Communications

Continuing conversations towards the horizon

RINA ZAZKIS, AMI MAMOLO

In the November Issue of *For the Learning of Mathematics* two responses to our article "Reconceptualizing knowledge at the mathematical horizon" were published, by Foster (2011) and by Figueiras, Ribeiro, Carrillo, Fernández and Deulofeu (2011). We appreciate the opportunity to continue the conversation. It is always interesting to read different interpretations of our ideas, particularly when they may not coincide with our intended meaning. We take this opportunity to clarify our position.

On knowledge at the mathematical horizon

In a nutshell, building upon the work by Ball and Bass (2009) and on Husserl's notion of horizon, we consider teachers' use of mathematical subject matter knowledge—acquired in undergraduate studies—in teaching situations as an instantiation of knowledge at the mathematical horizon (KMH). We explored KMH in terms of concepts (inner horizon), connections between concepts (outer horizon), and major disciplinary ideas and structures (outer horizon) as applied to ideas in elementary or secondary school mathematics. Our article focused on one particular aspect of teachers' knowledge, and illustrated how such knowledge can influence pedagogical choices.

On peripheral knowledge

Foster (2011) introduces a construct of peripheral mathematical knowledge. This is indeed a novel construct that sparked our interest and motivated us to consider our examples through such a lens. We invite Foster to elaborate further, taking into account Ball and Bass's (2009) description of knowledge at the mathematical horizon as "a kind of peripheral vision" (p. 5), which is in accord with our interpretation of "inner horizon" (Zazkis & Mamolo, 2011, p.11). We are interested in teaching situations where such knowledge is used in "cushioning and supporting the learner's mathematical trajectory upwards" (Foster, 2011, p. 24).

As an example of peripheral knowledge, Foster suggests "mathematical trivia"; however, we see a need to distinguish "trivia facts" from examples based on mathematical algorithms or theorems. The former belong to the category of "knowing that" (*e.g.,* Foster knows that $2^{10}=1024$), and the latter to "knowing how" (*e.g.,* Foster knows how to generate quadratic equations that factorise). The latter, we believe, belongs to the realm of pedagogical knowledge when it is used by teachers in choosing practice problems for their students.

On the pentagon

Mrs. White was introduced in our article through an example in which she used her KMH to check students' work when counting the number of triangles generated by all the diagonals in a pentagon. Foster attributed to Mrs. White a conjecture regarding the number of triangles in an *n*-sided polygon. However, Mrs. White did not make any conjectures. She made a specific determination in one particular case, noting a relationship between rotational symmetry and the number of triangles appearing in the figure. Foster introduced a conjecture that attempted to generalize Mrs. White's reasoning and provided numerous examples to refute it. He explained that his suspicion of the falsity of the conjecture was affirmed when he "noticed that sometimes more than two lines pass through a point… [and] triangles would be lost" (p. 25). In contrast to Foster's approach, we believe that when focusing on the hexagon, Mrs. White would have noticed that as a result of rotation, some triangles (exemplified in Figure 1) are mapped onto themselves. Therefore, unlike the case of a pentagon, not every triangle will have 6 congruent images.

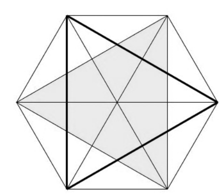

Figure 1. Two triangles—one shaded, one bolded—map onto themselves by 120° rotation.

Figueiras *et al.* (2011) noted that Mrs. White's knowledge of symmetry helped her to solve the pentagon problem, yet they were critical of her for imposing a particular strategy on her students, expressing a "doubt whether this way of using advanced knowledge […] allows teachers to build on students' knowledge or to interpret alternative solution paths implicit in students' answers" (p. 26). As we did not observe Mrs. White and learned about her approach only from her report, we suggest that the only data-driven conclusion is that Mrs. White directed students to systematic counting, which is an important problem-solving strategy, and which is, in our opinion, definitely attainable by 9 year olds. Figueiras *et al.* go on to note that advanced mathematical knowledge may be important for teaching, but that it is not enough to deal with the complexities of teaching mathematics. We fully agree. Our article illustrates how KMH "can contribute to teachers' instructional choices and be potentially beneficial for students' learning" (Zazkis & Mamolo, 2011, p.13). We doubt Figueiras *et al.*'s suggestion that introducing students to such a strategy would reinforce the preconception that problem solving is unattainable.

On the exponent (-1)

Figueiras *et al.* suggested that "Students' confusion about $1/f(x)$ and $f^{-1}(x)$ is well known and the teacher should have heard about it in any course about teaching analysis" (p. 27). It is possible that Spanish teachers are better educated than

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teachers in Canada, as we are unaware of any institution offering as part of their secondary school teacher education a course on "teaching analysis". In our informal survey, only 2 out of every 10 teachers explained the situation similarly to Miss Mauve, referring to inverse with respect to an operation. The most frequent response was that the case of exponent (-1) is not the only place where the same symbol is used for different purposes and the interpretation is context-dependent. Perhaps a survey of Spanish teachers would show a different trend and we invite our colleagues to share their results. Further, we do not claim that Miss Mauve was surprised about the mistake; we only reported her skillful care of the situation.

On horizon knowledge again

Foster (2011) writes: "the teacher's horizon knowledge could be regarded as mainstream mathematical knowledge [...] There is no reason to believe that the results referred to would be of more use to a mathematics teacher than to any other user of mathematics" (pp. 25-26). We maintain, based on our teaching and research experience, that the knowledge of mathematical connections and disciplinary structure exemplified in our article is not "mainstream". Furthermore, we see such knowledge as particularly useful for mathematics teachers, while it may also be helpful to other users of mathematics such as statisticians, engineers, carpenters, or bankers, should they decide to engage in instructional activities.

Figueiras *et al.* (2011) write: "they articulate all their reflection around the premise that mathematical teaching problems, and thus theoretical outcomes in the field of mathematics education, should be subordinated to the problem of teachers' learning of advanced mathematics" (p. 28). We were quite surprised to see such a statement. Indeed, we agree that "our professional task of teaching mathematics to primary and secondary students, as well as to future elementary and secondary school teachers, requires a much broader perspective on the nature of knowledge" (Figueiras *et al.,* 2011, p. 28). We reiterate that our article explored one particular aspect of teachers' subject matter knowledge and we made no suggestions regarding "theoretical outcomes in the field of mathematics education." Our article exemplified several teaching scenarios in which KMH is useful, presented a refined perspective on horizon knowledge, and argued that extended experiences of learning mathematics can be functional in teaching.

References

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Thinking about a mathematics for mathematics teachers

MERCEDES GARCÍA, VICTORIA SÁNCHEZ

Two comments from the November 2010 issue of *For the Learning of Mathematics* prompted this communication. One of them was made by Mason, who points out that:

> Many recent papers are so heavily theory-laden in the opening sections that by the time I get to the substance I have forgotten exactly which parts of which theory are actually being employed, and, indeed sometimes it is not even very easy to detect this. It seems to me that often only tiny fragments of theoretical frameworks are called upon. (Mason, 2010, p. 8)

In the other, Kajander and colleagues affirm that:

> Accepting the possibility that math for teaching is a particular mathematics sub-discipline, like math for engineers or carpenters or citizens, requires the dismissal or at least questioning of the premise that the math that teachers need will be a subset of mathematics that mathematicians value. (Kajander *et al.*, 2010, p. 56)

We agree with Mason that sometimes too much theory is included in mathematics education research papers. Nevertheless, we also believe that on many occasions, theory is lacking in the design of mathematics teacher education programmes. It is precisely the adoption of a theoretical framework that allows us to answer the question put forward by Kajander and colleagues:

> Can mathematics for teaching be viewed simply as the overlap of the two realms [the realm of academic mathematics and the realm of mathematics teaching in schools], or is it something that has yet to emerge, something that will encompass the full realities of both of its parent realms? (Kajander *et al.,* 2010, p. 56)

In this communication, we outline a proposal that responds to this question.

Developing a proposal of mathematics for teachers

In the design of our own teacher education programme, we have taken a situated perspective with respect to mathematics. This perspective leads us to think of mathematical knowledge as being what a teacher should know in order to be able to join the mathematics teacher community as a full member (Sánchez & García, 2008, 2009). For us, this implies the identification of intrinsic mathematical knowledge.

We characterize intrinsic mathematical knowledge as knowledge that considers mathematical elements from a dual point of view, integrating "operational" and "structural" aspects (Sfard, 1991), as well as the mathematical competence used in analysing these elements. This competence makes possible the "packing and unpacking" (Ball, Thames & Phelps, 2008) of the mathematical elements. People who