Communications

Solid geometry in the works of an iron artisan

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Ethnomathematics is a research field where the cultural relativism of mathematics is stressed. The term 'ethnomathematics' is credited to the Brazilian mathematician and educator D'Ambrosio (1988) who defined it as follows:

> We call Ethnomathematics the art or technique of understanding, explaining, learning about, coping with and managing the natural, social and political environment, relying on processes like counting, measuring, sorting, ordering and inferring which result from well-identified cultural groups. (p. 5)

Later, Bishop (1988) identified three new environmental sources of mathematics: playing, designing and locating. Looking again at the literature (Pinxten, 1997; Macpherson, 1987; Gerdes, 1999) shows that designing, locating, geometrical thinking, the conceptualization of space and even one's cultural world-view are closely linked.

Pinxten (1997) has studied the relationship between spatial conceptions, language and world-view in the Navajo. He found that the spatial geometrical thinking of Navajo is very different from the Western culture way of thinking. Macpherson (1987), working in an Inuit school of Northern Canada, found, through the works of an Inuit boy called Norman, that he had a considerable ability to deal with shapes and space without using numbers. Macpherson stresses the difference with Western culture when she says:

> From my cultural perspective which includes a highly developed numerical conceptualization used to describe features of shape, space and size, it is almost impossible to separate the descriptor (the number) from the feature (shape, space and size). (p. 25)

Gerdes (1999) has found interesting aspects of solid geometry in examining some African handcrafts like the funnel. In these works, Gerdes goes beyond the discovery of frozen mathematics and develops new mathematical knowledge with his students.

Iron and solid geometry

I am fond of designing wooden and iron toys. One day, in building a truck (see Figures 1 and 2), I needed a 28 × 14 × 10 (inches) skeleton of a right parallelepiped. I cut the edges out of iron tube and asked Luis González, an iron artisan, to weld them together. As a first step, Luis built two 28 × 10 (inches) independent rectangles. He then put them vertically along the 28-inch side on a table in such a way that the two rectangles were contained in parallel planes. Then he slightly connected the two rectangles using the four last

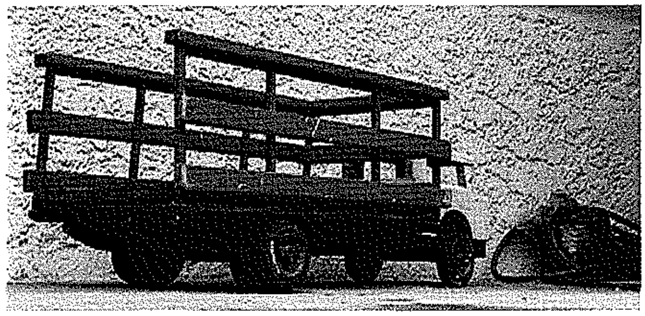

Figure 1: The wooden toy truck

edges. I observed each step attentively because I wanted to know how he was going to get the right parallelepiped. Finally, he slightly deformed the solid until two diagonals, those that began in the opposite vertices of one of the two 28 × 10 rectangles, became of the same length, thus the right parallelepiped was ready.

The knowledge required to build the skeleton for the right parallelepiped is not in the mathematical high school curriculum in Venezuela. Only some regular solids are built using nets and then without examining relationships of the cube or the right parallelepiped and their diagonals.

Luis never got a degree beyond high school. In this case, as Ferreira (1991) says, there is "mathematics encoded in the know-how" (p. 160)

Unfreezing mathematics

In what follows, a mathematical exploration is presented in order to unfreeze mathematics. This activity can be developed by advanced high school students or future teachers.

The problem is to construct a right parallelepiped from sticks using any method. After examining the different students' answers we can develop the following tasks

I. To build a right parallelogram as Luis did.

II. State Luis's procedure as a mathematical proposition.

III. Try to demonstrate the proposition

Figure 2: Angelo and the dog Peek in the toy truck
(Photographs by Gualberto Torrealba.)

IV. Luis was sure that at least two opposite sides of the hexahedron were rectangles. What happens if we eliminate this condition in III and all the sides are simply parallelograms?

Twist a cube, keeping two opposite sides parallel.

How many sides does the new solid have?

Can we come back to the original cube by equating diagonals?

Can we deform a cube in order to get a non-convex dodecahedron?

Acknowledgments

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References

Bishop, A. (1988) *Mathematical enculturation: a cultural perspective on mathematics education*, Dordrecht, Kluwer Academic Publishers.

D'Ambrosio, U. (1988) 'A research program in the history of ideas and in cognition', *ISGEM Newsletter* **4**(1), 5-7. (A compilation of these newsletters was published in 1992 by Patrick Scott, Albuquerque, USA.)

Ferreira, E. (1991) *Etnomatematica: in memorias del primer congreso Iberoamericano de educacisn matematica*, document number 42 in the collection *Enseqanza cientmfica y tecnolsgica*, Paris, UNESCO, pp. 160-165.

Gerdes, P. (1999) *Geometry from Africa: mathematical and educational explorations*, Washington, DC, MAA.

Macpherson, J. (1987) 'Norman', *For the Learning of Mathematics* **7**(2), 24-26.

Pinxten, R. (1997) 'Applications in the teaching of mathematics and sciences', in Powell, A. and Frankenstein, M. (eds), *Ethnomathematics: challenging eurocentrism in mathematics education*, Albany, NY, SUNY Press, pp. 373-401.

Three worlds and the imaginary sphere

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Gray and Tall (2001) recently suggested that mathematics can be split up into 'three worlds': the embodied, the proceptual and the axiomatic. They claim that objects from each of these worlds are formed, and reasoned about, in significantly different ways. During the course of this article I consider the three world's theory and present the view that, as currently described, it is not sufficient to characterise all mathematical objects.

In doing this, I discuss the investigations of Johann Heinrich Lambert into non-Euclidean geometry, and claim that, *for Lambert*, the so-called 'imaginary sphere' does not fit comfortably into any of the three worlds. Through a discussion of an example from the classroom, I suggest that, looking from the point of view of certain learners, many objects may not satisfactorily fit into Gray and Tall's worlds.

Lambert and his imaginary sphere

Johann Heinrich Lambert, a Swiss mathematician who lived in the eighteenth century, was highly influential in his day. His scope of influence was wide, ranging from philosophical ponderings on rationalism to the proposal that the Milky Way was finite. In mathematics, he famously proved that π and e were irrational. I shall be discussing his work relating to non-Euclidean geometry (Gray, 1989, is the source for the historical detail).

Ever since Euclid published his *Elements*, in around 300BC, mathematicians had been attempting to prove his fifth postulate using only the other four. The so-called parallel postulate was noticeably more complicated (and less obviously true) than the others, leaving mathematicians down the ages dearly wishing to reclassify it from postulate to theorem. However, in the 2000 years that passed between the time of Euclid and the 1800s, nobody had managed to produce a correct proof.

Instead of trying to construct a direct proof, Lambert went about the task somewhat differently. Following the work of Saccheri, he showed that there were three possibilities.

1. If there is a triangle with angle sum < 180°, then every triangle has angle sum < 180°

2. If there is a triangle with angle sum = 180°, then every triangle has angle sum = 180° (this is standard Euclidean geometry)

3. If there is a triangle with angle sum > 180°, then every triangle has angle sum > 180°

Lambert attempted to show that cases one and three were contradictory, which would imply that Euclidean geometry - and therefore the parallel postulate - was 'true'. He managed to do this with the first case, but deriving a contradiction with the third proved more difficult. We now know that this task is impossible. Mathematicians had to wait until the work of Bolyai and Lobachevskii (in the 1820s) who paved the way for Beltrami (in 1868) to show that the third hypothesis was as consistent with Euclid's first four axioms as the second.

Although Lambert did not successfully find a contradiction, he did manage to derive some rather interesting results. In particular, he found that the area of a triangle (with angles α, β and γ) in the third case was proportional to $\pi - (\alpha + \beta + \gamma)$.

Recalling that the area of a triangle on a sphere of radius r is: $r^2 (\alpha + \beta + \gamma - \pi)$, he noticed that letting $r = i$ gives: $i^2 (\alpha + \beta + \gamma - \pi) = \pi - (\alpha + \beta + \gamma)$, which is the formula Lambert derived for the angle sum under the third hypothesis. He considered this for a while before making the following observation.

> I want to say: if of two triangles one has a greater area than the other then the angle sum of the first triangle is smaller than that of the other [...] I should almost therefore put forward the proposal that the third hypothesis holds on the surface of an imaginary sphere. (Fauvel and Gray, 1987, pp. 518-520)

Gray (1989) comments on the significance of this remark:

> it marks [Lambert] as a correct and inspired thinker [...] To enter the land which Lambert's vision was the first to descry was to take mathematics another hundred years. (p. 75)