STEPPED TASKS FOR COMPLEX PROBLEM SOLVING: TOP-DOWN-STRUCTURED MATHEMATICAL ACTIVITY

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In this article we introduce Stepped Tasks—top-down-structured mathematical activities aimed at developing students' ability to solve complex mathematical problems independently. Complex problem solving and self-regulation are among the major 21st century skills (Pellegrino & Hilton, 2012), and Stepped Tasks are especially designed to encourage solving complex mathematical problems in a self-regulated process. Pellegrino and Hilton stress that there is no consensus between researchers on the concepts of self-regulation and self-regulated learning. Stepped Tasks adopt the definition of self-regulation as a skill that allows achieving a set goal through reflection and self-evaluation of progress made towards achieving the goal, together with careful decision making regarding the steps to be taken. Using Stepped Tasks, students are directed to solve a complex mathematical problem, developing their problem solving skills and strategical reasoning as well as advancing skills of self-regulated learning.

The invention of Stepped Tasks is based on the following observations:

First, the goal of mathematical instruction is to develop students' ability to solve problems independently. Self-regulated problem solving is based on students' own step-by-step decision making process that promotes a deliberate and thoughtful problem solving process. Promoting self-regulated problem solving is not an easy task for teachers due to the heterogeneity of mathematics classrooms: self-regulation in the context of complex problem solving is linked to strategic reasoning and students' meta-cognitive awareness (Schoenfeld, 1992) and thus is a function of the level of students' mathematical knowledge and skills; therefore the mathematical challenge embedded in the tasks should differ for different students. Additionally, common instructional practice associated with teaching complex problem solving involves a bottom-up structure of mathematics lessons that presumes solving problems in a sequence of tasks with an increasing level of difficulty. In this way, the most challenging problem appears at the end of the problem sequence. There is a danger that bottom-up approaches to problem solving may 'defuse' the problems so that they no longer present a challenge, and thus no longer present an opportunity for learners to develop the ability to solve problems independently. Stepped Tasks are especially designed to encourage self-regulated complex problem solving in mathematics that develops students' strategic reasoning and meta-cognitive awareness.

Second, effective learning opportunities in mathematics classrooms are connected to challenging mathematical tasks that teachers have to devolve to their students. A challenge is a difficulty that an individual is able and motivated to overcome. In this context, mathematically challenging tasks require students to tackle an approachable difficulty which they are willing to overcome. Mathematical challenge is relative to students' mathematical potential: a problem that is too difficult for one student may be so easy as to be uninteresting for another. This puts teachers in the position of needing to reduce the level of challenge for some students without simplifying the problem to the point of making it trivial. We use the term 'varying mathematical challenge' to refer to decreasing the level of mathematical challenge that a task entails while ensuring that the task remains challenging for the students (Leikin, 2019). Stepped Tasks support teachers' capacity to conduct complex problem solving through varying mathematical challenge.

Third, an example of mathematical tasks that allow varying mathematical challenge is open tasks. 'Multiple solution tasks' are open-start ones, since they require solving a single problem using different problem solving strategies (Leikin, 2007). When solving multiple solution tasks students implement strategies that fit their knowledge and the number of solutions performed by the students also corresponds to their proficiency in problem solving. Problem-posing tasks are both open-start and open-end: students can pose problems of different types and different levels of complexity using different problem-posing strategies, in accordance with their knowledge and skills (Silver, 1994). The tasks' openness allows varying mathematical challenge by allowing students to perform the tasks at a level appropriate to them. However, open problems, while inherently challenging, present a different risk: when problems are too open for students, students may be unable to solve the problems and become frustrated.

Thus, neither a bottom-up structure of problem solving nor completely open problems guarantee the development of students' problem solving abilities. In contrast to open tasks, the Stepped Tasks introduced here are designed to develop students' strategic reasoning through self-regulated learning.

Introducing Stepped Tasks

A Stepped Task is a mathematical activity that includes a complex mathematical problem, called the 'target problem', which is accompanied by paths that include 'steps' of

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different levels of mathematical challenge. Each step includes a number of problems with a reduced level of challenge with respect to the previous step(s). Stepped Tasks' design is based on changes to the problem characteristics, including the following:

1. Conceptual density of a problem, which is determined by the number of concepts and their properties required to solve the problem (Silver & Zawodjewsky, 1997);
2. The length of the solution or logical chains in a proof; and
3. The level of mathematical knowledge and skills required to solve the problem.

The top-down approach to problem solving implemented in Stepped Tasks is considered a goal-oriented one, since the goal of the problem solving process is explicitly presented to the participants at the beginning. Top-down and bottom-up approaches are directed at similar goals, but are different in terms of the ways in which they achieve them. Top-down teaching starts with the target task, which is the main goal of the mathematical activity. Solvers have to uncover the necessary problem solving strategies and mathematical concepts, which are not presented explicitly, and find the meaning of the problem by applying their own knowledge and skills. That is why the top-down approach is mostly student-regulated. Bottom-up teaching is more teacher-directed and focuses on ways of decoding and simplifying each component of a problem. The bottom-up teaching approach lacks an emphasis on learning the complete picture. An analogy can be made to completing a puzzle in which a solver has to complete a given picture by searching independently for pieces of the picture. This is in contrast to completing a puzzle with guidance from a parent or a friend who knows the picture, with the picture appearing as if by magic with the aid of a more experienced individual.

When students tackle a Stepped Task they are first introduced to a *target task* that requires solving a complex mathematical problem P, which is the goal of the activity. The students are allowed to solve this target problem with or without using a number of steps that include other—less complex—problems, the solutions to which can lead to solving the target problem.

Figure 1 schematically represents the structure of a Stepped Task. It starts with a target task, "Solve problem P." If P is too difficult for the students, they are provided with an opportunity to solve problems at Step-1 (P1.1 … P1.*k*), each of which is less complex than P. The problems at Step-1 are not necessarily sub-problems of P. However, solving the set P1.1 … P1.*k* evokes the use of concepts and tools relevant for P. After solving problems at Step-1 students are presumed to be able to solve the target problem. If the problems at Step-1are still too difficult, students can solve problems at Step-2 and then solve either the target problem or Step-1 problems.

Students can be engaged with the Stepped Tasks individually and decide on the problem solving path appropriate to them. Alternatively, students can work with Stepped Tasks in a collaborative learning setting in which a group of students makes a joint decision about moving among the steps. In collaborative learning settings the Stepped Task can be used by more knowledgeable students who can provide their own scaffolding for students who are struggling. Additionally, Stepped Tasks allow bottom-up implementation if teachers prefer to use this mode of implementation.

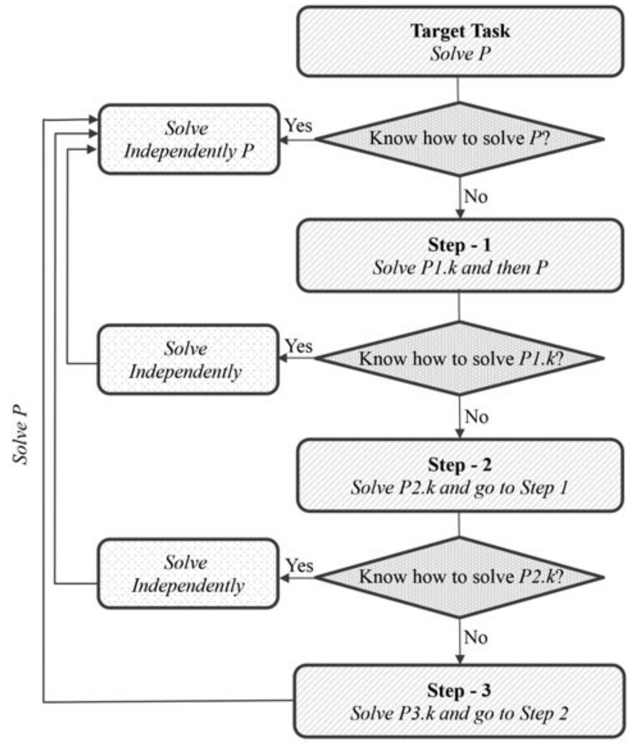


Figure 1. Stepped Task structure.

Stepped Tasks—examples

In this section, we describe two Stepped Tasks. The first task is a geometry task that involves proving a property of altitudes in triangles. It is intended for an upper-level high school class. The second task is an algebra task appropriate to middle school mathematics. The two tasks are used here to illustrate different ways of 'stepping'.

Stepped Task 1: Altitudes and angle bisectors in a triangle

Stepped Task 1 presents students with the Target task of solving the Geometry Problem (GP) shown in Figure 2 (overleaf).

The main principle in stepping this task is *decrease of conceptual density.* The GP is of high conceptual density: its solution(s) requires the integration of a number of important definitions and theorems. One of the possible proofs requires (a) identification of cyclic quadrilaterals, (b) using the theorem 'the sum of opposite angles in a cyclic quadrilateral is 180°', (c) knowing that the hypotenuse of a right triangle is the diameter of its circumscribed circle, (d) using equality of inscribed angles that lie on the same chord in a circle and (e) using similarity of triangles can significantly shorten the length of proof (by applying logical symmetry—proving 'in the same way'). Note here that there are other ways of solving the GP, however, the choice of the suggested solution strategy was due to its elegance in the eyes of the task designers.

At Step 1, problem GP-1 removes the requirement to draw a connection between the GP and the identification of cyclic quadrilaterals. This connection between the GP and cyclic quadrilaterals is challenging and usually constitutes a significant pitfall for students attempting to solve the GP. At the same time, the requirement to find the number of circles (see GP-1 in Figure 2) makes GP-1 challenging in itself, and provides students with motivation to identify cyclic quadrilaterals.

At Step-2, problem GP-2.1 removes the requirement to find the number of cyclic quadrilaterals. Problem GP-2.2, in addition to GP-2.1, asks students to find similar triangles in order to shorten the solution, and thus removes conceptual density by providing students with an additional tool for performing the proof. Additional stepping in Steps 1 and 2 is performed by providing the students with GeoGebra applets that do not provide hints (Figure 3). Rather, they allow an investigation process in a dynamic environment that leads students to deepen their understanding of the problems and discover properties that are essential for proving. As mentioned above, the investigation tasks are open and ultimately challenging for students.

Step 3 introduces auxiliary constructions that do not appear in Steps 1 and 2 and removes the stage of searching for cyclic quadrilaterals which is essential for Steps 1 and 2.

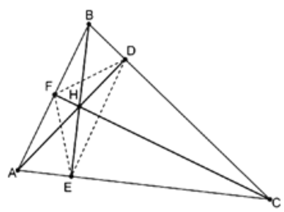
In sum, Steps 1 and 2 remove conceptual density by reminding students about concepts and their properties that can be used when proving. Step 3 reduces mathematical challenge by providing auxiliary constructions. At all the stages students are asked to solve problems which are challenging for them. They activate their working memory and strategic and logical reasoning when connecting the problem to appropriate theorems and designing the proof.

Even when the complexity of the problems is decreased by stepping, the problems at the different steps remain challenging since the problems are open with regard to the problem solving strategies that students can use. We recommend that students decide for themselves whether to move among the steps; this way, they can regulate the level of challenge of the problems they work with. However, step choice can also be performed collaboratively by students or with the teacher's guidance; in any case, the stepped tasks remain directed at the development of students' strategical reasoning and independent problem solving. Note that the GP is borrowed from the high-level mathematical curriculum, which is aimed at the top 15% of students in high school. In general, Stepped Tasks can be developed for other levels of mathematics as well.

Target task: *Solve GP*

GP:
Given: Segments AD, BE, CF are altitudes in the acute triangle ΔABC. (fig. 2.a)

Prove: The altitudes in ΔABC are angle bisectors in ΔDEF.



If proof is completed, find an additional proof

fig 2.a

If needed go to Step 1

Step 1 – Solve problem GP-1 and then GP

GP-1: In the given triangle, find quadrilaterals that can be inscribed in a circle (fig. 2.a).
How many inscribed quadrilaterals did you find?
Are there additional inscribed quadrilaterals (fig.2.a)?

Return to target task
or *If needed go to Step 2*

Step 2 - Solve problems GP-2.1, GP-2. 2 and then GP

GP-2.1: In the given triangle, find quadrilaterals that can be inscribed in a circle (fig 2.a).
Did you find six inscribed quadrilaterals?

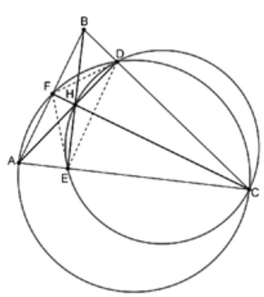
GP-2.2: Find all the pairs/trios of similar triangles in the picture

Return to target task
or *If needed go to Step 3*

Step 3 - Solve problems GP-3.1, GP-3.2 and then GP

GP-3.1: Prove: quadrilateral AFDC can be inscribed by a circle. (fig. 2.b)

GP-3.2: Prove: quadrilateral EHDC can be inscribed by a circle. (fig. 2.b)



Return to target task

fig 2.b

Legend: GP – geometry problem

Geo-Gebra applets are available for investigation

Figure 2. Example of a Stepped Task in geometry.

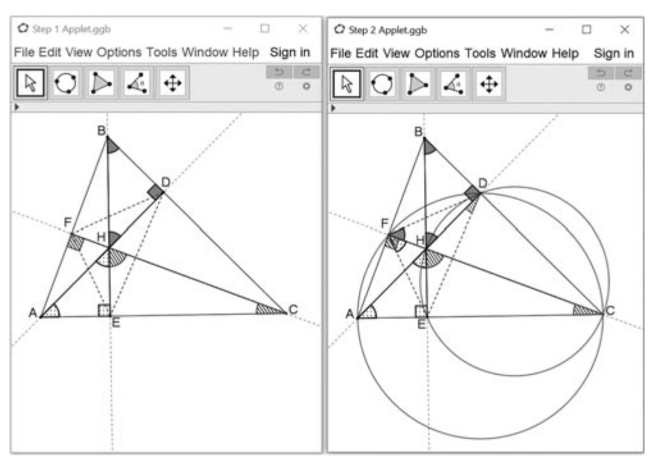


Figure 3. Applets for GPs: Step-1 (left) and Step-2 (right).

AP:

Given: Pictured are the graphs of functions $f(x), p(x), h(x), t(x), g(x)$ (fig. 4.1)

1. Write expressions for the parabolas $f(x), p(x), h(x), t(x), g(x)$ to obtain the same picture. Check your work in Geo-Gebra.
2. Can you find other expressions for the parabolas $f(x), p(x), h(x), t(x), g(x)$?
3. Write parametric representations of $f(x), p(x), h(x), t(x), g(x)$ that represent a general solution to the problem.

If needed go to Step 1 (see appendix 1)

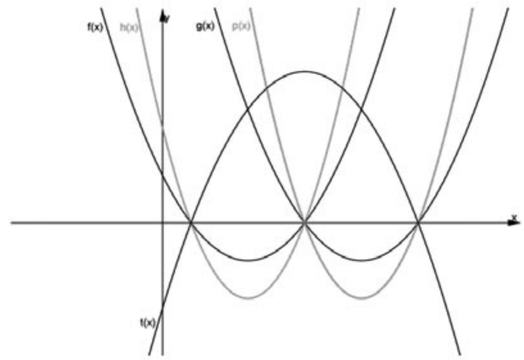


fig 4.1

Figure 4. Target task in Stepped Task 2.

Stepped Task 2: Parabolas

The Algebra Problem (AP) in Figure 4 is a Target task appropriate to middle school mathematics. We invite readers to solve this task and design an appropriate Stepped Task. Our proposed Steps are given in Appendix 1.

Top-down versus bottom-up structure

While we encourage teachers to use Stepped Tasks in a top-down structure we were happy to discover that Stepped Tasks allow flexible use by teachers. Teachers are advised to use Stepped Tasks as a mediating tool in their efforts to guide students by providing advice on varying mathematical challenge.

To date, we have developed over 60 Stepped Tasks for high level mathematics in 10th, 11th and 12th grades as part of the national Steps-to-5 project. We conducted a design experiment in two focal schools where all the teachers who teach mathematics at high level volunteered to implement three Stepped Tasks in each high-level class, under our team's guidance. In parallel, Stepped Tasks were implemented in communities of practice of mathematics teachers who taught mathematics at a high level (Leikin & Aizik, 2020). Based on 400 reports submitted by the teachers we found that the teachers implemented the stepped tasks either in *top-down structure* as we suggested or in *bottom-up* structure, which is more familiar to teachers. In a top-down structure, the teachers present students with either the target task or Step-1 problems, depending on how complex they believe the target task would be for the students in that particular classroom. In the bottom-up structure the level of complexity of problems is raised gradually, starting from Step-3 problems (see Figure 1) through Steps 2 and 1 until reaching the Target task. We also observe implementation of Stepped Tasks in a combined structure, in which at the beginning of the activity different students are assigned problems at different steps. While some teachers' decision to convert Stepped Tasks to bottom-up structure (in 22% of the reported cases) raised concerns that the suggested approach may be impeded, we found that providing teachers with flexibility regarding the mode in which they can implement the stepped tasks increases teachers' willingness to use them, and eventually the teachers develop confidence in using the tasks in a top-down structure as well.

The teachers based their choice of initial step level on their familiarity with students' mathematical achievements, with their primary goal being to match the level of learning to the students' level. For example, teacher Michal (all names are pseudonyms) reported :

> *Michal:* I assigned groups of students based on the students' mathematical abilities and the social interactions in the class. Students in different groups (strong students, good students, and less good students) were assigned different cards [*steps*].

Some teachers, such as Rachel, reported that they directed students to different levels according to the students' problem solving progress:

> *Rachel:* All of the students got a card at the highest level [*the target task*]. I laid out the cards [*different steps*] in three piles and allowed students to take them freely. […] But they didn't. So I walked around and gave them appropriate cards [*steps*].

These distinctions are linked to the balance between *student-regulated* and *teacher-directed* modes of problem solving. In student-regulated activity the students are provided with autonomy when solving the assigned problems with or without using the steps. In a teacher-directed mode the teacher determines the steps at which some or all students solve problems at different stages of the activity. Pure student-regulated implementation was identified in 30% of the reports. Teachers who used this approach provided the students with a Stepped Task including the target task and all the steps (in electronic or paper form). It should be noted that implementation of Stepped Tasks increased teachers' tendency toward student-regulated problem solving and decreased their tendency to use *pure teacher-directed mode.*

The teachers explained that they use Stepped Tasks in order to encourage students' independent work, to challenge students with high abilities, and to allow students to use the steps when they experienced a certain level of difficulty.

Sometimes teachers chose to stop the students' work during the lesson in order to have a discussion on their progress or on a problem they encountered, and then returned to student-regulated work. For example, Hila said in an interview:

> *Hila:* Everyone chose to start at the advanced level, and they essentially guided themselves. Students worked individually or in pairs and chose the steps appropriate for them. After 10 minutes [*of students' work*] [...] I invited a student to present her results to the whole class and the discussion focused on the logical structure of her results [...]. Then everyone continued working.

Penny worked differently and described her lesson as follows:

> *Penny:* I [...] explained how to work with Stepped Tasks and whoever wanted, could start at the hardest level. I didn't direct them, and surprisingly—or not—everyone chose to start at the advanced level. But later, there were some students who moved to the easiest card.

Our analysis of the teachers' reports and the lessons demonstrates that teachers who use Stepped Tasks find that the tasks enrich their instruction and support their skills. Moreover, the teachers report that they design additional stepped tasks by themselves after implementing the Stepped Tasks developed by the Stepped Tasks team. However, some teachers admitted that Stepped Tasks require "changing one's state of mind" and "almost revolutionary changes in the teaching strategies and skills". Thus, only 21% of the sessions employing Stepped Tasks were based on students' self-regulated problem solving process as initially planned by our team. The causes for this difference may be rooted in the teachers' educational backgrounds and the students' attainment levels. A deeper understanding of these causes is subject to future investigation.

A concluding note

The top-down structure of learning is familiar in teaching languages and in management, but this structure is less well known in the teaching and learning of mathematics. Through their acquaintance with the target tasks, students should become aware, at the beginning of the lesson, of where they are supposed to be at its end. The students can then decide whether they are able to attain this target by themselves or whether they need help in decomposing the complex problem into smaller and more approachable problems.

In spite of our initial intention to promote implementation of Stepped Tasks in a top-down structure, teachers implemented them in different modes. Over time, our attitude to the different modes of implementation changed from disappointment to satisfaction. We realised that these modes are related to teachers' goals, experiences and the stages of learning at which the Stepped Tasks are used. For example, the top-down structure is mostly used in lessons concluding a topic, as self-evaluation tasks. Stepped Tasks in top-down structure are used for developing problem solving skills in individual or cooperative learning settings. Interestingly, we realised that the bottom-up structure appears to be more suitable for lessons opening a topic, when teachers prefer gradually raising the level of mathematical complexity of problems that are new to the students. We also found that implementation of the Stepped Tasks in the mixed mode is usually linked to teachers' perception of students' mathematical potential in respect to the suggested task.

Because target problems are complex curricular tasks, solving Stepped Tasks in a self-regulated manner allows students to perform self-evaluation of their competencies. Stepped Tasks entail teachers' trust in students' ability to make decisions about what level of challenge is most appropriate for them in each particular problem, and require them to transfer the responsibility for students' learning to the students themselves. Moreover, implementation of the Stepped Tasks in the top-down structure requires teachers to have a deep understanding of mathematics as well as of the theory behind varying mathematical challenge.

We suggest that Stepped Tasks are an effective research tool that can shed light on students' self-regulated learning, for instance, via examination of students' problem solving processes using Stepped Tasks. Moreover, Stepped Tasks seem to be effective for the development of students' collaborative skills and learning motivation. These assumptions are a subject for future research.

Acknowledgments

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Appendix: PARABOLAS TASK

Target task: *Solve AP*

AP:

Given: Pictured are the graphs of functions $f(x), p(x), h(x), g(x), t(x)$ (fig. A.1)

1. Write expressions for the parabolas $f(x), p(x), h(x), t(x), g(x)$ to obtain the same picture. Check your work in Geo-Gebra.
2. Can you find other expressions for the parabolas $f(x), p(x), h(x), t(x), g(x)$?
3. Write parametric representations of $f(x), p(x), h(x), t(x), g(x)$ that represent a general solution to the problem.

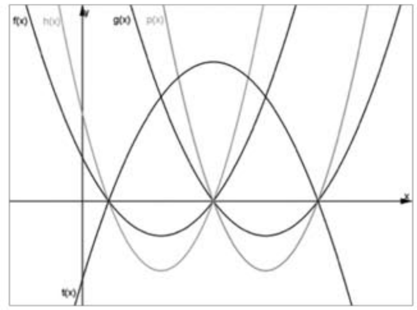

fig A.1

If needed go to Step 1

Step 1 – Solve problem AP-1 and then AP

Given: Pictured are the graphs of functions: $f(x), p(x), t(x)$ (fig. A.2)

1. Write expressions for the parabolas $f(x), p(x), t(x)$ to obtain the same picture. Check your work in Geo-Gebra.
2. Can you find other expressions for the parabolas $f(x), p(x), t(x)$?
3. Write parametric representations of $f(x), p(x), t(x)$ that represent a general solution to the problem.

Return to target task or *If needed go to Step 2*

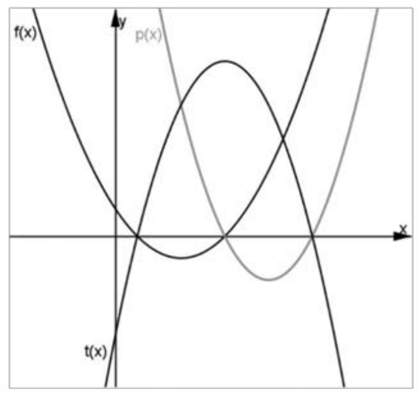

fig A.2

Step 2 - Solve problems AP-2 and then AP

Given: In the two figures below are the graphs of functions: $f(x), h(x)$ (fig. A.3a); $h(x), p(x)$ (fig. A.3b)

1. Write expressions for the parabolas $f(x), h(x)$ to obtain the picture in fig. A.3a and expressions for the parabolas $h(x), p(x)$ to obtain the picture in fig. A.3b. Check your work in Geo-Gebra.
2. Can you find other expressions for the parabolas $f(x), h(x), p(x)$?

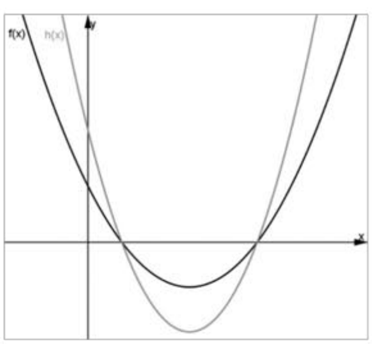

fig. A.3a

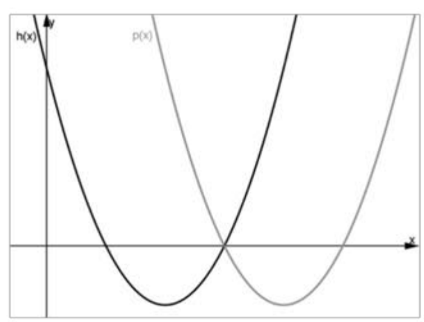

fig. A.3b

Return to target task

Legend: AP – algebra problem