Surprises in Mathematics Lessons

KAZUHIKO NUNOKAWA

In this article, I draw on a range of Japanese work and unify it under an integrating theme of 'surprise', to help gain insight into a characteristic of mathematics lessons which make students interested in mathematical ideas Hersh (1997) took the standpoint that mathematics is a social-cultural-historic reality. From that standpoint, mathematics is considered to be *outer* to individuals, while it is considered *inner* to larger societies or communities (p. 17)

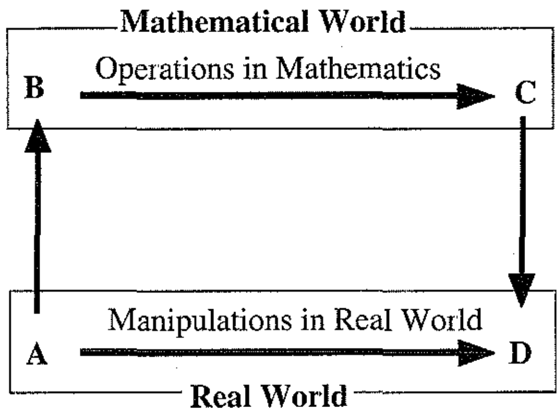
Mathematics, like the other social institutions, has been created by human beings and cannot be easily changed by only one person or one classroom in a school If we, as mathematics teachers, expect our students to learn such mathematics outer to themselves, surprise may be a critical factor in whether students would be interested in mathematical ideas This issue is also important from the standpoint that learning is increasing participation in a certain practice (e.g. Lave and Wenger, 1991; Sfard, 1998). Usually, people want to participate in practices which seem interesting to them.

Osawa's (1996) practice suggests that surprises are one of the factors which can elicit students' interest. In his lessons, new information about baton passes in the relay was generated using mathematical ideas Students were surprised that such information could be generated by mathematics:

> At first, I doubted whether mathematics could deal with issues in sports. But I was very surprised because baton passes could be done very smoothly and the time got shorter than before I was happy It was fun. So, I want to try it again with other tasks. (p 251; *my translation*)

Here, a kind of surprise made the student interested in mathematical activity. This comment reminds me of the notion of 'magical' moments in classrooms (Barnes, 2000). While magical moments in Barnes' article are concerned with an understanding of mathematics itself, the surprise observed in this student's comment seems related to an understanding of the value or power of mathematics. If feelings of surprise can also be considered a kind of magical moment, it may be expected that such surprises can positively influence students' attitudes toward mathematics and make them interested in mathematical ideas

When examining Japanese mathematics lessons, it can be found that there are many lessons which seem to incorporate this 'surprise' element. Moreover, as shown in Osawa's practice, those lessons tend to deal with relationships between mathematics and real-world situations Thus, in this article, I attempt to analyze such lessons with 'surprises' by referring to one schematic model consisting of the 'mathematical world' and the 'real world' (see Figure 1).



A: Initial Information in Real World
B: Mathematical Formulation
C: New Information in Mathematics
D: New Information in Real World

Figure 1

This model is often used to represent problem-solving processes (making number sentences or expressions, calculating or solving the expressions to get answers, and interpreting them based on the real situations) or knowledge construction processes embedded in real settings (Schroeder and Lester, 1989)

By taking account of its similarity to mathematical modeling processes, this model can be used in wider contexts, such as exploring relationships between school mathematics and real situations (Nunokawa, 1998) or analyzing the role of transformations of algebraic expressions (Miwa, 1996). It will be used in this article to integrate some lessons with respect to 'surprises' and to characterize them

In using this diagram here, the notions of both the real and mathematical worlds need to be considered as relative. In other words, the 'real world' in the model will refer not only to everyday situations, but also to worlds which are less abstract than the mathematical systems to be learned in the lessons or which are more real to the students.

For example, the natural-number world can be considered real to most high school students who study algebraic expressions Geometric figures are mathematical entities and calculating their areas with some formulae can be said to be operations in mathematics. However, such operations can be taken as relatively more real and direct in the context of proving a certain conjecture about areas.

1. Surprises caused by gaps between A and D

Inoue (1993) planned and implemented twenty-nine lessons around eighth-grade geometry aiming at arousing the students' intellectual curiosity and intrinsic motivation. In the last four lessons, he used an instructional resource called 'Hatome-Kaeshi' (see Figure 2), which is made of thick paper in the shape of a quadrilateral. It can be cut along three lines and separated into four pieces.

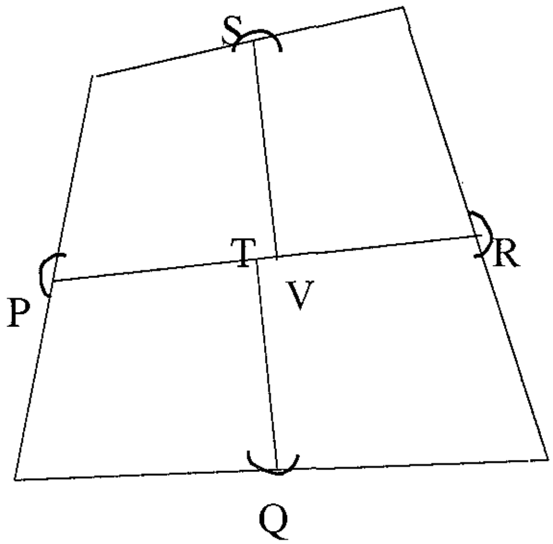


Figure 2

Here, the points P, Q, R and S are the midpoints of the sides and the lines QT and SV are perpendicular to PR. Moreover, these four pieces are tied together at P, Q, R and S with rubber bands. When turning over each of the four pieces, we get a rectangle regardless of the shape of the original quadrilateral.

In the first lesson, Inoue presented one 'Hatome-Kaeshi' and asked the students why it could become a rectangle. After examining the reason for this, he also asked them to make a similar 'Hatome-Kaeshi', but this time one which would become a parallelogram when each piece was turned over. The students presented and discussed their solutions to this task at the end of the first lesson and during the second lesson. In the third and fourth lessons, they made 'Hatome-Kaeshi's' which would become other shapes.

Manipulating such instructional materials made of thick paper can be considered more realistic to students than checking the reasons why the pieces resulted in rectangles or parallelograms by reference to geometrical facts. In other words, the former involves manipulations in the real world, while the latter requires logical operations in the mathematical world.

Inoue reported that the students made loud comments like, "Ah, that's a rectangle" or "Why can it become a rectangle?" when he demonstrated a 'Hatome-Kaeshi' at the beginning of the first lesson (p. 115). This implies a gap between the states A and D in Figure 1 and the realisation of that gap might produce a kind of surprise in the students. Simple operations like turning over each piece changed arbitrary quadrilaterals into more regular shapes like rectangles or parallelograms. It may be such a fact, based on manipulations in the real world, that elicited the students' surprise. This surprise facilitated the next phase of the lessons, namely uncovering the mechanism of 'Hatome-Kaeshi' geometrically.

Checking the reasons why the materials could become more regular shapes involves uncovering the mechanism behind this fact which produced such surprises. This check was done in the flow of A → B → C → D in Figure 1. The original state was characterized in geometrical terms (e.g. midpoints, perpendicular) (A→B). New information was introduced by geometrical reasoning (e.g. the sum of interior angles of quadrilaterals is 360°, a quadrilateral all of whose interior angles are 90° is a rectangle) (B→C).

The results in the real world were explained according to new information in the mathematical world (e.g. the pieces can fit together because the sum of the interior angles of the original quadrilateral is 360°) (C→D). This flow through the mathematical world bridged the gap between the initial and end states in the real world. The learning tasks, where the students made new 'Hatome-Kaeshi's', can be seen as giving control over or encouraging predictions of results in the real world through mathematical reasoning (e.g. using mathematical reasoning to determine how to make a 'Hatome-Kaeshi' which would become a parallelogram). The above discussion shows that Inoue's lessons were implemented to cover all the possible flows in Figure 1, i.e. A→B→C→D and A→D. The surprise produced by the gap between A and D was a trigger for further learning.

A task based on the Fibonacci sequence (Kotou, 1988) can result in a similar story in classes. In this task, students are asked to choose some single-digit numbers arbitrarily and write them horizontally in a row. Then they are asked to take a single one-digit number (here 4) and write it under each number in the first row. Students are to start adding vertically the first and second numbers and then the last digit of this sum is placed immediately below, creating a new row.

Here is an example.

1	3	5	2	6	8
4	4	4	4	4	4
5	7	9	6	0	2
9	1	3	0	4	6

They then continue the same calculation for the second and third rows, the third and fourth rows, and so on.

When the numbers in the seventeenth row are written, an interesting thing happens: all of them are the same number. Although the initial two numbers are selected arbitrarily (whichever starting number from the first row and the constant number in the second row), the result is rather neat. There is a gap between the arbitrary choice of initial numbers and the same result all along the seventeenth row: it is expected that this gap will produce student surprise. Whether such a result holds generally can be examined by checking all the combinations of the initial single-digit numbers and constant second row numbers, i.e. 81 cases.

However, consideration of this task by means of algebra can make it possible to uncover the underlying mechanism Expressing the initial numbers as x (the particular row one starting entry) and y (the constant second row entry), students can examine how such numbers appear in the middle results of the calculation and how they work in the result of the seventeenth row. Proofs for this task should place their emphasis not upon *whether* the interesting fact always holds, but upon *why* such a fact happens. This is consistent with Hanna's (1995) idea of 'proofs that explain' (see also Hersh, 1997, pp. 59-61). [1]

If making actual calculations with concrete numbers is more realistic to secondary students than algebraic proofs, the surprise mentioned above can be considered to be generated by the gap between the initial and end states in the real world (A and D in Figure 1). Uncovering the mechanism of the curious result using algebra can be interpreted as following the flow of A→B→C→D In other words, mathematical reasoning can fill the gap which generates students' surprise in the real world. Furthermore, algebraic explanation can give new information about other results of the calculations For example, the algebraic expression of the seventeenth row may imply that the sum of this row will be determined by seven times the second number The expression of the eighteenth row shows that the sum of this row will be determined by seven times the sum of the first and second starting numbers Such information corresponds to the control aspect mentioned in the previous example

Yashiki (1995) dealt with a kind of Japanese game 'Janken' in lessons on probability at ninth grade. In 'Janken', each player displays one of the three choices (rock, paper and scissors) by a gesture The player showing rock beats scissors, scissors beats paper and paper beats rock. Yashiki asked each pair of students to play 'Janken' one hundred times and record the choices each player selected each time After collecting the students' data, he called on two students who had made biased choices, i.e. one of them selected scissors and paper more often and another selected scissors more often. He predicted that the latter student would lose less often when playing the former, and verified this prediction by letting them actually play 'Janken' ten times

After demonstrating such an interesting consequence, Yashiki explained the underlying probability reasoning and showed how to predict that consequence based on probability. Through this demonstration, he expected his students to feel that "probability reasoning can be used in their everyday life" (p. 57) Many students may think that results of 'Janken' are influenced by chance and cannot be determined in advance. There is a gap between having two people play 'Janken' and making a reliable prediction about that game The fact that the teacher could make such a prediction might generate student surprise.

This lesson seems structured around magic and showing its trick. The fact that a reliable prediction could be made is a curious phenomenon in the real world and implies a gap between the states A and D in Figure 1. Inferring a result of the game based on probabilistic reasoning proceeds via operations in the mathematical world. It has the same flow of A→B→C→D and shows how the reliable prediction D was made. Thus, these lessons tell a story similar to Inoue's (1993) lessons, in the sense that a gap between the states A and D elicits students' curiosity and then mathematical reasoning bridged this gap.

2. Surprises caused by gaps between C and D

Matsumiya and Yanagimoto (1995) reported on ninth-grade lessons where the students were asked to find out the distance which could be seen from the skyscraper in their city and from Mt Fuji, the highest mountain in Japan. These lessons were the second part of a project titled 'Mathematics about skyscrapers', and the students were interested in them most among the three themes in the project (p. 81)

This theme consisted of two lessons. In the first lesson, the students came up with the formula for finding out the distance which could be seen from a building or a mountain. Based on the Pythagorean theorem, the furthest visible distance d can be expressed as $d = \sqrt{x^2 + 2rx}$, where x is the height of a building or a mountain and r is the radius of the earth (p 79) After that, they found out the range that could be seen from the skyscraper in their city and from Mt Fuji

In the second lesson, the students noticed that, if there was an overlapping of the areas that could be seen from two different places, one of them could be seen from the other. Finally, the students recorded on maps the ranges that could be seen from various places and made calendars using those maps.

Some students commented on these lessons as follows: "I didn't think that I could see such a wide area of the city from that building", "I didn't know that we could see such distant places" and "I was very surprised, because I noticed that we could see further than I had expected" (p. 81; *my translation*). These comments seem to show gaps between the mathematical results they found and the expectations they had had before If we consider these expectations to result from everyday reasoning without mathematics, surprises observed here can be generated by the gap between the states C and D in Figure 1

This surprise might support the students' feeling that mathematics provides them with new information about the real world. One student wrote that he wanted to go to the places they had determined based on their calculations and check how accurate their decisions were This comment suggests that implementing the flow of A→D once again may be important, even after mathematical results were obtained. Such a flow can cause students to re-examine their expectations and mathematical results, and bridge the gap between C and D. [2]

In his lesson about division of fractions at sixth grade, Tsubota (1996) also seems to have made use of a gap between mathematical results and student expectations in the real world (pp. 106-115) But the gap played a slightly different role from that played by the gap in Matsumiya and Yanagimoto (1995). Tsubota presented this problem to his students:

> When cutting a $5\frac{1}{4}$ m rope into $\frac{3}{4}$ m ropes, how many $\frac{3}{4}$ m ropes can we obtain?

The students solved this problem at once and answered "seven ropes" After that, he changed the condition of "$\frac{3}{4}$ m" in the problem into "$\frac{1}{5}$ m". The students began to calculate and noticed that something was strange, partly because they could not interpret "$\frac{1}{4}$" in their result of "$26\frac{1}{4}$" well (Some students considered this to be the length of the remaining rope, $\frac{1}{4}$m.) Then the students discussed why they felt something was strange and, during this discussion, they made it clear that "$\frac{1}{4}$" was not the length of the remaining rope, but its proportion to a shorter rope. Finally, Tsubota explained this idea using a diagram to make it clearer

In this case, the flow of A→B, i.e translating the situation into a division of fractions, was implemented smoothly by the students. The teacher also made this flow smooth by presenting an easier problem first. Tsubota reported that the students could divide fractions with no difficulty Since asking for the number of shorter ropes in a long rope is a typical situation for measurement division, the students could imagine what this situation looked like. They might have expected that the answer was a whole number like '*n* ropes' and, if there were a remainder, it should be shorter than $\frac{1}{3}$m.

Thus, the gap in this lesson lay between these expectations in the real world and the calculated result This gap made the students feel strange and generated surprise, expressed via comments like, "Why didn't the same approach lead to the number of ropes?" The teacher expected in advance that his students might well experience "an unexpected situation" (p. 108; *my translation*) at this point of the lesson. He aimed in this lesson to re-examine "the meaning of the calculation" during the ensuing discussion triggered by this gap

At the end of the lesson, the teacher complemented the students' discussion to bridge this gap more firmly. In the diagram, he used shorter ($\frac{1}{3}$m) ropes which were combined one by one in his explanation Concerning this diagram, Tsubota emphasized the importance of "showing the process of making the diagram" (p. 113; *my translation*). This corresponds to implementing more realistic manipulations of this situation, although there was a difference between 'cutting' and 'combining'. The teacher seems to have tried bridging the gap between states C and D by demonstrating the realistic manipulation A→D more intentionally

3. Surprises caused by gaps between A and B

Matsushita (1991) analyzed the lesson where the number of turns of a toilet paper roll was explored (Katsuno *et al*, 1991), using a model similar to Figure 1 This lesson was planned for a mixed class of junior and senior high school students First, the teacher presented the task to find out the number of turns of a toilet paper roll that was 65m long Then the students were asked to make conjectures three times:

(i) based on mere intuition;

(ii) based on the information about the radii of the paper roll and its core;

(iii) based on observing the unfolding of the paper for 50 turns.

Next, the teacher introduced the formula for the sum of an arithmetic progression and the number of turns was calculated using this formula. When cutting the toilet paper along a line and extending it (see Figure 3), its side shape becomes a trapezoid, and the difference between the lengths of the *n*th and (*n*+1)th sheets is always $2x\pi$ (where x is the thickness of a sheet).

According to Katsuno *et al*., the total length of a toilet paper L can be expressed as $L = (n/2)(2\pi r+2\pi R)$, where n is the number of turns, r the radius of the core, and R the radius of the whole toilet paper roll. After calculating the number of turns, the toilet paper was unfolded completely and the number of turns was counted by the students Finally, a roll of paper tape was handed to each group of the students, and only its number of turns was told to them The students calculated the length of the tape using the same formula and then checked their result by measuring the actual length of the tape.

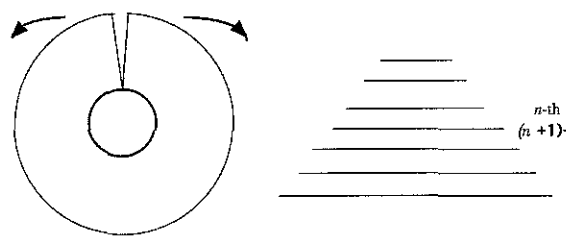


Figure 3

Matsushita (1991) mentions one of the characteristics of this lesson:

> This lesson is planned for the students to appreciate that problems in the real world can be solved more effectively if we use mathematical knowledge. (p. 71; *my translation*)

This seems reflected in the students' own comments (Katsuno *et al*, 1991, pp. 30-33) For example, "I noticed that we can translate any situation around us into mathematics and calculate it", "It was mysterious and interesting that we can find it referring to the area of trapezoid", "It is wonderful that mathematics is useful for familiar things". In addition to this, they report that, when the formula was introduced, the students could not envisage how the formula could be related to the situation. These comments imply that there was a gap between states A and B in Figure 1. In other words, the students were not sure that the problem in the real world *could* be translated into mathematics, nor that mathematics *could* be used for solving this problem.

In the latter part of this lesson, the teacher tried to bridge the gap After calculating the predicted number of turns, he actually unfolded the toilet paper and counted the number of turns with the students When they observed that the result of this experiment was the same as their prediction inferred by mathematics, the students raised a cheer and clapped their hands. They also commented that they were happy to know that the actual number of turns was the same as the result of their calculation. Their applause and comments meant that fitting with the real world played an important role for bridging the gap between states A and B.

In this lesson, the fit of mathematics and the real world generated the students' surprise, which made the students appreciate the value of mathematics

The last task in this lesson strengthened the validation of the mathematical translation A→B, by asking the students to compare the predicted lengths they calculated with the result of their actual measurement of the paper tapes. Also in this lesson, both flows A→B→C→D (i e. mathematical translation and operations) and A→D (i.e. realistic manipulations) were used and they supported each other

Kashihara (1993) presented the map of one expressway in Japan and asked his ninth-grade students to find out its total length. They approximated the road by lines and circles, found out their lengths on the map and calculated the actual lengths using the map's reduced scale After calculating the total length, the teacher gave them the information he had obtained at the expressway office. In this lesson, it can be seen as a flow of A→B→C→D to approximate the road by lines and circles and calculate the total length using this mathematical model. Presenting the length that the expressway office told the teacher may correspond to a realistic manipulation in Figure 1, although the actual measurement was implemented by the office itself.

According to Kashihara, the students commented as follows: "I was surprised because we could measure that distance using length of arcs we studied in mathematics", "It is great to resolve such situations by mathematical formulae, even if the road has winding curves", "I didn't think that we could calculate things including such curves, but I noticed that we can resolve various situations using calculation formulae we have studied" (p. 109; *my translation*). This implies that the students felt a gap between the complicated shape of the road and an application of mathematics they knew and that they were surprised to know that the length of winding road could be translated into mathematics (i.e. the gap could be bridged) In this lesson, the students found the gap between states A and B at first, and then the gap was bridged by the fit of the two flows A→B→C→D and A→D.

The students in Osawa's (1996) lessons (which I mentioned in the introduction to this article) wrote similar comments: "At first, I doubted whether mathematics could deal with issues in sports. But I was very surprised because baton passes could be done very smoothly and the time got shorter than before." This also implies the gap at the transition from state A to state B. Osawa used the 'baton-pass problem' of a relay (Matsumiya and Yanagimoto, 1995; Yanagimoto *et al*, 1993) in his ninth-grade lessons.

First, the students gathered the data of lap times of each student After expressing the preceding runner's motion as a linear function and the subsequent runner's motion as a quadratic function, each student found out his/her own 'mark-point' through mathematical operations (to identify the y-intercept of a linear function so that it can be tangent to a quadratic function, using graphic calculators). Here, a 'mark-point' is a mark on the track and, when the preceding runner reaches it, the subsequent runner starts Finally, the students actually ran a class relay referring to their own mark-points and experienced that the total time could get shorter. As the above comment shows, this experience seems to validate applying mathematics to such situations. In other words, the gap between A and B was bridged by the combination of new information obtained through B→C with the realistic treatment A→D It generated a kind of surprise, which in turn impressed them with the "usefulness of mathematics" (p. 252).

Deguchi (1997) implemented the lessons titled 'coins and statistics' in eighth-grade classes. After discussing the production numbers of each type of coins, they focused on 1-yen coin (the lowest value coin in Japan) and examined the variation in its production numbers during the last forty years using samples the students brought. First, each student examined it using twenty coins he/she had brought. Then, each group gathered the data of each student to make a graph for one hundred coins Finally, the class gathered the data of each group to make a graph for all the coins (about five hundred coins) and inferred the variation in production numbers from that graph

After that, the teacher presented the graph of actual production numbers of 1-yen coins and asked his students to compare it with the graph they had made. They repeated the same work for 10- and 100-yen coins. When finishing the work for 10-yen coins, the teacher introduced the idea of a sampling method and asked in what kinds of situations it could be used. At the end of the lesson, the teacher explained the variation they had found by relating it to social phenomena (e g appearance of supermarkets, introduction of the sales tax).

Some students commented after the work for 1-yen coins: "It is great to be able to find out approximate numbers using only 500 coins", "I didn't think we could find it because there is a big difference between hundreds of millions and 500. So, I was surprised that our own result was not so different from the graph" (pp. 187-188; *my translation*). Since the sampling method is a kind of mathematical operation, these comments imply that there was a gap at the transition from A to B In other words, the students thought that small samples could not show the whole picture The students became surprised when this gap was bridged, i e. small samples could show us rough sketches of the whole picture. Knowing the actual production numbers directly (e g. asking experts or reading reports) can be considered a kind of realistic manipulations in this case. The validation of the transition from A to B was supported by the fit of the information obtained through B→C with the information through A→D Bridging the gap elicited the students' surprise, and the surprise supported the value of mathematics

4. Surprises caused by gaps between B and C

This type of gap can be interpreted as a feeling that a certain operation in the mathematical world seems impossible. Such gap is observed in Ikeura's (1996) lessons.

She reported on fifth-grade lessons about areas of circles. In talking about the formulae for areas they had learned, the teacher asked them whether they could make a formula for a circle's area Since applying formulae to find out the areas is more mathematical than more direct methods, like counting unit squares (Nunokawa, 1998), the teacher's question can be considered to be one about the transition B→C (B: trying to calculate a circle's area using only the given information, C: finding the area as a result of calculation). It asked about the possibility of mathematical operations for finding out a circle's area (e.g. transforming a circle into another shape whose area can be calculated using formulae), not the value of the area itself

She reminded the students of how the other formulae had been derived, and asked whether a circle could be transformed into other figures The students reacted to this with comments like, "we cannot do it with circles", "we can't, because it has a curve". Ikeura (1996) wrote that the students' resistance was greater than she had expected (p. 8) This implies that there was a gap between states B and C When the class found the formula for the areas of circles, the students clapped their hands This is the point when the gap was bridged

The formula for the areas of circles was explained by cutting a circle along radii into sixteen equivalent sectors and transforming it into a parallelogram. This transformation is a kind of proof of that formula and can be seen as a part of the flow B→C. But the teacher did not introduce this idea from the outset. Through discussion with the students, it was noticed that approximate areas could be found using polygons, i e adding up areas of small triangles. Then, they calculated the area of a circle with a 5cm radius using this idea. In this work, they also noticed that, as the number of vertices of a polygon increased, the value of the area was refined One student used a 360-gon based on this idea

At this point, the teacher introduced the idea of transforming a circle into a parallelogram Finally, the students appreciated the formula when they noticed that it was essentially the same as the expression based on a 360-gon And this comparison was triggered by the fact that the value obtained by the equivalent transformation coincided with the value calculated from a 360-gon.

Finding values of small triangles' areas and adding them up is more realistic than finding a formula, in the sense that it is a numerical approach to a specific circle So, this comparison was also supported by information in the real world. The mathematical formula was established moving back and forth between the mathematical and real worlds. This is consistent with the model presented by Schroeder and Lester (1989). Through such comparison, the gap between B and C was bridged

Nakagawa (1997) reported on fourth-grade lessons about areas, where he spent time having his students think about various shapes with an area of $2cm^2$. One student thought that there had to be a square with the area of $2cm^2$, because such a square had to appear during squashing a (2cm) x (1cm) rectangle into a (1cm) x (2cm) one continually He divided a (2cm) x (1cm) rectangle into smaller squares and rectangles, and rearranged them to look for a $2cm^2$ square. He also tried to find out a number whose square could become 2, by calculating 1.0 x 1.0, 1.5 x 1.5, and 1.2 x 1.2. Failing in these attempts, a student wrote, "I thought that I could find it, but I couldn't" (p. 169; *my translation*).

Although his images, which were more realistic, supported the existence of a $2cm^2$ square, he could not find an equivalent transformation (i.e. permitted mathematical operations in this lesson) that would make a $2cm^2$ square, nor could he find a number whose square would be 2 It can be said that the gap for this student occurred in the flow of B→C. When his classmate showed a 2 cm^2 square afterwards, that student listened to the presentation with his body thrust forward: that is, he was surprised at the idea that made it possible to bridge this gap.

5. Some features common to the examples

First, teachers planned their lessons carefully so that students could feel surprises easily. In the lessons of 'Hatome-Kaeshi' and 'Janken', the manipulations in the real world were simple but they generated unexpected consequences. If the manipulations of a "Hatome-Kaeshi" had been rather complex, the students might not have been so surprised to see that it became a very regular shape. In the lessons about skyscrapers, a building in their city was adopted. This is important, not only because it was familiar to the students, but also because the students could have their own expectations about how far they could see from that building. The gap between these expectations and the mathematical result elicited the students' surprise.

In the lesson on fractions, Tsubota (1996) asked in advance for a division whose answer was a natural number, which was consistent with the students' expectations. The main problem was then presented by means of a minor change to the initial problem. In the examples of toilet paper rolls, the expressway and the baton pass, the treated phenomena were so complex that the student might doubt that mathematics could even be used to deal with such phenomena. In addition to this, although the tasks can be approached in more direct ways (e g. unravel a whole toilet paper roll and count the number of turns), it would be troublesome for students to implement those approaches.

In the lessons on coins, the coins were taken to the class by the students, not by the teacher It can be said that the fact that the teacher could not be controlling the coins strengthened the students' feeling of surprise Ikeura (1996) asked her students not for the area of the circle, but for a mathematical formula Nakagawa (1997) used the area '$2cm^2$', which is not a square number. Such tasks may facilitate the occurrence of gaps and the fact of those gaps or bridging those gaps in the mathematical world surprised the students

These lessons seem to use the students' expectations for eliciting their surprise, perhaps because surprises occur when facts do not fit with their expectations. Adopting simple manipulations in the real world, selecting complex real phenomena, using materials students provide and asking questions consistent with their expectations - these can be seen as ideas for intensifying students' expectations. Osawa (1998) allowed his students to model a situation with a linear function, although it can more appropriately be expressed with a quadratic function. This linear model was more consistent with the students' expectations about the situation. The gap between the linear model and the real phenomena forced the students to re-examine their model. His lessons used the students' expectations explicitly.

Second, combinations of the real and mathematical worlds were used in the above lessons All the arrows of Figure 1 appeared in each lesson Moreover, manipulations in the real world played an important role in validating the mathematical processing, even after it was explained and implemented in the mathematical world This applies to all examples in Section 3. A diagram presenting realistic manipulations was used in the lesson about fractions in Section 2, after the students noticed the meaning of division during their discussion. The students checked whether their new 'Hatome-Kaeshi' really became the intended shapes by turning its pieces, after they had made that 'Hatome-Kaeshi' based on the mathematical mechanism they found. The students wanted to go to the building and check whether he could see the area they had determined mathematically.

6. Concluding remarks

In this article, some Japanese lessons were analyzed using the model shown in Figure 1, focusing on surprises and gaps students may feel. The following four types of lessons were found:

(a) unexpected consequences in the real world are explained with mathematics (section 1);

(b) a gap appears between mathematical results and expectations in the real world, which triggers a re-examination of mathematical operations or expectations in the real world (section 2);

(c) mathematics is used in inferring something about the situation where mathematics does not seem to be applicable, and mathematical inferences are confirmed through manipulations in the real world (section 3);

(d) mathematical operations, which seem impossible at first or which cannot be realized despite it seeming possible, are considered and then are developed referring to manipulations in the real world.

	Gap between	Bridging gap	Surprise	Used mathematics	Implication
(a)	A and D: in a manipulation in the real world	by means of mathematical explanations of mechanisms	generated by the gap (unexpected phenomena in real world)	familiar to students	power of mathematics for explaining some phenomena
(b)	C and D: mathematical results and expectations in the real world	by means of mathematical explanations of mechanisms	by means of re-examining real-world expectations or mathematical knowledge	familiar to students	power of mathematics to generate new information or new meaning of mathematical knowledge
(c)	A and B: possibility of applying mathematics to situations	by means of real-world phenomena which support the mathematical results	generated by bridging the gap (unexpected coincidence of two types of results)	unfamiliar to students in some cases familiar in other cases	power of mathematics for solving realistic problems
(d)	B and C: in an operation in the mathematical world	by realizing a mathematical operation referring to the real world	generated by bridging the gap (unexpected operation in the mathematical world)	new to students	appreciation of new mathematical knowledge

Figure 4 The four types identified in the lessons

Stigler and Hiebert (1999) claim that most Japanese lessons they observed tell a consistent single story (p. 64). The analysis in this article suggests that, in some lessons, teachers designed such stories in advance so that the stories can attract students' attention, elicit their interests and illuminate the value of mathematical ideas.

The author does not insist that all mathematics lessons should include such surprise However, since there are successful lessons where student surprise plays an important role and students can express their interest, it can be one aspect in planning mathematical lessons to consider whether surprise or gaps can be helpful in the lessons The analysis in this article implies that, in order to use 'surprises' in mathematics lessons, teachers need to plan a lesson so that a certain gap will happen

As shown in Figure 4, the surprises found in the above lessons are related to the gaps in two ways In some lessons, the surprises are generated by the gaps themselves (e g why did such phenomena happen?) In the other lessons, bridging the gaps generated the surprise (e.g. It is great that such a thing is possible!). Thus, how to relate surprises to expected gaps needs to be considered in planning lessons.

The analysis also shows that generation of gaps is closely related to students' expectations: something contrary to their expectations could well elicit student surprise. This implies that it is important to take into account what kinds of expectations students have, how to raise their expectations and how to come up with unexpected situations.

Finally, the analysis suggests that bridging gaps is important for convincing students and making them appreciate the value of mathematical ideas. The way to bridge gaps also needs to be considered in planning lessons: for example, whether the mathematics treated in a lesson is familiar to students is related to this aspect (see type (a) and (b)). Unfamiliar mathematics may not become a means of explaining something. In such a case, more familiar evidence (e g real data) is necessary for showing the validity of mathematical approaches and results (see type (c)) I hope that mathematics lessons designed in such a manner produce magical moments concerning the power or value of mathematical ideas and result in students being interested in them.

Notes

[1] In explaining this mechanism with algebra, it is more convenient not to worry about the condition that only the last digit is written down. When expressing the first and second number as x and y, the third number can be described as $x + y$, the fourth $x + 2y$, and so on The number in the seventeenth row can be expressed as $610x + 987y$, which implies that the last digit of this number does not depend on x.

[2] Matsumiya and Yanagimoto (1995) report that the teacher showed a videotape to the students (p. 79) But it is not clear that the videotape was of the landscape visible from the building concerned

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