativity, for instance, environments having the core features of problem-based learning and mathematical problem solving (Silver, 1997; Leikin & Pitta-Pantazi, 2013), the flashes of creativity may remain rare without the introduction of emotional and intellectual triggers. As we know, competition and pressure for high grades are emotionally stressful factors. However, as Mason, Burton and Stacey (2010) point out, in order to master mathematical thinking one has to control emotional and psychological aspects of problem solving and learn to turn them to ones advantage. A mathematics classroom that is not only demanding, but also provides students with a safe place to compete, to test their abilities, to approach challenging problems repeatedly, despite local failures, may produce interesting flashes of creativity.

We suggest that facilitation of flashes of creativity by introducing accessible yet non-orthodox mathematical problems (*cf.* Barbeau & Taylor, 2009) into environments which are tolerant of students' originality might be a more realistic pedagogical goal than 'creativity development'. In this context, a question of personalization arises. Namely, which characteristics should mathematical problems possess in order to trigger flashes of creativity in individuals of different personality traits? Finally, it seems that Nika's decision to be creative in a way that is acceptable in a particular socio-mathematical environment led her to a redistribution of intellectual and emotional resources, which in turn induced meaningful learning of mathematics. In our view an interplay between voluntary and spontaneous characteristics of splashes of mathematical creativity demands further attention.

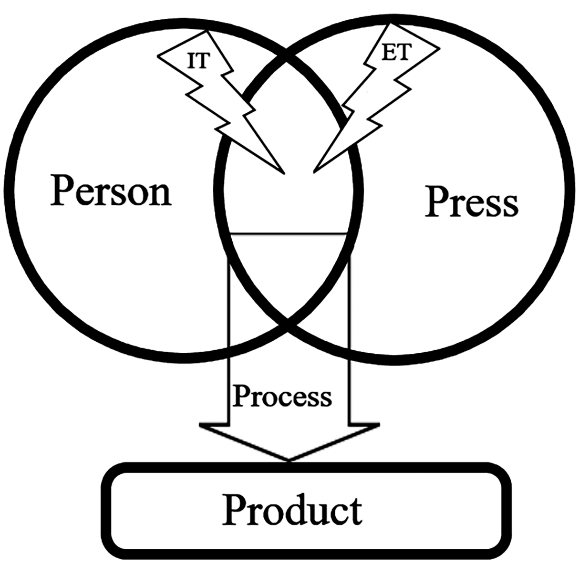


Figure 5: Emergence of a flash of creativity (ET and IT stand for emotional and intellectual trigger respectively).

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Note

[1] The source of the task is Sharygin (1982), p.10, problem 33.

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