An Artefactual Approach to Ancient Arithmetic

IRENE PERCIVAL

Can artefacts from a civilisation long dead help arithmetic come alive? This sounds contradictory, yet my recent work with young students seems to suggest that this can indeed be the case. More than a decade ago, the French *Institutes for Research on the Teaching of Mathematics* (the IREMs) published a set of papers (Fauvel, 1990) promoting the use of historical documents to 'humanise' mathematics classes – but these articles focused on students in secondary education. On the other hand, several excellent texts (e g. Reimer and Reimer, 1992) have recently been published to facilitate the introduction of historical mathematics to the elementary classroom, but they make little or no reference to actual artefacts. My work with elementary school students combines the documentary approach of the IREMs with two other artefactual approaches: students' own construction of objects and documents imitating those studied and using ancient calculating devices, albeit in modern reconstructions.

I have explored these approaches in several series of enrichment classes during the past three years. At first, my motivation was to complement the work on ancient civilisations covered in social studies classes by twelve- and thirteen-year-old students in their final year of elementary school in British Columbia, Canada. These classes cover many aspects of life in the ancient world, but mathematics is rarely mentioned. I considered this to be a regrettable omission, so I read several social studies texts in order to be able to relate the mathematics, particularly the arithmetic, of ancient civilisations to the students' other knowledge. A detailed account of such an integration was the basis of my M.Sc. thesis (Percival, 1999). Since then, I have also taught this material to students in other grades, in both large and small groups, and have found a high level of interest in the number systems used by other civilisations.

"So much stuff, it could be ….. numbers"

Egypt and Mesopotamia often seem to head the list when ancient civilisations are mentioned and my courses are no exception. We began 5000 years ago at the start of the Egyptian dynasties with the macehead commemorating Pharoah Nārmer's victories in battle (Budge, 1926/1977, p. 5).

I explained the nature and purpose of this object to my students and challenged them to 'find the numbers'. Almost immediately, I was told that "there's so much stuff there [in the bottom right corner], it could be to do with numbers". I was surprised how quickly the students reached this conclusion: the numbers they use every day belong to a ciphered system and yet they had no difficulty assuming the tally nature of an older system. Does this reflect earlier school work in tallying or is it really an intuitive idea? The tally nature of so many early number systems would suggest the latter. Later in the session, we looked at several problems taken from Chace's transliteration of the Rhind Papyrus (1927/1979) and the students were able to identify other hieroglyphics whose repetition marked them as possible number symbols.

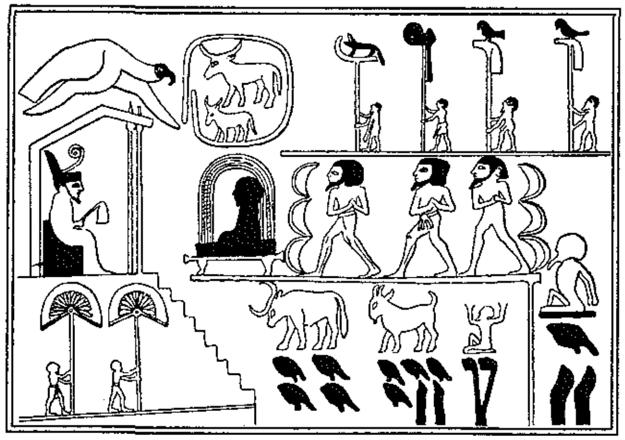

Figure 1 Nārmer's Macehead

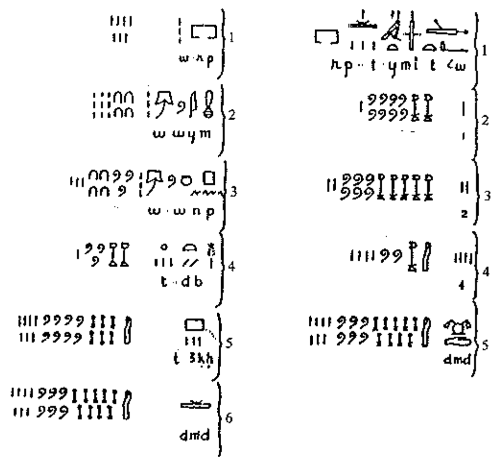

Figure 2 Problem 79 from the Rhind Mathematical Papyrus

For some of the civilisations studied, such artefacts enabled the students to determine the value of individual symbols. Although the problem from the Rhind Papyrus shown in Figure 2 (p. 137) could perhaps have led to conjectures for these values, a fairly high level of mathematical sophistication would be needed to justify such guesses. My students were able to see powers of seven in the left-hand column when they knew the meaning of the symbols, but the length of time taken to do so suggests that this pattern was not sufficiently obvious to confirm speculations on the value of the symbols. However, the idea of students acting as 'mathematical archaeologists' was fundamental to my course, so I designed a puzzle in which six Egyptian numbers had to be paired up with their modern equivalents: a very rough approximation to the Rosetta Stone. This led to some valuable conjecturing and, in one case, a heated discussion broke out as to the value of a particular symbol.

One student suggested that it meant 100,000, but two others, who rarely had the courage to contradict the first in standard mathematical tasks, each produced arguments designed to show that it represented 100. This Egyptian work, which was new to all of them, seemed to place the students on an equal footing and the voices of the second and third students held such a ring of confidence that the first student finally backed down.

The contrast between the modern place-value system and the tally-like Egyptian system made this matching puzzle a revealing exercise and the resulting pairs of notations provided useful data when I later asked the students to focus on the similarities and differences between the two systems. These comparisons led to a discussion of ciphered numbers as opposed to the repeated marks of the tally system and also helped solidify the students' understanding of place value and the role of zero in our present numeration system.

As mentioned above, the problem shown in Figure 2 provided an interesting exercise in pattern recognition, particularly as the fourth line (left-hand side) contains a mistake by the ancient copyist. Discussion of this error, in which only three spirals were drawn instead of four, not only highlighted an advantage of a ciphered number system, but also focused on the difficulty in interpreting ancient documents, thus combining the mathematical and 'archaeological' aspects of the course.

Decoding is a task which appears to interest most students. At its most elementary level, it merely requires implementing a given one-to-one correspondence between two sets of symbols, but even this can be a valuable mathematical exercise for young students. However, in my ancient arithmetic course, the decoding often required more sophisticated thinking, as is illustrated by the next step of the Egyptian work. I showed the students the original hieratic version (p. 137) of the Rhind Papyrus problem (see Figure 3) they had previously decoded from hieroglyphics (see Figure 2), and asked them to match the 'Egyptian hand-writing' with the transliteration.

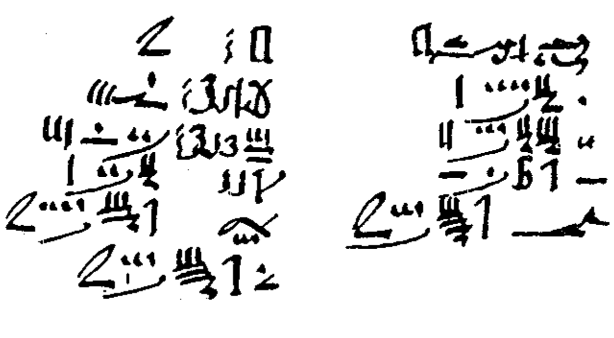

Figure 3 Original hieratic script of Problem 79

The students were able to determine the value of the individual ciphered symbols and to consider how these symbols had developed from the earlier hieroglyphics. One observant boy noted that eight hundred was represented by four dots with a line underneath, with similar notations for six hundred and two hundred, and shouted excitedly "they're all times'd by two". Other students investigated how the seven single lines of the hieroglyph for seven could have developed into something rather like the modern numeral 2. The ensuing discussion highlighted the human actions involved in the development of mathematics. This humanist viewpoint is rarely considered by either students or teachers, but is a major focus of my course.

The students enjoyed deciphering the ancient documents and were intrigued by the imagery of the hieroglyphics. They produced a variety of explanations as to what the symbols represented, based on their knowledge of Egyptian civilisation. Many of their suggestions were close to those suggested by the 'experts', but the 100,000 symbol remained 'a parrot' to them, despite being told Menninger's comment that:

> the Egyptian symbol for 100,000 is a tadpole, such as wiggle by in countless numbers in the mud of the Nile when the water retreats within its normal banks after the yearly flood. [...] Its name, *hfn*, also means 'innumerable' (1969, p. 122)

Elsewhere (Irons and Burnett, 1995), we read that:

> the 100,000 symbol represented the number of tadpoles in a mudhole (p. 13)

This led to the question of 'which mudhole?'. The students decided it would be that of 'a camel's hoofprint', and this proposal was followed by some valuable work on estimation, which did indeed suggest that 100,000 tadpoles would fill a volume of approximately that size. This direction was not one which I had planned, so I have to admit to feelings of relief when the estimation gave reasonable values! Relating the mathematical symbols to the flooding of the Nile, and the camel's hoofprint, again reinforced the 'human-made' nature of mathematics: unlike our own number symbols whose original significance is mostly lost in antiquity, the Egyptian symbols clearly show signs of human intervention in their origins.

It could be argued that reading and writing hieroglyphics is nothing new, that it is an integral part of any social studies course about Egypt. However, I feel that a study of *mathematical* hieroglyphics is particularly beneficial, as students can then read relatively large sections of Egyptian text, a difficult task for prose passages which usually contain a much wider variety of symbols.

When I first taught this course integrating arithmetic into the Egyptian part of the social studies curriculum, I gave my students dry reeds from which to make their own sheet of papyrus. Concurrent with this practical work, they learnt Egyptian methods of addition, subtraction, multiplication, division and fractions and were able to compose hieroglyphic calculations to copy onto their personal 'Egyptian artefact'. This home-made papyrus proved to be a powerful motivator. Although I had asked each student to prepare one question to write upon it, one boy (an excellent mathematician, but with a reputation for never wanting to do any written work) asked, "Can I put two problems in? I might even have room for three if I write really small." His finished papyrus included five of his calculations, covering the complete arithmetic range from addition to fractions, all written with great care.

Pedagogically, it can be seen that an artefactual approach to Egyptian arithmetic can be successful for two reasons. On the one hand, ancient documents which are 'written in a foreign language' stimulate the students' curiosity and appeal to their fascination with decoding. On the other, the students' creation of their own artefacts provides the motivation to learn the arithmetic techniques to write upon them and to take considerable care in producing a final product.

"One doesn't always mean one"

I started the Babylonian sessions by giving my students three different views of ancient tablets. First, we looked at pictures of complete tablets densely covered in tiny indentations: these were accompanied by a scale marker and the students were amazed that anything so small could be written or read. Next, I showed them the imitation tablet which I had constructed from 'playdoh' (a clay-like substance which hardens when baked), on which Babylonian numerals had been inscribed with a suitably carved chopstick. Although this was far bigger than the genuine Babylonian objects, it gave the students a three-dimensional experience to complement the pictures they had seen. It also served to show what I expected them to produce at the end of the unit! Finally, I gave them transcriptions of the front and back of a multiplication tablet and it was these that they explored in detail (Aaboe, 1964, p. 7).

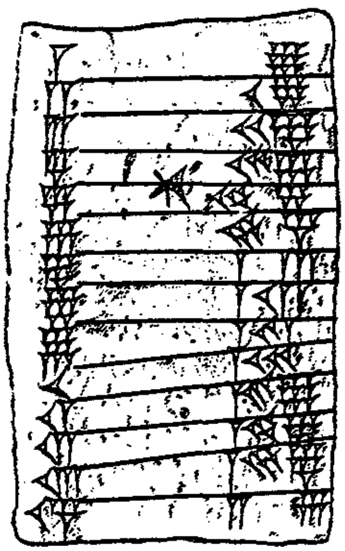

Figure 4 Babylonian multiplication table for 9 (front)

They quickly noticed the pattern of the symbols and had no difficulty in identifying the purpose of the tablet. However, there was general consternation about the break in the pattern after the Babylonian number fifty-four [sixth line, right hand side]. Although a few students quickly realised that 'one doesn't always mean one', with most groups I had to ask a few pointed questions about the significance of the digits in modern numerals, before the concept of a positional system was recalled. The idea of place value is introduced early in the curriculum, but sometimes takes years to become fully assimilated, so any work which reviews it in a different context has pedogogical value.

The students became competent at translating between Babylonian and modern symbols, so much so that I suspect that some of the Babylonian calculations they performed were actually carried out in modern notation, with the answer being turned back into the cuneiform symbols. Working in this complex base-sixty system proved difficult for many of them and they saw this double translation as an easy way out. Calculations were done with pencil and paper, but my social studies approach also required them to construct their own tablets: discovering how to use a chopstick to form the correct shapes proved to be an interesting exercise in practical geometry. The initial difficulty they experienced with this style of writing also helped them appreciate Boyer's suggestion that it might have been:

> the inflexibility of the Mesopotamian writing materials [...] that made the Babylonians aware that their two symbols for units and tens sufficed for the representation of any integer [... and led to] their invention [...] of the positional notation. (1968, p. 29)

In making their own multiplication tablets, some students marked the cuneiform symbols directly on the 'clay', using the emerging patterns to predict the next number. Others wrote the numbers in modern notation before translating each one, a much harder task but one whose familiarity provided security. Apart from the obvious link that this practical work has to the study of 'writing' in ancient Babylonia, working directly with 'clay' enabled students to correct their work simply by pressing the surface flat again. The ease with which errors could be amended was of crucial importance when the students made their own tablets: many students were caught up in the pattern of symbols and followed it past fifty-nine, the point at which the pattern should change to introduce symbols of the next place value. As with the Egyptian papyrus, most students were motivated to work carefully at this 'hands-on' arithmetic, pleased to have produced something that they could take home.

Although the number symbols in this unit do not appear to have the 'real-life' origins of the Egyptian hieroglyphics, a humanist element was available. Number tokens dating back ten thousand years have been found in several areas of Mesopotamia. Using 'actual' tokens made from baked playdoh, together with some soft playdoh to form 'envelopes' in which such tokens have been found, my students were able to re-enact transactions involving these concrete forms of number. We also discussed the evolutionary path from these number tokens through the Sumerian symbols to the cuneiform script of the Babylonians. Although this showed where the Babylonian base-sixty system came from, it merely pushed the 'Why sixty?' question back several thousand years. We discussed various answers that have been proposed, but the students remained as unsatisfied as the majority of mathematical historians

"It was OK to be 'in the red'"

I tried several different approaches to Chinese arithmetic, depending on the age and ethnicity of the students in the various groups. At first, I discussed only the rod numerals, using Chu Shih-Chieh's 'Arithmetic triangle' (Needham, 1959, p 135) as the document for translation

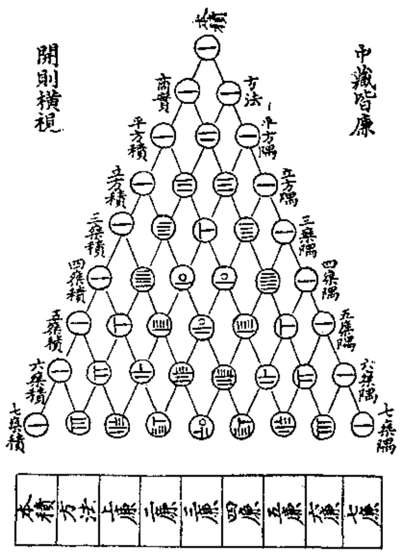

Figure 5 Chinese arithmetic triangle

The students were able to identify the first five positive integers immediately and this gave them sufficient information to deduce the value of the others from the symbolic patterns The symbols used in this document (dating from 1303) are not quite the same as those used more than a thousand years earlier Needham comments:

> [they] are so turned as to presuppose that the bottom of the triangle originally stood vertically on the left (p. 137)

However, the triangle investigation was sufficient to give students the concept of rod numerals. The counting rods from which these numerals derived were explored in a 'hands-on' fashion, with toothpicks placed on a paper 'counting board'.

A feature of this Chinese system of numeration which made it almost unique in the ancient world was its use of both positive and negative numbers, represented by red and black rods respectively. As one student remarked, "In ancient China, it was OK to be 'in the red'" Explorations with these coloured rods helped my grade seven students to clarify concepts of integers recently encountered in their regular mathematics lessons. Using the rods in my grade six class led these students, who knew of negative numbers only from weather reports, to discover the rules of addition and subtraction of integers for themselves. One student commented in a tone of wonder, "So subtracting negative three is like adding positive three?" That such important results developed from this practical work lends credence to the pedagogic belief that fundamental arithmetic concepts can be acquired or reinforced by studying ancient artefacts.

The students were more familiar with the abacus, the calculating device used by the Chinese many centuries later. However, even among the ethnic Chinese students in my classes, there were few who were competent in its use. All students spent some time practising addition and subtraction and, although they initially considered it to be rather a clumsy device, they later realised its advantage, which lies in producing the answer to an addition or subtraction simply by 'entering' the second number

The largest group I taught contained many Chinese students, so I decided to start with a document using the traditional ideographic numerals, expecting that a few of the students would recognise these. I selected the Chinese remainder theorem from the Sun Tzu *suan-ching* (Davis and Hersh, 1981, p. 189) and carefully outlined the numerals to make them easy to locate. I need not have bothered: almost all the Chinese students not only recognised the numerals, but were delighted to be able to provide a translation for their classmates, an example of multicultural co-operation at its best. They were also able to teach me that nowadays these numerals include a zero for missing place values, something that my history texts had not explained

For groups not including Chinese students, I designed a puzzle which required students to determine both the overall structure and specific symbols of the numeration system by matching two Chinese numerals to their modern equivalents. At first, they found this task difficult, but the 'aha' moment came when they noticed that certain symbols were repeated at similar positions in the two Chinese numerals: this gave rise to the conjecture that these represented the powers of ten and the problem was quickly solved thereafter

The 'named place-value' system that they discovered is the written equivalent of the spoken language forms, a natural consequence of the Chinese ideograms It provided the students with an example of numeration more advanced than tallying, but less advanced than the full place-value system in widespread use today, and was thus a useful step to students' understanding of modern numeration In fact, an article by Bohan and Bohan (1993) describes how grade four students developed ways to 'improve' the Egyptian method, which led them to the construction of something very similar to the traditional Chinese system, before moving on to another that was "just like our system!" (p. 86)

This artefactual approach highlights the close connection among the rod numerals, the counting rods and the abacus. Calculating with these three different methods, the students became aware that, in each case, a five-fold repeat of the action of placing a rod or bead was permitted before these five objects were replaced by one in a different orientation. Thus, even though the rods and beads formed part of a base-ten positional system, similar in many respects to that in use today, these Chinese artefacts show a subsidiary structure which is absent in our system

"So that's where our numbers come from"

As I could find no original artefacts for Indian arithmetic, this section is perhaps out of place in this article. However, I include a few lines here, since the presence of ethnic Indian students in my classes encouraged me to include the topic in my course To my delight, one of these students prepared himself for our Indian session by researching the ancient number symbols for himself. After seeing these, another student commented "so that's where our numbers come from" in a tone of great satisfaction

After a discussion of the origins of the Hindu-Arabic numeration system and its progress from India to Europe, the focus shifted to the methods of calculation included in the Veda, the ancient texts of the Hindu priests. The students investigated two distinct methods of multiplication, both of which were very concise: after I had talked about the Indian 'dust-board' used for writing, the students appreciated the need for compactness. Most students grasped these techniques quickly and several announced that they planned to use them in their regular mathematics classes. I was glad that I had not allowed the absence of artefacts to keep me from presenting material that was obviously of great significance to the students of Indian background.

"I'm worth the most!"

For Greek arithmetic, I used two ancient manuscripts The first was a Greek schoolboy's copy of the multiplication tables for two and three (Guéradu and Jonguet, cited in Ifrah, 1985, p 263)

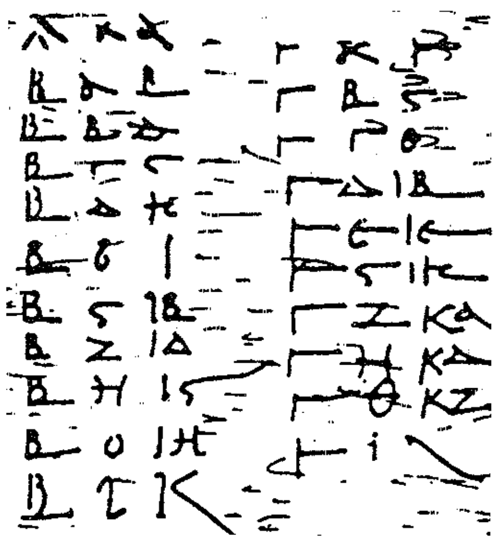

Figure 6 Greek multiplication tables for two (left) and three (right)

A few students managed to guess the nature of this document after noticing the repetition of the symbols B and Γ However, even though there were some comments that the B, H and K looked like our letters, most students were still rather surprised to hear that the Greeks used the letters of their ancient alphabet as numerals. This led to a discussion of how words and numbers were distinguished and the practice of *gematria* My students then calculated the value of their names, which of course gave rise to comparisons and the claim "I'm worth the most!" from the 'winning' student. Later, I gave them a list of the Greek alphabetic numerals, enabling them to perform a simple one-to-one decoding of the second document, a table of squares.

As with all these topics, an important purpose in introducing Greek numeration was to increase understanding and appreciation of our contemporary numeration system and the arithmetic algorithms it affords. Attempting calculations using the twenty-seven different Greek symbols soon made the students thankful for the conciseness of the modern ten-digit system.

Roman numerals are familiar to many students, so I did not spend much time on them, other than to point out that the symbols for 100 (C) and 1000 (M) predated the 'obvious' connection with the initial letters of the Latin words *centum* and *mille*. The students were relieved to hear that Roman numerals were not used for calculation, but merely to record the results of abacus, counting board or finger calculations: they had been anxiously contemplating the task of performing multiplication and division with these symbols. To emphasise this point, I showed them a Roman abacus I had constructed using ball-bearings and the ubiquitous playdoh, and the students compared it with the Chinese abacus they had used earlier in the course.

"They must have counted on their toes as well"

Although little is known of the methods of calculation used by the early civilisations in Central America, their methods of numeration have been determined from the few artefacts that survived the ravages of the Spanish *conquistadors*. I showed my students a page from the Dresden Codex (Ifrah, 1985, p. 427), which includes several numerals written using the base-twenty system employed by the Mayan people.

My students quickly identified the section containing numbers (shown in Figure 7) and had little difficulty determining the method of recording numbers up to nineteen. However, it was a while before even the best students made the leap to conjecturing a place-value system in which each group of symbols was but a single 'digit'.

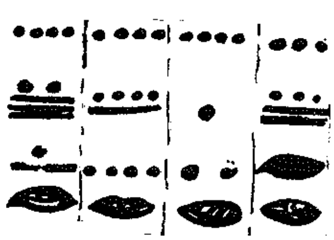

Figure 7 Mayan numerals from the Dresden Codex

Having agreed that it was a base-twenty system, the students decided that "they [the Mayans] must have counted on their toes as well", although the symbols suggest the use of sticks, beans (the local currency) and shells as counting aids. Consistent with the social studies approach of the course, the students constructed their own Mayan numerals, using toothpicks, coins and pasta shells: however, using these objects for addition and subtraction also provided the mathematical experience of working in a different base.

Our mathematical journey next took us further south into Bolivia and Peru. We looked at pictures of the knotted *quipu* of the Inca and the students were given the task of making one to record their school population by grade. They enjoyed this assignment, but many preferred making *chimpu*, a related method of numeration still found in parts of South America. Texts vary as to whether these involve tying knots in groups of strings or threading strings through shells: for simplicity, we chose to thread strings through 'shells' (pasta tubes), with the number of strings through any set of shells indicating the place value of the base-ten digit represented by the shell count. This reinforced the concept of place value and also provided a method of displaying statistical data very different from the charts commonly found in elementary classrooms

"It made me feel like I could do anything"

The study of ancient arithmetic often focuses on how such work can increase students' understanding of the contemporary numeration system and its algorithms As indicated briefly in the previous pages, I consider these benefits to be extensive and encouraged my students to compare each new number system or algorithm with all the others known to them at that time However, the primary focus of this artefactual approach is in developing a new attitude towards mathematics: increased understanding comes as a bonus.

Most students showed a keen interest in learning about different numeration systems and enjoyed the three-pronged artefactual approach to arithmetic, particularly when I managed to co-ordinate the classes with their social studies work on ancient civilisations. They commented that "it's interesting to see things that are so old" and enjoyed decoding the ancient documents: one enthusiastic student even gave the rousing endorsement that "it made me feel like I could do anything".

The 'hands-on' features of the course clearly appealed to them. Constructing their own artefacts not only gave them some practical experience in physically different writing methods used in a variety of times and places, but also provided an 'end-product' to take home after the session, quite a contrast from the page of exercises that is often the only tangible result of a mathematics class. The students also seemed to welcome the chance to learn how to use an abacus and various algorithms from other civilisations

Underlying this tripartite structure of looking at, creating and using artefacts lies an attempt to show how mathematics, or in this case arithmetic, is influenced by the civilisation using it:

- the change from the tallying hieroglyphics to the hieratic ciphered numerals, when the Egyptians started to write on papyrus rather than carving numerals in stone;
- the place-value system developed by the Babylonians, in order to reduce the number of different symbols;
- the brevity of the Indian algorithms, required by the small area of the dust-board;
- the knotted strings of the Inca, which provided a portable way of recording statistical information in a civilisation with no written language.

All these show the humanist, 'human-made' side of arithmetic, which is rarely mentioned in school texts I believe that this humanist element is the primary reason for the apparent success of this approach: children are naturally curious to know what they would have been doing if they had lived in another era, and this course satisfies their curiosity about how children in other times and places 'did their sums'.

One final observation: although a few students were apprehensive about the discovery work essential to my 'mathematical archaeologist' approach, it was popular with the majority. As one boy said:

> If you're told things, you can just as well be a computer that's told its instructions. But a computer can't figure things out, so [making your own discoveries] makes you feel you're a somebody.

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