The drop in numbers of participants that I mentioned in the first paragraph could have many different explanations: the proliferation of conferences, the cuts in funding in many countries, changed research policies, the loss of impetus in curricular reforms, and new forms of communication. The first ICME in Lyon (1969) was held in the wake, and on the eve, of great events in mathematics education: the modern mathematics movement, the foundation of the journals of mathematics education that are now the backbone of our community, and the societal changes after the second world war. In short, the first ICME was the concrete acknowledgement of the birth of a new community of researchers (in mathematics education) in a new educational setting. The context, however, has now changed.

So, finally, I look to the future. The affiliations of the various contributors to ICME-10 show that there is a lack of the voices of teachers. I am tempted to say that the format of ICME needs re-styling from its current planetary and ecumenical dimension to a more local and focused dimension. On the other hand, I cannot forget that ICME is organised under the auspices of ICMI, whose birth and aims are rooted in the ideals of communication, solidarity, and internationalism claimed in the journal *L'Enseignement Mathématique*. This journal (founded 1899) was the cradle of ICMI and has been its official organ since the beginning (Furinghetti, 2003).

Thus, after having complained about the too-rich program and the confusion provoked by too many people, I am ready to face it all again to experience whatever is going to happen at ICME-11. I like being part of the long-lasting, wonderful project underlying ICMI activities too much!

References

Furinghetti, F. (2003) 'Mathematical instruction in an international perspective: the contribution of the journal *L'Enseignement Mathématique*', in Coray, D., Furinghetti, F., Gispert, H., Hodgson, B. and Schubring, G. (eds), *One hundred years of L'Enseignement Mathématique, Monographie 39 de L'Enseignement Mathématique*, 19-46.

*The following three communications are comments inspired by two articles about algebra, particularly solving linear equations, by Brizuela and Schliemann (pp. 33-41) and by Dickinson and Eade (pp. 41-47), which appeared in FLM **24**(2). (ed.)*

Solving linear equations – why, how and when?

ABRAHAM ARCAVI

I would like to organize the thoughts these two articles provoked in me as the answers to three questions, which are not directly addressed by them, but are certainly implicit in most of what they describe and claim.

Solving linear equations – why?

> It was with the number puzzle that algebra seems to have taken its start. The desire of the early philosophers to unravel some simple numerical enigma was similar to the child's desire to find the answer to some question in the puzzle column of a newspaper [...] Such was the first algebraic problem (Number 24, in *Peet, Rhind Papyrus*, with slightly different translation) of Ahmes (*c.* 1550, BC), "Mass, its whole, its seventh, makes 19" [1]. (Smith, 1958, **2**, p. 582)

1. Many would disagree with the above claim about the origins of algebra. Sfard (1995) summarizes some of the arguments and counter-arguments about the origins of algebra, and points to the heart of this controversy as stemming:

> [...] not so much from different historical information as from the fact that they obviously have their own interpretation of the term *algebra*. (Sfard, 1995, p. 17)

We may attribute to Smith (based on the quotation above) an implicit view of algebra in which equations and their solutions are at its core. This would contrast with the view that seems consensual in mathematics education nowadays:

> Students in the middle grades should learn algebra both as a set of concepts and competencies tied to the representation of quantitative relationships and as a style of mathematical thinking for formalizing patterns, functions, and generalizations. (NCTM, 2000, p. 223)

Thus, according to the National Council of Teaching of Mathematics (NCTM), solving linear equations is not at the core of algebra, it is just one of the competencies listed – only at the end – of the NCTM's opening paragraph on algebra:

> In the middle grades, students should also learn to recognize equivalent expressions, solve linear equations, and use simple formulas.

So, a first negative answer to the question, 'why learn the solving of linear equations?', may be, certainly not because they are considered to be at the core of algebra (even if this is implicit in history treatises or school textbooks).

2. Regardless of the epistemological or historiosophical controversies about algebra and its origins, there is a rich history of methods for solving linear equations. This history tells us about methods such as 'false position', 'double false position', 'method of the scales' and *regula infusa* (see, for example, Smith, 1958; Boyer, 1985; Charbonneau and Radford, 2002). Students also generate methods of solution (*e.g.*, counting techniques, cover-up and undoing as described, for example, in Kieran, 1992, p. 400).

The creation of solution methods both in history and by students leads me to consider further parallels, in an attempt to see in the evolution of this topic an example of the assertion that "ontogeny recapitulates phylogeny". This phrase was coined in 1866 by the biologist and philosopher Ernst Haeckel (1834-1919), and it summarizes the disputed claim that the development of a single embryo of a species retraces the evolutionary development of that species. In our case, it would mean that the development of a mathematical idea for individual students retraces the evolution of that idea through the history of mathematics. Such parallels would indeed constitute an answer to our question, "Why?". A possible point of contact between historical methods such as 'the rule of false' and 'trial and error', which are natural for some students, seems, initially, to be "guessing" followed by "adjusting". However, the similarities seem to end there. Whereas the rule of false treats the guesses quite systematically by means of proportional reasoning, students' adjustments to guessed solutions seem to be guided by their number sense or lack of it.

If we pursue the comparison further, we will realise, for example, that history does not tell us about techniques such as undoing or cover-up, and students usually do not re-invent the rule of false. Therefore, we may conclude our brief and oversimplified attempt by claiming that solution methods generated throughout history are quite different from the usual methods generated by students. Consequently, we cannot assert that a reason for the study of linear equations is based on or inspired by parallels between history and psychology – these parallels do not seem to exist.

However, I do propose to look at the historical and psychological perspectives in a wider sense. A very human characteristic is apparent with this topic – problems requiring solution of linear equations, if properly and timely posed, can appeal to our curiosity and playfulness (see Smith, above, p. 582). This was the case in history, and it is the case with most students. Therefore, I offer a positive answer – the topic has the potential to become an attractive motivational entry point to algebra. Moreover, students are capable of generating idiosyncratic and informal answers and, by doing that, possibly developing self-confidence and a less menacing image of mathematical activity.

3. Connecting to what seems natural and aligned with students' intellectual curiosity, predilections and capabilities is in itself a very important consideration, but not the only one. Why else would we, as mathematics educators, place importance on the study of solving linear equations?

The two articles comment that the topic helps to highlight and support the learning of central ideas of algebra. Which ideas?

Brizuela and Schliemann describe students engaging with problems that:

> could be represented by an expression that included unknown and variable amounts on both sides of the equal sign, thus satisfying the constraints set for a particular vision of algebra as involving the use of algebraic syntax. (p. 39)

Elsewhere in their article they claim that:

> young students can learn to use algebraic-symbolic notation meaningfully […] while exploring problems in open-ended rich contexts. (p. 34)

This seems to be their main reason to pay attention to the solving of linear equations – connecting and using symbols in sense-making ways to represent and solve problems.

Dickinson and Eade (2004) make a similar point – by learning to solve linear equations with a powerful model, students can, with proper support, have the opportunity "to attach meaning to standard solution strategies" (closing words, p. 47).

Both articles, in their own ways, seem to agree that solving linear equations is an appropriate topic to act as a bridge between informal thinking and more formal symbolic ways of expression and solution. This in itself may justify the choice for such a topic as a starter for algebra. However, this choice does not imply that learning to solve linear equations is the only or even the preferred entry point to algebra.

Alternative approaches are based on and respect the same fundamental idea of bridging the formal and the informal. Yet, the claim is (see NCTM, 2000) that the learning of algebra would be better served by postponing the learning of solution methods of linear equations and subsuming them to the "big ideas": variation, generality and functionality (*e.g.*, Chazan and Yerushalmy, 2003; Arcavi *et al.*, 1990).

Solving linear equations – how?

> Visualization is the
>
> - ability,
> - the process, and
> - the product
>
> of
>
> - creation,
> - interpretation,
> - use of, and
> - reflection
>
> upon
>
> - pictures,
> - images,
> - diagrams,
>
> in
>
> - our minds,
> - on paper, or
> - with technological tools,
>
> with the purpose of
>
> - depicting and communicating information,
> - thinking about and developing previously unknown ideas, and
> - advancing understandings. (Arcavi, 2003, p. 217).

There seems to be agreement on the power of visualization in the teaching and learning of mathematics in general, even within highly symbolic topics (*e.g.*, Arcavi, 1994; Hershkowitz *et al.*, 2001). In this spirit, even without explicitly saying so, the two articles on solving linear equations seem to provide a first straightforward answer to the question "how?" – go visual!

And we can read more into them. For example, we realise that visual means are used not only as simple illustrations to depict a problem, but also as powerful "cognitive technologies" (Pea, 1987). Cognitive technologies are defined either as amplifiers of human mental capacities or as tools with which one can do new things that were impossible to perform before having such tools. Thus, a refinement of the first answer to "how'? would be – provide powerful cognitive technologies, rather than mechanical (and possibly meaningless) drilling techniques.

Borrowing constructs from two seminal research studies in physics education, I propose a further elaboration of our previous two answers to "how?":

> *Anchoring conceptions*: "[student] preconceptions that are largely in agreement with currently accepted theory" [2] (Clement, 1993, p. 1241)

> *Intermediate models*: "causal models represented at an intermediate level of abstraction" (White, 1993, p. 178) with which to introduce students to new ideas.

I would briefly propose that, in our case, the relevant *anchoring conceptions* (or intuitions) seem to emerge naturally by "*acting out* the problem" (Brizuela and Schliemann, 2004, p. 35, original emphasis) and would consist of the spontaneous "matching up" of amounts and "canceling" of equal terms. Matching up and canceling are procedures central to the solving of equations. The *intermediate model*, in our case, would be the number line [3], which can be considered as mid-way between verbal/mental solution methods and their full symbolic representation. Thus, if one properly harnesses students' anchoring intuitions and uses well-designed ad-hoc intermediate models, meaningful learning is likely to occur, as described in the two articles.

A concern about the long-term goal of reaching expertise may arise. This goal would consist, at least partly, of how to cause informal solution methods to evolve and become the automatic, efficient and flawless ways used by experts. In other words, is there any danger that intermediate models become permanent and thus impair the learning of the more general formal method? More studies of students beginning algebra are needed to investigate whether or how intermediate models serve as good springboards for further learning.

Solving linear equations – when?

More than forty years ago, Bruner proposed a startling hypothesis:

> […] any subject can be taught effectively in some intellectually honest way to any child at any stage of development. (Bruner, 1975, p. 33)

Bruner's hypothesis should be handled with care. A crucial point in his argument is that to be fully respectful of the children's ways of thinking:

> explanations based on the logic that is distant from the child's manner of thinking and sterile in its implications for him [… are] futile. (Bruner, 1975, p. 38)

So, "some intellectually honest way" may imply intensive and complex "didactical transposition" (Chevallard, 1985) in order to meet students' ways of thinking.

The two articles on linear equations are concerned with teaching the solving of linear equations to students younger than is usually the case and their attempts to do so can certainly be considered as honest.

Having granted that, some concerns still arise, *e.g.*

> Are our ways to evaluate what is distant from (or close to) the child's modes of thinking within a specific mathematical topic adequate?

> To what extent is the student's good performance on certain tasks a reliable indication that meaningful learning occurred in the whole topic?

> Couldn't it be that, in the process from the informal graphical equation solving to the teacher's introduction of symbols (see Brizuela and Schliemann's didactical process), students lose some of the meanings? May we delude ourselves by over-interpreting some of their actions? (I am not claiming this is the case in the study described, but it is certainly a worry worthy of intensive scrutiny.)

However, even when our concerns are justified, the conclusion is not necessarily an outright rejection of Bruner's hypothesis. I think that we need to undertake an painstaking theoretical (epistemological-mathematical) and empirical monitoring of how far we can stretch what is taught to students younger than usual.

Moreover, we may need to consider problems and costs. One practical concern is that when one teaches something, the time invested comes at the expense of other topics that are not taught. Another concern related to the cognitive/affective domain refers to a "spiral" curriculum (whose virtues are advocated by Bruner in the same chapter as the quotation). When, in later grades, students return to equations, they may develop a feeling of *déjà vu* typical of spiralling back to a topic (even if it was barely met before). This feeling creates the illusion of knowing, preventing, perhaps, a meaningful re-connection and learning in the new. [4]

One way to address these concerns would be the following. Playing with models for solving linear equations does not necessarily have to be followed immediately by a symbolic formalization – this could be left for later years (a slightly different version of "spiralling"). The main idea is tacitly to spread seeds of incipient algebraic thinking (in all its nuances and not only for solving linear equations) throughout the study of arithmetic, in the spirit proposed, for example, by Blanton and Kaput (2003). In such an approach, the ideas of algebra would be met implicitly, yet the symbols, the syntax and the overt procedures would be postponed.

This approach opens up a whole new terrain of potential disagreements by those among us who are concerned with respecting the mathematics no less than respecting students' ways of learning. A complaint is often heard that the informal/intuitive handling of mathematical ideas, which is not immediately followed by a symbolic formalization, may be educationally valueless and even harmful. I overheard some mathematicians and mathematics educators claiming that they would prefer to postpone (or even to take out from the curriculum altogether) the teaching of certain topics, if the aim was for informal treatment only.

In this respect, Sfard (2003) points to the unsolvability of central dilemmas:

> In our attempts to improve the learning of mathematics, we will always remain torn between two concerns: our concern about the learner and our concern about the quality of the mathematics being learned. Because of this ever-present tension, we are repeatedly thrown from one extreme solution to another. (p. 386)

Colophon

Perhaps, one general lesson to be learned (relevant to our topic, but also applicable beyond it), is to face questions and dilemmas in all their depth and scope, to sharpen them, and to continue the systematic search for more answers. A cautionary note is due here: even when many questions may be solved by research, still others are strongly imbued by deep epistemological beliefs and philosophical standpoints. Thus, we should continue our dialogues within ourselves and with all people genuinely concerned with mathematics education who are willing to learn the trade and enter the fray. We need to be mindful in order not to confine ourselves to the voicing of strongly held beliefs – such a practice only serves the pushing of the pendulum into either one of the extreme positions of the past.

I hope this communication serves to fuel the continuation of a productive dialogue which unfolds subtleties and complexities and avoids extremes.

Notes

[1] $x + (1/7)x = 19$

[2] As opposed to misconceptions.

[3] Other intermediate models worth discussing in this context are the *Lab Gear* (Wah and Picciotto, 1994, p. 211) and a proto-symbolic approach (Bruckheimer and Arcavi , 1999).

[4] A further discussion can be found in the papers, and subsequent reactions to them, at the Research Forum on "Early Algebra" in van den Heuvel-Panhuizen, M. (ed.), 2001, *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*, Utrecht, The Netherlands, Freudenthal Institute, **1,** pp. 129-159.

References

Arcavi, A. (1994) 'Symbol sense: informal sense-making in formal mathematics', *For the Learning of Mathematics* **14**(3), 24-35.

Arcavi, A. (2003) 'The role of visual representations in the teaching and learning of mathematics', *Educational Studies in Mathematics* **52**(3), 215-241.

Arcavi, A., Friedlander, A. and Hershkowitz, R. (1990) 'L'algèbre avant la lettre', *Petit X* **24**, 61-71.

Boyer, C. (1985) *A history of mathematics*, Princeton, NJ, Princeton University Press.

Brizuela, B. and Schliemann, A. (2004) 'Ten-year-old students solving linear equations', *For the Learning of Mathematics* **24**(2), 33-40.

Bruckheimer, M. and Arcavi, A. (1999) 'Of discs and cubes and magic squares: a sort of algebra', *The Australian Mathematics Teacher* **55**(1), 17-20.

Bruner, J. (1975, thirteenth edition, original 1960) *The process of education*, Cambridge, MA, Harvard University Press.

Charbonneau, L. and Radford, L. (2002) 'Crafting an algebraic mind: intersections from history and the contemporary mathematics classroom', in Proceedings of the 24th Annual Meeting of the Canadian Mathematics Education Study Group (CMESG/GCEDM), May 26-30, 2000, Montreal, Canada, Université du Québec à Montréal, 47-60. (also at http://laurentian.ca/educ/lradford/GCEDM.pdf)

Chevallard, Y. (1985) *La transposition didactique*, Grenoble, France, La Penseé Sauvage.

Chazan, D. and Yerushalmy, M. (2003) 'On appreciating the cognitive complexity of school algebra: research on algebra learning and directions of curricular change', in Kilpatrick, J., Martin, W. and Schifter, D. (eds), *A research companion to principles and standards for school mathematics*, Reston, VA, NCTM, pp. 123-135.

Clement, J. (1993) 'Using bridging analogies and anchoring intuitions to deal with students' preconceptions in physics', *Journal of Research in Science Teaching* **30**(10), 1241-1257.

Dickinson, P. and Eade, F. (2004) 'Using the number line to investigate the solving of linear equations', *For the Learning of Mathematics* **24**(2), 41-47.

Hershkowitz, R., Arcavi, A. and Bruckheimer, M. (2001) 'Reflections on the status and nature of visual reasoning – the case of the matches', *International Journal of Mathematics Education in Science and Technology* **32**(2), 255-265.

Blanton, M. and Kaput, J. (2003) 'Developing elementary teachers' "algebra eyes and ears"', *Teaching children mathematics* **10**(2), 70-77.

Kieran, C. (1992) 'The learning and teaching of school algebra', in Grouws, D. (ed.), *Handbook for research on mathematics teaching and learning*, Reston, VA, NCTM, pp. 390-419.

National Council of Teachers of Mathematics (2000) *Principles and standards for school mathematics*, Reston, VA, NCTM.

Pea, R. (1987) 'Cognitive technologies for mathematics education', in Schoenfeld, A. (ed.) *Cognitive science and mathematics education*, Hillsdale, NJ, LEA, pp. 89-122.

Sfard, A. (1995) 'The development of algebra: confronting historical and psychological perspectives', *The Journal of Mathematical Behavior* **14**(1), 15-39.

Sfard, A. (2003) 'Balancing the unbalanceable: the NCTM Standards in light of theories of learning', in Kilpatrick, J., Martin, W. and Schifter, D. (eds), *A research companion to principles and standards for school mathematics*, Reston, VA, NCTM, pp. 353-392.

Smith, D. (1958) *History of mathematics,* **2**, New York, NY, Dover Inc.

Wah, A. and Picciotto, H. (1994) *Algebra, teacher's edition*, Mountain View, CA, Creative Publications.

White, B. (1993) 'Intermediate casual models: a missing link for successful science education?', in Glaser, R. (ed.), *Advances in instructional psychology,* **4**, Hillsdale, NJ, LEA, pp. 177-252.

Early algebra: perspectives and assumptions

BARBARA DOUGHERTY

In FLM **24**(2),

> *Dickson and Eade* describe how a number line for solving linear equations can be used with 11-year-old children.
>
> *Brizuela and Schliemann* present a function approach to ten-year-old students as part of a method for solving linear equations.

Both articles suggest that students were successful with these methods, not only in applying them but understanding the rationale behind them as well.

Certainly, the authors of these articles present compelling arguments for placing algebraic topics earlier in the elementary mathematics curriculum. The methods for solving linear equations presented here are appropriate for younger children and offer the opportunity to gain an understanding of what solving an equation means.

However, both articles make the assumption that students understand equalities, quantified relationships, and units. Without these three understandings, the approaches suggested by the authors would result in either a rather algorithmic application of a function or a diagrammatic method for solving equations. Additionally, the articles intimate that solving linear equations is an add-on to what could be considered a traditional curriculum. That is, students' background in mathematics in the earlier grades would have focused on number and operations presented in a manner consistent with a conventional approach.