An Idea from Jakob Bernoulli for the Teaching of Algebra: A Challenge for the Interested Pupil*

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1. The effective method of Jakob Bernoulli for power sums: Motivation and pattern

> *What in former epochs occupied the mature minds of men is in later times made accessible to boys*
>
> *Hegel*

In Jacob Bernoulli's famous book on probability—in his *Ars conjectandi* (the art of conjecting) from 1713—we find a readable discussion of the binomial numbers. Derived from this we find formulas for evaluating the power sums—the Bernoulli polynomials. Jakob is obviously not a little proud of this achievement:

> *"Using this table I have, within half a quarter of an hour, found that the 10th powers of the first thousand numbers sum to*
>
> *91 409 924 241 424 243 424 241 924 242 500*
>
> *From this we see how hopeless is the effort which Ismaël Bullialdus has shown in writing his very extensive 'Arithmetica Infinitorum'. He has not achieved much, as he has calculated only the power sums for c = 1 to c = 6, with the greatest effort. This is just a part of what we have reached in one page."*

$$S(n) = \sum_{j=1}^{n} j = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$S(n^2) = \sum_{j=1}^{n} j^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

$$S(n^3) = \sum_{j=1}^{n} j^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2$$

$$S(n^4) = \sum_{j=1}^{n} j^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$$

$$S(n^5) = \sum_{j=1}^{n} j^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

$$S(n^6) = \sum_{j=1}^{n} j^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$$

$$S(n^7) = \sum_{j=1}^{n} j^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}n^2$$

$$S(n^8) = \sum_{j=1}^{n} j^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$$

$$S(n^9) = \sum_{j=1}^{n} j^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4 - \frac{1}{12}n^2$$

$$S(n^{10}) = \sum_{j=1}^{n} j^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - 1\,n^7 + 1\,n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$$

Bernoulli polynomials

Even less sensible than the calculation of the insulted Ismaël Bullialdus—acquainted with Desargues, Mersenne, Roberval—is the blind summation:

$$S(1000^{10}) = 1^{10} + 2^{10} + 3^{10} + \ldots + 1000^{10}$$

The binomial theorem would scarcely reduce the amount of calculation. But the Bernoulli polynomial dramatically reduces the work, and the calculating is fun!

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$$
\begin{aligned}
\frac{1}{11}\ 1000^{11} &= 90\,909\,090\,909\,090\,909\,090\,909\,090\,909\,090\,\tfrac{10}{11}\\
\frac{1}{2}\ 1000^{10} &= 500\,000\,000\,000\,000\,000\,000\,000\,000\,000\\
\frac{5}{6}\ 1000^{9} &= 833\,333\,333\,333\,333\,333\,333\,333\,333\,\tfrac{2}{6}\\
-\ 1\ \ 1000^{7} &= -\ 1\,000\,000\,000\,000\,000\,000\,000\\
+\ 1\ \ 1000^{5} &= 1\,000\,000\,000\,000\,000\\
-\frac{1}{2}\ 1000^{3} &= -\ 500\,000\,000\\
+\frac{5}{66}\ 1000^{1} &= 75\,\tfrac{50}{66}\\
\hline
S\,(1000^{10}) &= 91\,409\,924\,241\,424\,243\,424\,241\,924\,242\,500
\end{aligned}
$$

Also, in the era of the computer — as in Bernoulli's time — the search for such effective rules (algorithms) for solving problems already posed is a main occupation of mathematicians.

In school one can find many occasions to study an algorithm with respect to its effectiveness We have tried to find certain recipes to make concious step-by-step improvements. One example is the teaching of the so-called conjugate theorem. With a sufficiently extensive table of squares (like that of Crelle), you can solve multiplication tasks easily by subtraction:

$$(n+m)(n-m) = n^2 - m^2$$

which can be written more directly

$$a \cdot b = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2$$

2. Remarks on the teaching proposals for grade 10 (or 9)

The attempt to teach by examples claims much preparation and care. But it preserves the pupils and the teacher from the worst that can happen the deadly slumber of the eternal repetition of a canalized curriculum

Wagenschein

Jakob Bernoulli was interested in sums obtained from the *Pascal triangle* We start by taking certain sums out of the simple *multiplication table* as an opportunity to ask for fast calculating methods. One of our results will be for instance, that the product

$$\left[\frac{n(n+1)}{2}\right]^2$$

is fairly good for the value of $\sum_{i\ j=1}^{n} i \cdot j$

On the way to this, pupils knowing the usual algebraic techniques should be completely armed, so they may also use the Bernoulli polynomials (in the beginning without proofs). And they will use this algebraic means so frequently that this teaching proposal could be titled, "A family of problems for the general repetition of basic algebraic techniques".

This teaching proposal (besides leading to other facilities) could also serve as a preparation for "harder matters" (calculus, linear algebra, probability) in the higher grades. For pupils coming from different classes, put together for the purpose of equalizing individual differences in knowledge and skill, and with the goal of deepening and widening the basis for the "harder matters" to come, it could serve as an "exemplary" approach, in the Wagenschein meaning.

As advantages of this teaching proposal we can count

- the unity and strength of the Bernoulli method of solution;
- the unity of the content (the multiplication table carries a coherent system of problems so to speak within itself);
- the importance of the results for the later integral calculus;
- the possible adaption to very different levels

The proposal, sketched in sections 3-7, is made for usual class teaching. Therefore we avoid there the sum (Σ) signs, so as not to give the quick reader an impression of the elegance of the formulas, thinking that this is "higher mathematics" which has no place in secondary school. For a working community of interested pupils, however, we would not hold back such elegance. In a free working group, released from pressure, the problem field can be so widely exposed, that the connection between the higher arithmetical sequences and the Bernoulli polynomials and the Pascal numbers appears. "The wonderful properties of the figurate numbers", traced by Jakob Bernoulli, where "eminent secrets from the whole of mathematics are hidden ." may thus be obtained by interested pupils.

In sections 9 and 10 we sketch a teaching procedure for such a group The "normal" teaching procedure has a weakness which we can nevertheless appreciate. It does not lift away from the average pupil the burden of indulging in intensive mathematical thinking. Though much effort is made to facilitate the creative exploration of the material through appropriate exposition and illumination, the burden is still there. But our suggestions are not for specially gifted pupils only.

Many schools have programmable calculators or computers at their disposal. This proposal should actually offer material for clever uses in this connection. Here we leave the necessary didactical considerations to the teacher, fearing that otherwise our notice would take the baroque form of a Bruckner symphony, which can only open itself after repeated listening when we have learnt to withstand its heavenly length.

3. The pattern example LEDIA as a starting point

Each problem that I solved became a rule which served afterwards to solve other problems

Descartes

The multiplication table offers many kinds of summation problems. We start by calculating the sums along the *le*ft-running *dia*gonals.

·	1	2	3	4	5	6	. . .
1	1	2	3	4	5	6	. . .
2	2	4	6	8	10	12	. . .
3	3	6	9	12	15	18	. . .
4	4	8	12	16	20	24	. . .
5	5	10	15	20	25	30	. . .
.	.	.	.	.	.	.	

$$\text{LEDIA } 5 = 1\cdot 5 + 2\cdot 4 + 3\cdot 3 + 4\cdot 2 + 5\cdot 1$$

We would probably not calculate

$$\text{LEDIA } 10000 = 1\cdot 10000 + 2\cdot 9999 + 3\cdot 9998 + \ldots + 10000\cdot 1$$

with paper and pencil, but with a programmable calculator or computer. In calculating LEDIA 10^6, will the calculator boil over?

What do we have to tell it when handling such big numbers? For the program we must express the "structure" in a much better way: with variables.

With a little intuition and with routines from elementary algebra, we can arrange this in a much simpler way.

$$\text{(1) LEDIA } n = 1\cdot n + 2\cdot(n-1) + 3\cdot(n-2) + \ldots + (n-1)\cdot 2 + n\cdot 1$$
$$= (1+2+3+\ldots+n)\cdot n - (1\cdot 2+2\cdot 3+\ldots+(n-1)\cdot n)$$

At both ends of the expression the reformulating was blocked up for a while. With the summation sign we would not have observed this.

The first part will be replaced with a simple formula by using the Bernoulli polynomial. In the second part, it comes out that the summands are nearly square numbers

$$k(k+1) = k^2 + k$$

Then we have

$$\text{LEDIA } n = \frac{1}{2}n^3 + \frac{1}{2}n^2 - (1^1+2^2+\ldots+(n-1)^2) - (1+2+\ldots+(n-1))$$
$$= \frac{1}{2}n^3 + \frac{3}{2}n^2 + n - (1^1+2^2+\ldots+n^2) - (1+2+\ldots+n)$$
$$= \frac{1}{2}n^3 + \frac{3}{2}n^2 + n - \frac{1}{3}n^3 - \frac{1}{2}n^2 - \frac{1}{6}n - \frac{1}{2}n^2 - \frac{1}{2}n$$

$$\text{(2) LEDIA } n = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$$

Now the calculating of LEDIA 10^6 has become child's play, and we have, like Jakob Bernoulli, fun by sensible calculating.

$$\text{LEDIA } 10^6 = \frac{1}{6}(10^{18} + 3\cdot 10^{12} + 2\cdot 10^6)$$
$$= 1\,000\,003\,000\,002\,000\,000 : 6$$
$$= 166\,667\,166\,667\,000\,000$$

When n is not a power of 10, we would prefer the factorized expression

$$\text{(2a) LEDIA } n = \frac{1}{6}n(n+1)(n+2)$$

This is easily obtained by a quadratic completion:

$$n^2 + 3n + \frac{9}{4} - \frac{9}{4} + 2 = (n + \frac{3}{2} + \frac{1}{2})(n + \frac{3}{2} - \frac{1}{2})$$

We should like to state an intermediate result here:

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + (n-1)\cdot n = (1+2+\ldots+n) - \text{LEDIA } n$$

$$\text{(3) } 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + (n-1)\cdot n = \frac{1}{3}n^3 - \frac{1}{3}n$$

or

$$\text{(3a)} \qquad = \frac{1}{3}(n-1)n(n+1)$$

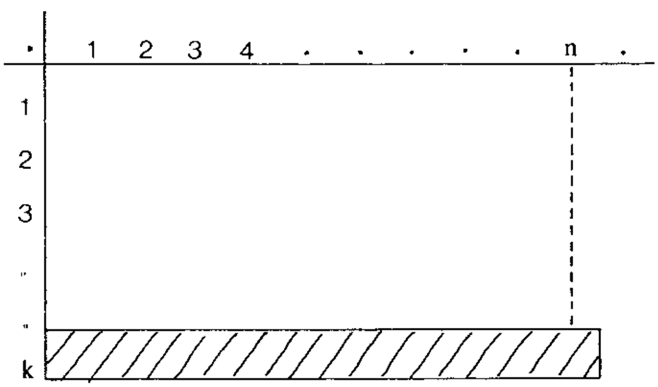
4 More sums from the multiplication table

It is not the knowing but the learning that gives the greatest pleasure.

Gauss

In complete analogy with our pattern example we consider

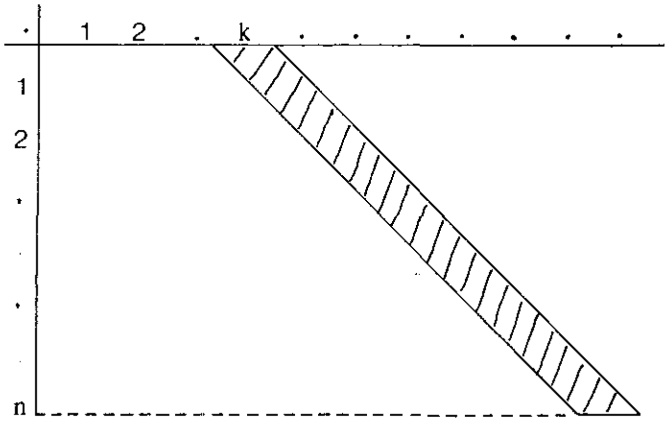
4.1. *Line sums*



$$\text{(4) } k \text{ LIN } n = k\cdot 1 + k\cdot 2 + k\cdot 3 + \ldots + k\cdot n \quad \text{(def)}$$

$$\text{(5) } k \text{ LIN } n = \frac{1}{2}kn(n+1)$$

4.2. *Right-running diagonal sums*



(6) k RIDIA $n = 1 \cdot k + 2(k+1) + 3(k+2) + \ldots + n(k+n-1)$ (def)

$$= \frac{1}{2}n(n+1)k + (1\cdot 2 + 2\cdot 3 + \ldots + (n-1)n)$$

and with (3a) this gives

(6a) k RIDIA $n = \frac{1}{6}n(n+1)(3k+2n-2)$

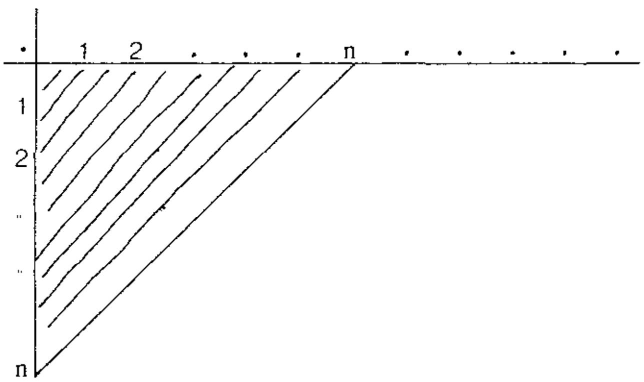
Especially we have of course

(6b) 1 RIDIA $n = 1^2 + 2^2 + 3^2 + \ldots + n^2$

$$= \frac{1}{6}n(n+1)(2n+1)$$

(We observe, that by this deduction of (6b), the snake swallows its own tail!)

4.3. *Triagonal sums.*



(7) TRI n = LEDIA 1 + LEDIA 2 + ... + LEDIA n (def)

$$= (\frac{1}{6}\cdot 1^3 + \frac{1}{2}\cdot 1^2 + \frac{1}{3}\cdot 1) + (\frac{1}{6}\cdot 2^3 + \frac{1}{2}\cdot 2^2 + \frac{1}{3}\cdot 2) + \ldots + (\frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n)$$

$$= \frac{1}{6}(1^3 + 2^3 + \ldots + n^3) + \frac{1}{2}(1^2 + 2^2 + \ldots + n^2) + \frac{1}{3}(1 + 2 + n)$$

With the help of the actual Bernoulli polynomials it follows

(7a) TRI $n = \frac{1}{24}n^4 + \frac{1}{4}n^3 + \frac{11}{24}n^2 + \frac{1}{4}n$

The factorization of the polynomial is done by the ordinary technique. One integral zero point is quickly obtained: -1. Then instead of using a method of division (no longer taught?), a display of unknown coefficients can be made

$$n^3 + 6n^2 + 11n + 6 = (n^2 + \alpha n + \beta)(n + 1)$$
$$= n^3 + (\alpha + 1)n^2 + (\alpha + \beta)n + \beta$$

By comparing, this makes

$$n^3 + 6n^2 + 11n + 6 = (n^2 + 5n + 6)(n + 1)$$

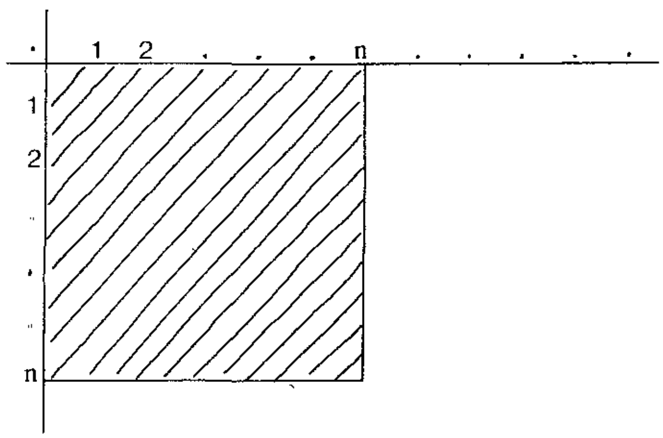
With the same method, or with quadratic completion, it follows at last

(7b) TRI $n = \frac{1}{24}n(n + 1)(n + 2)(n + 3)$

5. The Bernoulli polynomial for cubes falls into our hands

I love mathematics, not only because of its usefulness in technology, but also because it is beautiful
Rozsa

We also calculate the sums of all the numbers in a square of the multiplication table. This gives a little surprise



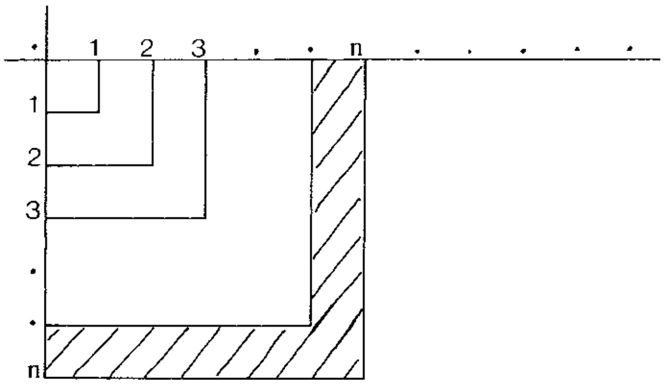
While we have short expressions for the line sums, we calculate

(8) SQUA $n = 1\cdot$LIN $n + 2\cdot$LIN $n + \ldots + n\cdot$LIN n (def)

$$= (1 + 2 + \ldots + n)\cdot\frac{1}{2}n(n + 1)$$

(8a) SQUA $n = [\frac{1}{2}n(n + 1)]^2$

The square, however, can easily be split up into its marginals ("gnomons").



(9) $$\text{MARG}\, n = 1 \cdot n + 2 \cdot n + \ldots + n \cdot n + n(n-1) + n(n-2) + \ldots + n \cdot 1 \quad \text{def}$$
$$= 2n(1 + 2 + \ldots + n) - n^2$$
$$= n^2(n + 1 - 1)$$

(9a) $$\text{MARG}\, n = n^3$$

The marginal sums are just cubic numbers, and here is the little surprise: with some algebra, nearly without effort, we get the Bernoulli polynomial for the sum of the cubic numbers

(10) $$1^3 + 2^3 + \ldots + n^3 = [\frac{1}{2}n(n+1)]^2 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}$$

6. The Bernoulli polynomials for $\Sigma\, j$ and $\Sigma\, j^2$ respectively

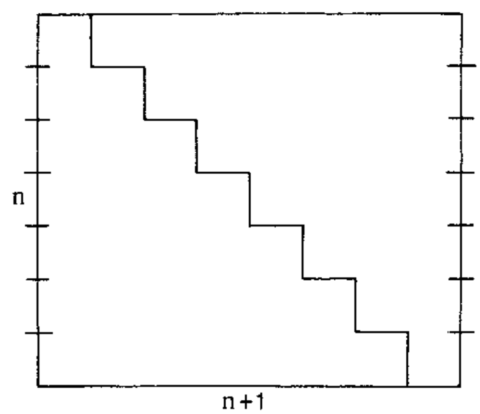
Don't swear by the name of your teacher but bring forth a proof!

Proverb from antiquity

Until now we have tested them with a few numbers, but used them without proving. Jakob Bernoulli has given directions for proof. Now we try to deduce these Bernoulli polynomials

$$\sum_{j=1}^{n} j = 1 + 2 + \ldots + n = \frac{1}{2}n(n+1) = \frac{1}{2}n^2 + \frac{1}{2}n$$

will be understood by the stair picture



The sum of squares is a harder nut than the corresponding sum of cubes. We are not ashamed to confess to the pupils that we don't see any straightforward way. So out of the need we make a virtue!

The pupils should also learn to read written mathematics with understanding, shouldn't they? For this kind of work the teacher should find some selected passages as a reading section, with special emphasis on their being understandable. Or better — several sections on the same theme, calling on different skills.

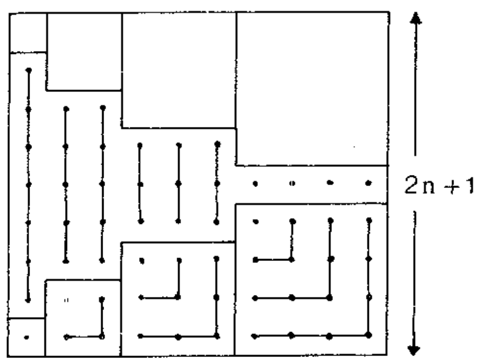
One reading section can, for instance, develop the Bernoulli polynomial from the starting point

$$\sum_{j=1}^{n} (j+1)^3 = \sum_{j=1}^{n} j^3 + 3\sum_{j=1}^{n} j^2 + 3\sum_{j=1}^{n} j + n.$$

Another will bring in the recursive argumentation

$$\sum_{j=1}^{n+1} j^2 = \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$$
$$= \frac{1}{6}(n+1)(2n^2 + 7n + 6)$$
$$= \frac{1}{6}(n+1)(n+2)(2n+3)$$

A third starting point is the picture



$$3 \cdot (1^2 + 2^2 + 3^2 + \ldots + n^2) = \frac{1}{2}n(n+1)(2n+1)$$

7. A reading passage from John Wallis

Even in the mathematical sciences, our principal instruments for discovering the truth are induction and analogy

Laplace

A fourth reading passage on this same theme (simple Bernoulli polynomials) emphasizes less the finite power sums, but more the connecting limit values. We will guide our *ordinary pupils* some steps along the way of John Wallis (1616-1713). On the way they discover through well-planned numerical calculation some patterns: the simple Bernoulli polynomials, and quite easily still more: the usable approximation

$$\sum_{j=1}^{n} j^k \approx \frac{1}{k+1}n^{k+1} + \frac{1}{2}n^k$$

We let our *ordinary pupils* read a page from John Wallis' *Arithmetica infinitorum* [1665]. It may be found in Struik's source book (See Appendix). This text gives enough material for a conversation among other things about the circumstances of the sums of cubes (Prop.40). Would not our pupils — accustomed to formula language (?!) — easily note

$$\frac{0^3 + 1^3 + 2^3 + \ldots + n^3}{n^3 + n^3 + n^3 + \ldots + n^3} = \frac{1}{4} + \frac{1}{4n}$$

$$1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4n}(n+1)(n+1)n^3$$

(10a) $1^3 + 2^3 + \ldots + n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \ldots$ (lower powers of n)

Do they really understand the Wallis text? Given the task of handling the sums $1+2+\ldots+n$ and $1^2+2^2+\ldots+n^2$ respectively according to his pattern we put them on trial.

From the results

$$\frac{0 + 1 + 2 + \ldots + n}{n + n + n + \ldots + n} = \frac{1}{2} \quad ; \sum_{j=1}^{n} j = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$\frac{0^2 + 1^2 + 2^2 + \ldots + n^2}{n^2 + n^2 + n^2 + \ldots + n^2} = \frac{1}{3} + \frac{1}{6n} ;$$

$$\sum_{j=1}^{n} j^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \ldots \text{ (lower powers of } n)$$

our ordinary pupils rise with Wallis even higher

$$\sum_{k=1}^{n} j^k = \frac{1}{k+1} n^{k+1} + \frac{1}{2}n^k + \ldots \text{ (lower powers of } n)$$

This gives useful approximations for power sums, important results for the coming integral calculus

Using this result for the sum

$$1^{10} + 2^{10} + \ldots + 1000^{10}$$

we need, instead of Jakob Bernoulli's half a quarter of an hour, only a fraction, and we get

$$\sum_{j=1}^{1000} j^{10} \approx \frac{1}{11} \cdot 10^{33} + \frac{1}{2} \cdot 10^{30} = \frac{2011}{22000} \cdot 10^{33}$$

= 91 409 100 000 000 000 000 000 000 000 000 correct to 5 digits

This approach to approximation, emphasizing limits of sequences, without doubt will prepare the ground for the calculus course.

8. A historical remark

> *Archimedes will be remembered when Aeschylus is forgotten because languages die and mathematical ideas do not*
>
> *G.H.Hardy*

Trying to discover the truth with the help of induction and analogy, you need—by small conjectures and harder patterns as well—a good intuition, a good nose. Wallis in his *Arithmetica infinitorum* goes quickly from the limit value just mentioned to the utmost generality of the Cavalieri result

$$\int_0^1 x^k\,dx = \frac{1}{k+1} \quad \text{for all } real \text{ numbers } k \neq 1$$

"Thereupon, he plunged into a maelstrom of numerical work, and with fine mathematical intuition to guide him in his interpolations, arrived at the infinite product for π that bears his name" (Struik):

$$\frac{3\cdot3\cdot5\cdot5\cdot7\cdot\ldots\cdot(2n+1)(2n+1)}{2\cdot4\cdot4\cdot6\cdot6\cdot\ldots\cdot2n\cdot(2n+2)} \cdot \sqrt{1 + \frac{1}{2n+1}} < \frac{4}{\pi}$$

$$< \frac{3\cdot3\cdot5\cdot5\cdot7\cdot\ldots\cdot(2n+1)(2n+1)}{2\cdot4\cdot4\cdot6\cdot6\cdot\ldots\cdot2n\cdot(2n+2)} \sqrt{1 + \frac{1}{n+2}}$$

What we now try to sketch with a few strokes is a paradigm showing that induction is an exceptional tool for discovering relationships.

Wallis knew $\int_0^1 (x - x^2)^{1/2}dx = \frac{\pi}{8}$, from the area of the circle.

His nose immediately picked up the scent:

$$\int_0^1 (x - x^2)^n dx = ?$$

Calculate, for $n = 0, 1, 2, 3, \ldots$ (?). Generalize.

$$\int_0^1 (x - x^2)^n dx = \frac{(n!)^2}{(2n)!(2n+1)}$$

With a mental jump

$$\frac{\pi}{8} = \frac{(\frac{1}{2}!)^2}{(2\cdot\frac{1}{2})!(2\cdot\frac{1}{2}+1)}$$

To understand $n!$ as $1\cdot2\cdot3\cdot\ldots\cdot n$ is a too narrow setting. Widen it! But how??? With a "free" variable k, which in turn will be killed by a limit process

$$n! = \frac{k!\,k^n\,(1+\frac{1}{k})\cdot(1+\frac{2}{k})\cdot\ldots\cdot(1+\frac{n}{k})}{(1+n)(2+n)\ldots(k+n)}$$

$$n! = \lim_{k\to\infty} \frac{k!\,k^n}{(1+n)(2+n)\ldots(k+n)}$$

The rest is not worth mentioning; calculate it!

The French, especially Fermat, criticized Wallis. They claimed an unfailing proof of this result which was so easily obtained. They claimed a complete mathematical induction (or a variant of this). Jakob Bernoulli worked through Wallis' *Arithmetica infinitorum* and gave such a proof of the power sums.

The use of mathematical induction in a simple case can be understood by *normal pupils* too, as long as we avoid its confusing logical subtleties

9. A steeper path for a working community of interested pupils

> *A great discovery solves a great problem but there is a grain of discovery in the solution of any problem. Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery.*
> *Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime*
>
> *Polya*

The teacher will not let "the young mathematicians" fade out but invites them to more demanding work on the same theme. First let them study a text considering Pascal numbers and recursive methods. In German this can be found in the textbook Stowasser/Mohry *Rekursive Verfahren*, Schroedel Verlag 1978, a book which has the stimulation of interest as its aim

Afterwards the pupils will be able to use recursive methods in simple cases, know the combinatorial meaning of the Pascal numbers and more than its two basic properties

(11) $\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$

(12) $\binom{n}{m} + \binom{n}{m+1} = \binom{n+1}{m+1}$

Our interested pupils will, after some calculating, find themselves discovering that certain sums in the multiplication table are connected with the Pascal numbers. The steep path goes by some trials to the conjecture

$$\sum_{i+j=n+1} i\cdot j = \binom{n+2}{3}$$

The recursive thinking must be supported by examples. The heart of the matter can be made clear without variables

$$\text{LEDIA } 7 = \sum_{i+j=7}^{n} i\cdot j = 1\cdot 6 + 2\cdot 5 + 3\cdot 4 + 4\cdot 3 + 5\cdot 2 + 6\cdot 1$$
$$= (6+5+4+3+2+1) + (1\cdot 5 + 2\cdot 4 + 3\cdot 3 + 4\cdot 2 + 5\cdot 1)$$
$$= \binom{7}{2} + \binom{7}{3} \text{ (induction hypothesis)}$$
$$= \binom{8}{3}$$

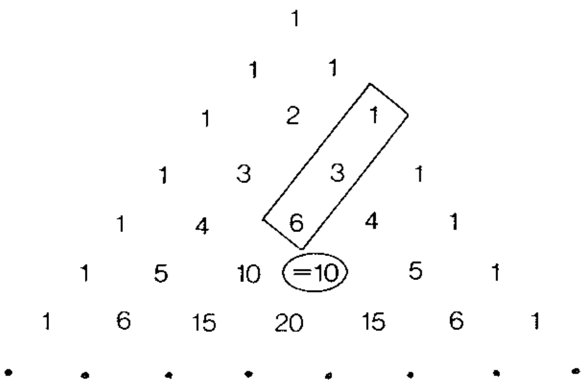
Here we have taken one step upwards. Further steps gives

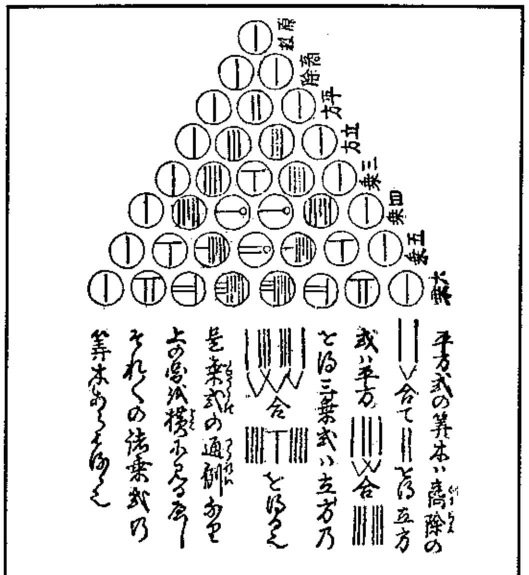
$$\binom{8}{3} = \binom{7}{2} + \binom{6}{2} + \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}$$

This will soon tempt us to generalize

(13) $\sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1}$

This is however the *main point in Jakob Bernoulli's deduction of the polynomials for power sums and also for the higher arithmetical sequences in general.* Bernoulli's main theorem follows immediately from (13) and vice versa. The relation of (13) to the Pascal triangle is illustrated by the following example.





Pascal triangle in Japan

From Murai Chūtzen's Sampō Dōshi-mon (1781), which also shows the Sangi forms of the digits

The recursive proof of (13) is trivial. We have

$$\sum_{j=k}^{n} \binom{j}{k} = \binom{n+1}{k+1} + \binom{n+1}{k}$$

according to the induction hypothesis

$$= \binom{n+2}{k+1}$$

according to (12).

The main point (13) was also stated by Blaise Pascal in his famous paper "Traité du triangle arithmétique...", Paris 1665, in an elegant way. Therefore we are surprised that Jakob Bernoulli had so much trouble to prove his main theorem: 5 lemmas, 4 pages.

Just like Bernoulli we fetch the Bernoulli polynomials from (13):

k = 1:

$$\binom{n+1}{2} = \sum_{j=1}^{n} \binom{j}{1} = \sum_{j=1}^{n} j$$

k = 2:

$$\binom{n+1}{2} = \sum_{j=1}^{n} \binom{j}{2} = \frac{1}{2}\sum_{1=1}^{n} j\,(j-1) = \frac{1}{2}\sum_{j=1}^{n} j^2 - \frac{1}{2}\sum_{j=1}^{n} j$$

From this it follows

$$\sum_{j=1}^{n} j^2 = \frac{2}{6}(n+1)\,n\,(n-1) - \frac{1}{2}(n+1)\,n$$
$$= \frac{1}{3}\binom{n+1}{2}(2n+1)$$

In the same way it follows for k=3, from

$$\binom{n+1}{4} = \sum_{j=1}^{n} \binom{j}{3}$$

$$\sum_{j=1}^{n} j^3 = \binom{n+1}{2}^2$$

etc.

Armed with the summation sign, the further sums in the multiplication table should offer no resistance to our young mathematicians.

"One has to care that the symbols are well suited to discoveries. Usually this is obtained when the symbols express some elements of the intrinsic nature of the concept – *then the mental work is reduced in a remarkable way."* [*Newton, quoted in Wussing, Isaac Newton*, Leipzig 1977)

The obstacles met in converting, for example, LEDIA *n* (Section 3) will be overcome easily.

$$\sum_{i+j=n+1} i \cdot j = \binom{n+2}{3}$$

$$\sum_{i+1\leqq n+1} i\;j = \binom{n+3}{4} = \sum_{k=1}^{n}\sum_{j=1}^{n} j\,(k-j+1)$$

$$\sum_{i,j\leqq n} i\;j = \binom{n+1}{2}^2 = \sum_{k=1}^{n}\sum_{j=1}^{n} k\;j = \sum_{j=1}^{n} j^3$$

etc.

To our young mathematicians we offer the "quadratic multiplication table" for a similar exploration.

·	1	4	9	16	.	.	.
1	1	4	9	16	.	.	.
4	4	16	36	64	.	.	.
9	9	36	81	144	.	.	.
16	16	64	144	256	.	.	.
.	.	.	.	.			
.	.	.	.	.			

The comparison of these two tables can raise interesting problems, as for example: Under what conditions will LEDIA *n* divide the corresponding diagonal sum in the "quadratic multiplication table"?

Afterwards the original text by Jakob Bernoulli from 1713, the chapter on Pascal numbers, should be excellent reading for our young mathematicians. Hopefully both the work and the quotations have highly motivated them. It is rewarding to look into the workshop of a master to find encouragement and ideas for further work.

To end, Jakob Bernoulli's beautiful results of the sums of higher arithmetical sequences, the ideas in a system of exercises on figurate numbers, may be appropriate. The chapter "Figurierte Zahlen" in the already mentioned book *Rekursive Verfahren* gives such a system.

The journal *Mathematiklehrer* (1/82) gave under the heading "Forschungsaufgaben" some starting paths into a great mountain landscape of problems: figurate numbers, Pascal and Leibniz numbers, Pell triangle, etc. Readers were challenged to journey into this landscape. This article can be seen as such a trip, trying to take the objective of "multirelated mathematics" (Freudenthal) seriously. We hope that readers will put the content of the article on trial, asking "Does it show some interrelationships between the different parts of mathematics?"

Pascal triangle, as it was first printed in 1527

Heading page of Arithmetik by Petrus Apianus. Ingolstadt, 1527. that more than a century before Pascal explored the properties of the triangle

APPENDIX

A reading passage from: J. WALLIS *Arithmetica infinitorum*, Oxford 1655 (quoted in: *A source book in mathematics*, 1200-1800, edited by D. J. Struik, Harvard Univ. Press, Mass. 1969)

Proposition 39 [1] Given a series of quantities that are the cubes of a series of numbers continuously increasing in arithmetic proportion (like the series of cubic numbers) which begin from a point or zero (say 0. 1 8, 27. 64. . .); we ask for the ratio of this series to the series of just as many numbers equal to the highest number of the first series.

The investigation is carried out by the inductive method as before We have

$$\frac{0+1=1}{1+1=2}=\frac{2}{4}=\frac{1}{4}+\frac{1}{4};$$

$$\frac{0+1+8=9}{8+8+8=24}=\frac{3}{8}=\frac{1}{4}+\frac{1}{8};$$

$$\frac{0+1+8+27=36}{27+27+27+27=108}=\frac{4}{12}=\frac{1}{4}+\frac{1}{12};$$

$$\frac{0+1+8+27+64=100}{64+64+64+64+64=320}=\frac{5}{16}=\frac{1}{4}+\frac{1}{16};$$

$$\frac{0+1+\cdots+125=225}{125+\cdots+125=750}=\frac{6}{20}=\frac{1}{4}+\frac{1}{20};$$

$$\frac{0+\cdots+125+216=441}{216+\cdots+216=1512}=\frac{7}{24}=\frac{1}{4}+\frac{1}{24};$$

and so forth

The ratio obtained is always greater than one-fourth or $\frac{1}{4}$ But the excess decreases constantly as the number of terms increases; it is $\frac{1}{4}, \frac{1}{8}, \frac{1}{12}, \frac{1}{16}, \frac{1}{20}, \frac{1}{24}$, There is no doubt that the denominator of the fraction increases with every consecutive ratio by a multiple of 4, so that the excess of the resulting ratio over $\frac{1}{4}$ is the same as 1 :4 times the number of terms after 0 etc.

Proposition 40 *Theorem* Given a series of quantities that are the cubes of a series of numbers continuously increasing in arithmetic proportion beginning, for instance with 0, then the ratio of this series to the series of just as many numbers equal to the highest number of the first series will be greater than $\frac{1}{4}$ The excess will be 1 divided by four times the number of terms after 0 or the cube root of the first term after 0 divided by four times the cube root of the highest term

The sum of the series $0^3 + 1^3 + \cdots + l^3$ is $\frac{l+1}{4}l^3 + \frac{l+1}{4l}l^3$. or, if m is the number of terms $\frac{m}{4}l^3 + \frac{m}{4l}l^3 = \frac{1}{4}ml^3 + \frac{1}{4}ml^2$ This is apparent from the previous reasoning.

If, with increasing number of terms, this excess over $\frac{1}{4}$ diminishes continuously, so that it becomes smaller than any given number (as it clearly does), when it goes to infinity, then it must finally vanish. Therefore:

Proposition 41. *Theorem*. If an infinite series of quantities which are the cubes of a series of continuously increasing numbers in arithmetic progression beginning, say, with 0 is divided by the sum of numbers all equal to the highest and equal in number then we obtain $\frac{1}{4}$ This follows from the preceding reasoning