UNSOLVED PROBLEMS IN PLANE GEOMETRY

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There are many charming unsolved problems in plane geometry [1]. The charm of these problems relies on two main facts: (i) anybody with an elementary education can understand their statement and (ii) if they are so accessible and yet remain unsolved, it is because they require the development of new techniques or viewpoints. Their statements may be elementary, but their solutions are not.

In teaching mathematics, these problems may be of interest at many educational levels. They allow students to put their knowledge in context with respect to the borders of knowledge of the scientific community. Such problems can be explored at many different depths. And students may also learn something about how research actually works.

I have worked with such problems in science courses in vocational training programs, in a wide range of courses at secondary school and in first year university courses. In every case students have been enthusiastically involved and, I hope, they have learned something.

Here I present one problem as an example: Kobon's Triangle Problem [2]. This problem is thought to have first appeared, with its definitive statement, in the Kobon Fujimura's 1978 book of puzzles.

> Determine the largest number $K(n)$ of (nonoverlapping, uncut) triangles that can be drawn using only n straight lines in the plane.

Obviously, it is not possible to draw any triangle with 1 or 2 lines. We can obtain 1 triangle with 3 lines. The reader may easily find a configuration with 4 lines and 2 triangles (see Figure 1) and another one with 5 lines and 5 triangles.

The sequence $K(1)$, $K(2)$, $K(3)$, … is called *Kobon's Sequence*. The terms $K(1)$, … , $K(9)$ are: 0, 0, 1, 2, 5, 7, 11, 15, 21. $K(10)$ is the first term of the sequence that is not known.

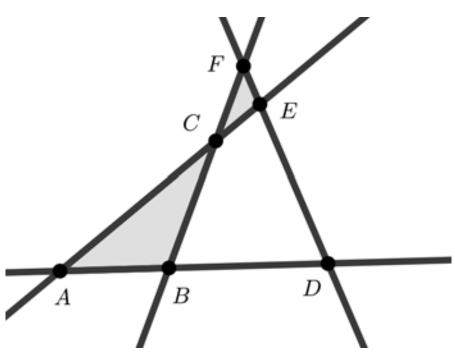

Figure 1. Four lines forming two triangles. We are not counting the triangle ADE because it is cut by the line through B, C.

There exist some general constructions that offer an arrangement of lines with a large number of triangles for each n. These give lower bounds for the numbers $K(n)$. One of the best general constructions known is the one by Füredi and Palásti (1984) (see Figure 2, facing page). It gives the lower bound:

$$\frac{n(n-3)}{3} \leq K(n)$$

But it is not easy to prove that a given configuration is optimal. A possible strategy, which has been used successfully for some values of n, is to find some upper bound $M(n)$ for the number $K(n)$ and then to find a configuration with n lines and $M(n)$ triangles. One of the results that have made this way of action possible is by Clément and Bader (2007), who found an acceptable upper bound for the numbers $K(n)$:

$$K(n) \leq \begin{cases} \lfloor \frac{n(n-2)}{3} \rfloor & \text{if } n \equiv 0,2 \mod 6 \\ \lfloor \frac{n(n-2)}{3} \rfloor & \text{in other cases} \end{cases}$$

Unfortunately, this strategy is not always satisfactory; frequently, there is a gap between the best known configuration and the best known upper bound. This is the case, for example, of $n = 10$. The best configuration known with 10 lines has 25 triangles. But the bound by Clément and Bader only guarantees that $K(10) \leq 26$. Maybe the bound is not tight enough, or it is possible to find a configuration with 10 lines and 26 triangles?

Some values of $K(n)$ are known for $n > 10$. The reader will find an updated survey concerning the state-of-the-art of this problem in Moreno and Prieto-Martínez (2021).

Notes

[1] See, for example, Croft, Falconer & Guy (2012) and Klee & Wagon (1991).

[2] Additional examples will be printed in subsequent issues, as space permits.

References

Clément, G. & Bader, J. (2007) *Tighter Upper Bound for the Number of Kobon Triangles.* Preprint available online at: https://oeis.org/A006066/a006066.pdf

Croft, H.T., Falconer, K. & Guy, R.K. (2012) *Unsolved Problems in Geometry: Unsolved Problems in Intuitive Mathematics,* Vol. 2. Springer Science & Business Media.

Fujimura, K. (1978) *The Tokyo Puzzles.* Charles Scribner's Sons.

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Klee, V. & Wagon, S. (1991) *Old and New Unsolved Problems in Plane Geometry and Number Theory.* Mathematical Association of America.

Moreno, J.P. & Prieto-Martínez, L.F. (2021) El problema de los triángulos de Kobon. *La Gaceta de la RSME* **24**(1), 111–130.