School Mathematics Theorems—an Endless Source of Surprise*

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The high school mathematics curriculum is basically a collection of concepts and of theorems relating the concepts one to the other. Some of the theorems we prove, then apply to solve problems, others we only demonstrate and then apply, still others we just apply right after their exposition. Sometimes we call the theorems rules, sometimes laws or facts, but all these are mathematically provable statements. In this paper I argue that all school theorems, except possibly a very small number of them, possess a built-in surprise, and that by exploiting this surprise potential their learning can become an exciting experience of intellectual enterprise to the students.

Mathematics at all levels is a boxful of surprises. Numerous books include all kinds of most interesting mathematical developments. [Gardner, 1956; Graham, 1968; Rademacher and Toeplitz, 1970; Steinhaus, 1983; to name just a few] However, school mathematics is largely perceived as a boring subject full of routines and pitfalls; a topic good for those smart ones who enjoy it for some unexplained reasons.

Part of the blame is on textbooks which in many cases fail to transmit anything of interest. Most often they present paragraphs of theory followed by sets of isolated exercises and disconnected problems, rarely incorporating an historical note, a game or a puzzle. Usually, no comment is made about the significance of a theorem or the elegance of a proof, leaving to the reader any kind of such impressions.

The curriculum itself bears also a part of the responsibility. For the majority of the students it is overloaded with algorithms and techniques, and usually it does not leave much room for the more fascinating parts of each topic. In considering what is still fundamental in the mathematics curriculum and what is not, the U.S. Conference Board of the Mathematical Sciences stated that: "Changes in the curriculum at this time could bring a new sense of vitality to K-12 mathematics" [CBMS 1982, p. 1]. The almost total exclusion of intriguing topics such as topology, graph theory, combinatorics, non-Euclidean geometries and even large portions of number theory is another aspect of the same problem. "... A number of topics should be introduced into the secondary curriculum and all of these are more important than, say, what is now taught in trigonometry beyond the definition of the trigonometric functions themselves." [ibid, p. 5]

Another part of the blame for the fact that mathematics becomes for many students the bitter pill to be swallowed for school survival purposes, rests on the teacher's shoulders. A bored tired teacher who does not convey enthusiasm about mathematics cannot hope for a different attitude on the part of the students. "The renewal of mathematics teachers' content knowledge, teaching skill, and enthusiasm for their work is clearly needed at all levels of education." [CBMS 1983]

How can we make school mathematics reflect the excitement embedded in mathematics? After all, we cannot afford the time to present extracurricular ideas, such as Long [1982] and others suggest, no matter how exciting and motivating these ideas truly are. It is the daily task of teaching ordinary school mathematics, its basic concepts and rules, to which this paper is addressed. It intends to analyze the surprise factors in school mathematics theorems, and to demonstrate how a major part of high school mathematics periods can be turned into an exciting experience by causing a surprise, which raises mathematical curiosity and motivates its learning. It takes more effort to plan such lessons. In some cases it is quite a challenge. Here is where teacher educators and curriculum developers enter the scene. Teaching materials which facilitate the employment of the surprise effect, and teacher education programs which encourage the search for lesson-planning towards it, are necessary in order to assist teachers in planning their daily lessons such that one lesson is not just more of the same. On the contrary, each lesson is different. Everyday there is something new to be learned — a better method to solve some problems, a shorter way to get at some results, an unbelievable connection between seemingly independent facts, a surprising exception to a familiar rule, etc.

The nature of surprise in mathematics

A person is surprised when something occurs unexpectedly, when it is in contradiction to expectation. Surely, in life some surprises make us happy, some don't. In the study of mathematics we are dealing with intellectual surprise: that is the discovery of some unforseen truth. It is necessary that the discovery be non-trivial: intuitively, it is not expected to be true, or may perhaps even be expected to be false. I claim that *every mathematics theorem is a statement of some non-trivial discovery*. This is a meta-statement which requires some kind of a proof. As it is judgmental it belongs to mathematics psychology rather than to

* This paper is based on a lecture in memory of Ms. Mary Cooper, the director of the mathematics laboratory at the University of Haifa

For the Learning of Mathematics 8, 3, (November 1988)
FLM Publishing Association, Montreal, Quebec, Canada

mathematical logic. We shall therefore not attempt a deductive proof... Instead we elaborate on its demonstration using a sample of school mathematics theorems. But first we try a general argument by negation.

Consider any mathematics theorem and the mathematician who was the first to declare it. Historically, we may not be able to trace this individual, but surely if there had been nothing new or exciting about his or her theorem, the mathematician who first published it would not have bothered, the mathematical community would not have been interested, and the theorem would not have found its way into the syllabuses of common mathematics courses around the world.

Before we turn to the demonstration of the surprise embedded in common school theorems, let us make one point clear. Intellectual surprise usually gives us a sense of fulfillment, an appreciation of some wisdom, a joy from its wittiness, and a drive to find some more. *Making mathematical findings appear unexpected, or even contra-expected, is the secret of teaching mathematics the surprise-way.* Such a way of teaching is not an end in itself, of course. It is however quite a promising means for achieving students' interest, which has for long been known to be positively correlated with successful learning.

Let us look at the nature of the intellectual surprises hoarded in school mathematics. We'll consider a sample of common high school mathematics theorems from across the curriculum to demonstrate the following ten types of mathematical surprise:

1. A common property in a random collection of objects;
2. A small change that makes a big difference;
3. Unexpected existence, and non-existence of the expected;
4. A rare property becomes generalizable (in more than one direction);
5. Analogies which prove not-analogous;
6. Plausible reasoning that fails;
7. Refutation of a conjecture obtained inductively.
8. The limit process yields an altogether new finding;
9. A single algorithm solves infinitely many problems;

and last but not least—

10. Mathematical paradoxes.

VARIOUS TYPES OF MATHEMATICAL SURPRISE

1. A common property in a random collection

Every single theorem can be turned into a surprise by considering the unexpected matter which that theorem claims to be true. Sometimes the theorem is so well known that it is hard to see the point. For example, what is so exciting about the claim that the sum of the interior angles of a triangle is 180 degrees?

To reach the surprise potential of a theorem it is usually helpful to assume we do not know it. Suppose we do not know that the sum of the interior angles of a triangle is 180 degrees. Would it be reasonable to suspect that *all* triangles, of *any* shape and size — equilateral, isosceles, scalene, acute-angled, right-angled, obtuse-angled, very large (in area) and very small, narrow and wide — must all have the *same* sum for their interior angles? Would it not take a novice by surprise to discover that the sum of the interior angles of triangles of various kinds is a constant? The task of finding the largest sum of the interior angles of an assortment of triangles may seem to some readers to be misleading, perhaps unfair. Practice has shown just the opposite. To those students who do not know anything about this theorem it sounds very reasonable to look for the maximum sum of the interior angles in such a variety of triangles. Surely, the results make their eyebrows rise. At first, they hesitate to admit that they have not found a maximum. Gaining confidence in the constancy, within the limits of measuring errors, is the main goal of this task.

To increase students' appreciation of this powerful claim, it is worthwhile, immediately following the task described above, to ask students to see if they can construct or draw a triangle *not* having this property. As long as they believe that there is one, they should continue to look for it. The frustration accumulated through trying in vain to construct a triangle with more or with less than 180 degrees is highly important in the process of internalizing the content of the theorem. We the teachers must be courageous enough to assign such tasks because they are at least as illuminative and as instructional as tasks in which success is guaranteed. There is a lesson to learn from this fruitless search!

Some of the students may at this point become eager to prove that the sum is a constant for all triangles. What a radical change from the initial intuitive feeling they had a little earlier when they willingly tackled the task of looking for a maximum sum.

2. A small change that makes a big difference

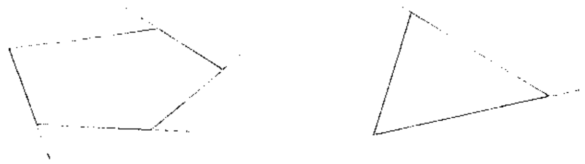
Based on the theorem of the sum of a triangle's interior angles, we can suggest an intriguing (over) generalization. Is the sum of the interior angles of a quadrilateral also 180 degrees? It takes only a quick investigation to get a counterexample and answer clearly: No. We can then attack it from the opposite direction: Is the sum at all the *same* for *all* kinds of quadrilaterals? Some ingenuity is required here to realize and then to prove that yes, indeed, there is a constant sum of 360 degrees.

We carry on the study of the sum of the interior angles of pentagons, hexagons, and so on, and eventually we realize that there is a pattern here. For *any* polygon the sum of the interior angles is a function of n, the number of sides: $S(n) = (n - 2)\ 180°$

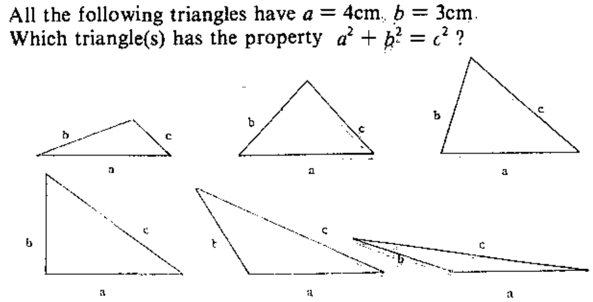
Now what about the sum of the *exterior* angles? The previous discussion may have set up an expectation that the *exterior* angles of polygons behave like the interior ones. Here awaits another surprise. Unlike the sum of the interior angles, the sum of the exterior angles is *not* a function of n. This sum is 360° for *all* polygons *independent* of the number of sides. Unbelievable — isn't it?

3. Unexpected existence, and non-existence of the expected

In order to create or to maintain surprise/tension we can sometimes make students search for something that exists



The sum of the exterior angles of any polygon is 360°
Figure 1



A systematic search for the Pythagorean property
Figure 2

when we suppose that they do not expect it to: sometimes, surprisingly, it becomes evident that there are infinitely many things of that kind. At other times we can make students look for something they believe they can find although it turns out that they can't. Moreover, we may be able to prove that nobody can find such a thing for it does not exist at all

Take for instance the Pythagorean theorem. Suppose we do not know it. What a challenge it would be to find a triangle that has the property that the square on one side (the largest, naturally) equals the sum of the squares on the other two sides. Just by making a sketch and looking at a few cases it would be very hard to believe that there is even one triangle like that. Why not allow students time for a search for such a triangle, using tools, of course, for drawing, measuring, and calculating areas. If they get too frustrated after a while, we can encourage them by promising that Pythagoras in the 6th century B.C. found that there are *infinitely* many triangles with this property. If they still don't find one, we can suggest a more systematic search through the family of triangles having two sides in common, say 3 cm and 4 cm, differing only in the angle between these two sides — let us say each having an angle 10 degrees greater than the other (see Figure 2). Another alternative is to give them a hint through a discussion of their findings up to that point, something like this: "Show me your triangles. Oh, I see, this one is an *acute-angled triangle* and your calculations show that the sum of the two small squares is *too large*. This one is an *obtuse-angled triangle*, and the sum of the two small squares is *too small*. Is there a third kind of triangle which you have not examined yet?

After a while, which may seem to last forever to the unoccupied teacher, one student will find such a triangle. That is, if the teacher is patient enough and believes in the value of the mathematical struggle students are going through while carefully refraining from spoon-feeding in order not to spoil the joy of discovery. After a while, then, another student will find a desired (right-angled) triangle, and then another student, and a few more. Finally, we may suspect that *all* right-angled triangles have this strange property, "all" meaning without even one exception. Some students may not be ready to "buy it" yet. To deepen their belief we can suggest they look for a right-angled triangle *not* having this property. They'll have to face the fact that nobody in class can find one. Can anybody else on earth? To be sure we must *prove* that the answer is no, but now they are ready to do it. They possess the experience and they have gained appreciation of the uniqueness of this property.

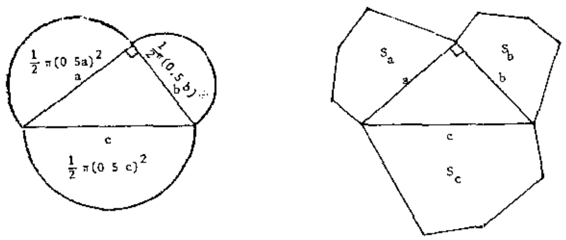
4. A rare property becomes generalizable (in more than one direction)

Let's push the above discussion one step further. Is there any triangle *other* than a right-angled triangle that has the sum of squares property? Students encouraged to work on that may become a little angry that they cannot find one, but perhaps there isn't one? This would prove correct! The path of surprise can be extended in two directions now (personally, I like the second better):

1. What about obtuse-angled triangles and acute-angled triangles? Is there a common relation between the sum of the squares on two sides and the square on the third? It is hard to believe, but yes, there is. The square on a side of any triangle is equal to the sum of the squares on the other two sides plus or minus "something". This qualitative generalization can in fact be gathered from the previous experimentation. (As we know, that "something" quantitatively equals twice the area of the rectangle formed by one of the two sides and the projection of the other on it.)
2. What about other polygons constructed similarly on the three sides of a right-angled triangle? Is the area of the figure constructed on the hypotenuse equal to the sum of the areas of the similar figures constructed on the other two sides? Or is the result only for squares? Surprisingly enough there is nothing magic about squares in this respect. The Pythagorean theorem can easily be extended to any similar figures erected on the sides of a right-angled triangle (see Figure 3)

5. Analogies which prove not analogous

It is a very good habit to interweave three-dimensional analogies in the course of teaching plane geometry: e.g., the locus of points equidistant from a given line in the plane is a pair of parallel lines, in space it is a cylinder. Through a given point on a given line there is one and only one line in



An extension of the Theorem of Pythagoras to semicircles and to similar pentagons
Figure 3

the plane perpendicular to the given line, but an infinite number of them exist in space, forming the unique plane perpendicular to the given line Many plane geometry theorems are naturally extendable to three-dimensional space. This gives us a great opportunity to introduce spatial relations. However careful examination is required of each generalization, as not all of them work according to expectations Consider, for instance, regular polygons.

As every geometry student knows, for all $n > 2$ there is one and only one regular polygon (up to similarity) having n congruent sides. In other words there are infinitely many types of (non-similar) regular polygons. It seems very reasonable to predict that the same will be true for the three-dimensional counterpart of regular polygons—regular polyhedra. How many regular polyhedra are there, then? Is there one polyhedron with n-gonal faces for each value of $n>2$? No. Not nearly as many. It is within reach of all ordinary high school mathematics students to prove that *no more* than five regular polyhedra exist Of these five, three are made out of isosceles triangles, one is made out of squares and one is made out of pentagons Five regular solids. That's it. No more. In fact, no less, either, but it takes higher mathematics to prove this part. [Dörrie, 1965]

Arithmetic and geometric progressions are analogous in many ways: There is a constant difference/quotient respectively between any two successive elements. Each element is the arithmetic/geometric mean respectively of its two adjacent elements:

$a_n = (a_{n-1} + a_{n+1})/2$ and $a_n = (a_{n-1}a_{n+1})^{1/2}$, respectively;

The nth element in each is:

$a_n = a_1 + (n-1)d$ and $a_n = a_1q^{n-1}$, respectively.

Now, having learned that the sum of the first n elements of the arithmetic progression is

$S_n = (a_1 + a_n)n/2$

it is highly suggestive that for the geometric progression the sum will be:

$S_n = (a_1a_n)^{n/2}$.

Testing this hypothesis brings about a surprise. How come that it does not work? A student who recognizes that the above is the expression for the *product* of the first n elements of the geometric progression and not of their sum has gained a deep insight into mathematical analogies.

6. Plausible reasoning fails

Let's look at another well known school theorem. Prime numbers become gradually more scarce The further we go along the number line the more chance there is that any natural number we encounter is divisible by a smaller one. There is nothing surprising about this fact.

Following the same logic, it is reasonable to conjecture that there is a place on the number line beyond which all natural numbers are composite. It is therefore highly surprising that the number of prime numbers is *not* finite. No matter how far we go along the number line, by going further still we will be bound to hit a number indivisible by each and every one of its predecessors (excluding 1). The proof itself is unique and very elegant and should not be neglected in high school. Leron [1985] shows a way to do it "softly".

7. Refutation of a conjecture obtained inductively

As with the case of prime numbers discussed above, there are many opportunities to make students cautious about generalizations from particular instances. Alongside practice in the highly important task of pattern recognition there is a high value to be obtained from meeting those parts of mathematics where conjectures fail. One counter-example and a whole building collapses. The following task illustrates the idea:

> Calculate the value $y = x^2 + x - 41$ for the following values of x:

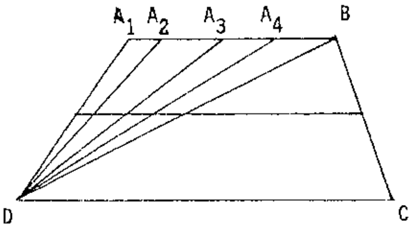
x	1	2	3	4	5	6	7	8	9	10	12	14	16	18	20	23	26	29	32	35	38
y																					

> Use a table of prime numbers (up to 2,000) to find out the values of x which yield a prime value of y.
> What conjecture can you make about the values y assumes for any value of x? ($x^2 + x - 41$ is a prime generating formula).
> Prove or refute you conjecture. (Substitute 41 to refute the conjecture.)

This example gives students a lot of practice in substitution in a quadratic form. It is therefore a demonstration of the surprise effect in a lesson that is devoted mostly to drill and practice. The wording of the last part of this task is also noteworthy. Any task involving a search for a pattern should end in this question, always leaving open both alternatives: *prove or refute* that your pattern is correct in general. Some patterns should be refutable, otherwise students get the wrong impression that plausible conjectures are always provable.

8. A limiting process yields a new finding

Suppose a class has just completed the study of the theorem stating that the line segment joining the mid-points of the non-parallel sides of a trapezium is parallel to the parallel sides, and its length equals half their sum. Now we show a trapezium $ABCD$ and start moving A towards B keeping C and D in place. (See Figure 4)



A limiting case of the midsides segment in a trapezium
Figure 4

As long as A and B are two distinct points, the above theorem remains valid. But what happens when A coincides with B? The trapezium becomes a triangle, and a theorem about trapeziums suggests a new theorem about triangles. Isn't that like magic?

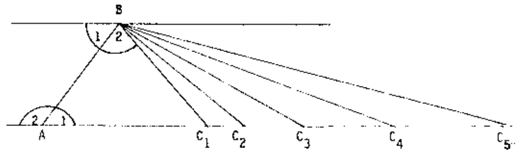
Sometimes the limit process suggests not only a new theorem but its proof too. Consider for instance the theorem stating that, in a triangle, an exterior angle is greater than any interior angle not adjacent to it: e.g., in triangle ABC we have angle $A_2 >$ angle B_2. Now, we move C to the right on line AC without moving A or B. (See Figure 5)

As C moves to the right, angle B2 gets bigger and bigger while angle A2 does not change; however, angle B2 remains always smaller than angle A2. No matter how far right C moves, the interior angle at B remains always smaller than the exterior angle at A. Nevertheless their sizes do get closer. Contradiction? No! The two angles become equal when C travels to infinity. At this moment the triangle "opens", BC and AC no longer intersect and angle B2 and angle A2 become alternating angles between the two parallel lines. We may speculate on a new theorem: "If a pair of alternate angles between two lines intersected by a third line are congruent, then the two lines are parallel." We also possess an indication of a proof by negation: we can show that as long as the two lines are not parallel we have a triangle in which the alternate angles are an exterior one and a non-adjacent interior one and which, therefore, can't be congruent.

9. A single algorithm solves infinitely many different problems

Any algorithm that works in general should be presented as a great achievement. Its existence must not be taken for granted. Mathematicians invest much of their time in looking for general solutions to avoid the complications one faces in parts of mathematics, such as integral calculus, in which general solutions are rare. Just the idea of looking for a general formula should be recognized as a brilliant idea for it may save time and energy. Finding one is an extraordinary surprise.

Suppose for instance that we did not have just one formula for *all* quadratic equations. Students would then have to experience solving one equation at a time, not even dreaming of a solution which could take care of all possible cases at once. Why underestimate the surprising fact that a general solution does exist? It is a real piece of magic in a way. There is no need to complete the square any more, no need to factorize. Put in numerical values for *a, b, c*, and get the output — two, one, or no real roots according to the value of the discriminant. What a relief! There is no single formula to solve all cubic equations and polynomial equations of higher degrees get more and more complicated. So let's teach quadratic equations with respect, not just technically. Students should work hard on particular quadratic equations before they learn about the formula. What a wonderful surprise and relief the formula will then constitute. Now they will appreciate the benefit they get from the algorithm, with all the extra information it provides about the roots — their number, their signs — before knowing anything about their values. The discriminant is some kind of "informer". It lets the quadratic-cat out of the equation-bag....



A limiting case of the exterior angle in a triangle theorem
Figure 5

10. A paradox — it is wrong, yet it makes sense

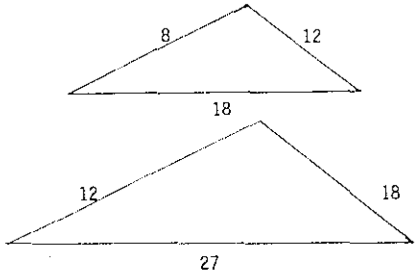
New knowledge which seems to contradict previous knowledge puts a thinking person in a conflict: "This is reasonable, yet logically impossible", or "This is logically valid, but I still can't believe it". Such situations are a great source of surprise in mathematics and have the potential to be irreplaceable learning experiences. For a long time after high school graduation students will retain these challenges and enjoy thinking about their reconciliation. Let's look at a few examples:

Let $S = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots$

Now, on the one hand
$S = (1-1)+(1-1)+(1-1)+(1-1)+(1-1)+ \ldots$
$= 0$
On the other hand
$S = 1 - (1-1) - (1-1) - (1-1) - (1-1) \ldots$
$= 1$
However,
$S = 1 - (1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \ldots)$
$= 1 - S$
Or in other words
$2S = 1 \Rightarrow S = 1/2$

Well, what *is* the value of S? Is it 0? or 1/2? Perhaps all three are equal?...

Isn't this a fascinating way to introduce the distinction between divergent and convergent sequences? This paradox has deep roots in the history of mathematics [Eves



Two non-congruent triangles with 5 congruent elements
Figure 6

p 135]. Understanding that this sequence does not have a sum to start with, and therefore it is meaningless to try and calculate that sum, is much more influential than ten reminders: "Be careful when dealing with infinite sums!"

My favorite conflict is a wonderful tool to emphasize the role of correspondence in triangle congruence/similarity theorems (Hadar & Hadass 1981). I ask my students to construct a pair of non-congruent triangles having *five* elements in one triangle congruent to five elements in the other (Elements mean sides and interior angles) Knowing my style, my students usually hurry to say that there aren't any. At this point I refer them to the constructive way may question is phrased, reassuring them that there is more than one pair. Still unbelieving, they start analyzing the problem, realizing that the two desired triangles can't have three sides and two angles congruent, hence they must have three angles and two sides congruent, and therefore they must be similar. Now they are much closer to a solution. (See Figure 6). Yet, it takes some deep insight to abandon the first feeling of impossibility. And it is all due to a fine point -- the five congruent elements are *not* in correspondence as required in the congruence theorems *

DISCUSSION

A surprise in every theorem

This paper suggests a meta-theorem. Unlike most meta-theorems which belong to mathematical logic, this meta-theorem belongs to the Theory of Mathematics Education which Steiner [1985] set rolling. It states that *every mathematics theorem is surprising*. "Every" is used here in the mathematical sense — that is to say, "all except possibly a finite number", meaning in our case, a small finite number. If the claim stated in the theorem were trivial it would be of no interest to establish it. A new mathematical theorem must say something unknown and nontrivial, othewise it cannot get published and it has zero probability of entering any syllabus. *Some* mathematicians must get a kick out of it.

The examples brought in this paper comprise a limited sample intended to give empirical inductive support to the general claim that, like all mathematics theorems, *any* school mathematics theorem presents some non-trivial truth. An inductive argument seems appropriate here because ours is an empirical discipline The theory of mathematics education has bearing upon mathematics, pedagogy, and psychology of mathematics. Claims referring to it need not take the form of deductive mathematical proofs.

The examples demonstrate how one can reach the surprise potential of a theorem one has known for years. It is usually helpful to pretend not to know it. Sometimes it helps to assume just the opposite, by negating one or more parts of the claim, considering the extent to which the negated version may make sense to a novice. Students can get into the habit of looking forward to the daily surprise. They had better learn to expect another surprise than another set of repetitive exercises.

It is the mathematics teacher's responsibility to recover the surprise embedded in each theorem and to convey it to the students. The method is simple: just imagine you do not know this fact. This is where you meet your students. Let them examine their expectations, and make them realize that they get new and very unusual results in every theorem.

Stimulating theorem presentations followed by responsive proofs compose the Stimulating Responsive Method. I have proposed this method elsewhere as an alternative to the guided discovery approach [Movshovitz-Hadar, 1987a]. Teaching mathematics as an endless chain of theorem-proof-theorem-proof interrupted here and there by exercises alienates a major part of the students. Instead, it is better to make students appreciate the great findings in each theorem by stimulating their intellectual curiosity, making them struggle with their expectations, and bringing them to an appreciation of and belief in the claim before they tackle its proof.

The classroom teacher must not be left alone in the battle with students' motivation to learn mathematics and their attitudes towards it. The challenge is in the hands of pre-service training as well as in-service training institutions To support the difficult daily task faced by the teacher there is need for the development of materials which incorporate wherever possible the built-in surprise potential every school mathematics theorem possesses

* For a more in-depth discussion of the role of paradoxes in mathematics education see Movshovitz-Hadar [1987b]

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The Bourbaki conception of a mathematic (singular) as a tapestry, woven by the able hands of the people who had learned the axiomatic method, was adopted almost universally soon after the war as world communications were restored

The crises at the turn of the century, and the plea for help from psychologists, led to an auto-psychoanalysis on the the part of mathematicians who looked at their deepest levels of mental structuration and found there "structures" and "relations" Venerable chapters of mathematics were abandoned and replaced by the study of underlying structures. As if fully grown trees had suddenly lost their appeal and only the water, the air, the nitrates, etc, that were needed for their growth had beauty, mathematicians were busy producing the machine tools that, it was presumed, would permit a close examination of these various components. The aesthetic experience was no longer acquired by gazing at the edifice, but by accumulating the bricks and mortar from which it had been made.

When those vast entities called "categories" had been spotted, it seemed that the "abstractors" had reached their goal; that from now on we could only go down into the "concrete" of entities constructed by choice and taste, much as artists use canvas and paint to produce pictures which only depend on the mood, know-how and opportunities of the particular painter.

A new tension was being created. The analysis of *most* mathematical notions led to some or other of the singled-out structures, and their behaviour could be forecast from the nature of these structures A universal and compelling quality of mathematics was still to be found in the nature of the structures, which now appeared to be mental structures, revealing themselves through a process of purification called abstraction.

However, besides this universal quality in the minds of mathematicians (of which they take advantage when they tell laymen that it makes them competent in all fields of rational study), the individual slant, the unpredictable twist was also let in; so that, whereas Hilbert in 1900 could voice on behalf of the body of working mathematicians the few most important challenges that lay in the future and would need the most determined efforts, in 1968 anyone's guess about the future of mathematics is equally valid. It could even become a trivial occupation for idle minds.

CALEB GATTEGNO