and takes place in Belgium, France, Italy, Luxembourg and Switzerland. It is organised by the Association Transalpine Mathematical Rally, [...], whose bylaws state, among other things: the Association Transalpine Mathematical Rally is a cultural association whose objective is to promote problem solving to improve the learning and teaching of mathematics by means of a comparison between classes. [...]" (our translation). More information can be found at the website.

[4] This institute designs and organises the national evaluation tests for the Italian school system. See https://www.invalsi.it/.

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Communications

'To Have or to Be' in mathematics education

CHRISTOPH ABLEITINGER

Erich Fromm wrote the book 'To Have or To Be' in 1976 as a socially critical work and it quickly became a bestseller. Its critique of consumption and capitalism is more relevant today than ever, precisely because of the painful awareness that the earth's resources are dwindling. In his work, Fromm explains how the pursuit of possessions, which has dominated Western culture for decades, leads to competition, greed, power relations, fear of losing possessions and ultimately to a society of notoriously unhappy people. He calls this the *mode of having*. Fromm contrasts this with the *mode of being,* in which possession does not play a decisive role. He refers to many philosophical and religious approaches that also propagate this attitude as life-enhancing and psychologically and emotionally healthy.

Fromm first explains this using the example of material possessions. He criticizes the throwaway mentality of Western society, the uncontrolled pursuit of private property and power. In our society, the identity of individuals is not determined by what they *are*, but by what they *have*. He does not demonize possession *per se,* but contrasts the *mode of having* with an attitude that, for example, focuses on maintaining existing possessions and is characterized, among other things, by a productive inner activity in which the use of one's own abilities and talents is central.

The difference between the *mode of having* and the *mode of being* is then discussed more broadly and also applied to other societal areas. Using the example of knowledge, the *mode of having* means acquiring as much knowledge as possible in order to pass an exam, for example. Conversely, the *mode of being* is about productive listening, about genuine interest in the subject. But more on that later. The difference between these two attitudes can also be seen in seemingly banal situations, for example when handling a beautiful flower. While picking the flower, taking it home and putting it in the vase, puts 'taking possession' in the foreground, *i.e. having,* the attitude of *being* means merely looking at it, enjoying it and being aware that other passers-by can also enjoy the flower if you do not pick it.

Fromm pleads for a radical humanism that really puts the human being at the center. For example, work should serve the true fulfillment of human needs and not industry and the economy as a capitalist end in itself. Society should be characterized by solidarity, reasonable consumption, cooperation with nature and the pursuit of human well-being. Personal relationships and inner fulfillment are more important than owning things. He makes suggestions for such a transformation, which ultimately has to begin with changing the inner attitude of the individual.

Personal preface

This book made a deep impression on me and I could not help but think about what this means for learning and teaching mathematics in schools and universities. I want to take Fromm's metaphor as a starting point and use it as a lens to look more closely at common practices in mathematics education. I do not want to judge or evaluate, start a revolution in mathematics education and force anything to change. The text also does not want to make any statement about how strongly the two attitudes of *having* and *being* are currently represented in mathematics lessons. It is only intended as an impetus to reflect on our attitude as mathematics educators and to illustrate Fromm's fundamental thoughts using selected examples of mathematics learning.

Elaboration of some examples

The *mode of having* could, of course, first of all be related to the acquisitive perspective ('mathematics as a product' to be acquired). Is mathematics education about learning solution methods as quickly as possible, which then can be used for solving problems of a certain type (*e.g.,* the calculation rule for adding two fractions or the differential calculus for determining local extrema)? Or is it about giving space to doing mathematics, understanding its inner structure, recognizing relationships? Then the focus is on thinking, investigating, arguing and justifying and thus 'mathematics as a process'. In Fromm's language, this supports the *mode of being.* Of course, no mathematics lesson is black or white in this respect, which of the two modes is represented to what extent is more likely to be located on a continuum in between.

However, this transfer of Fromm's concepts to the learning of mathematics is quite shallow and comparable to the reduced view that Fromm's philosophy is exclusively a critique of capitalism. Rather, his metaphor refers to many areas of society and also to individual actions and thoughts. The same applies to the transfer to learning mathematics. Therefore, I now want to dig a little deeper and also take a look at aspects of mathematics teaching where the application of Fromm's terminology is not immediately obvious. I bring examples on three different levels, starting out quite general and then becoming more and more specific.

The first example concerns interpersonal contact in the mathematics classroom. Dependent relationships can arise between students and teachers when teachers use praise and blame tools to lead the students to behavior that is desirable or that should be avoided. This often leads to an implicit competition among the students for the favor of the teacher in a capitalistic manner. Aside from that, there is also an asymmetrical relationship between those who praise or blame and those who are praised or blamed. The former *has* a certain power over the latter due to its evaluation (both *mode of having*). On the other hand, it has been proven that praise has a positive effect on motivation and thus also on competence development (Kamins & Dweck, 1999). This can contribute to self-development and joy and thus to the *mode of being.* I just want to point out here that we as mathematics educators have a responsibility for what we praise (the person, behavior, competences) and to what extent and with what goal we do it. Is the praise manipulative or is it characterized by inner joy being expressed? Ultimately, a goal of mathematical education can be to deal with topics, content, and problems—regardless of praise and criticism from others—because I am interested and enjoy them or the solution obtained contributes to personal or general well-being. Then I find myself primarily in the *mode of being.* If I stop educating myself without the praise of others, it shows that I was only interested in acquiring appreciation.

The second example concerns a meta-competence in doing mathematics, namely creativity. The school subject mathematics undisputedly has the potential for creative activities (Leikin & Pitta-Pantazi, 2013). One might spontaneously assign creativity to the *mode of being.* However, creativity is only an expression of *being* in Fromm's sense if the creative aspect (as an end in itself, for inner development and personal expression) is in the foreground (Lauster, 1990). If, on the other hand, it is understood as a sophisticated technique for product optimization and profit maximization and is thus instrumentalized for the success of a product, then even creativity serves to *have*. Creativity naturally plays a major role in mathematical problem solving (Pólya, 1973). The focus is on thinking, exploring, guessing and justifying and thus on the *mode of being.* Conversely, the acquisition and flexible retrieval of a repertoire of strategies and heuristic skills is a necessary prerequisite for creativity. The *mode of having* also plays a role here. The mathematical work itself cannot be assigned to one of the two modes in binary. One level above—*i.e.* when one asks for the motivation for problem solving—the two modes play a role again. Do you solve mathematical problems because you enjoy doing it? Or are you extrinsically motivated by the fact that you can compete with others in a problem-solving contest and get a medal at the end? Creativity as a useful prerequisite for problem solving and the process of finding a solution *per se* cannot be assigned to either mode in a binary manner. The inner attitude thus determines which of the two is pronounced and to what extent.

Now a final example at the level of a specific task: "Determine the value of p for which the equation $x^2 + px - 12 = 0$ has the solution set $L = \{-2; 6\}$" was asked in 2016 at the central school-leaving exam in Austria (BMB, 2016). This task can be solved efficiently by simply inserting $x = -2$ into the equation and determining p from it. In this case you would have received the full score, even without having gained any deeper insights. The focus here is on *having* and that is understandable during an exam situation. However, this task has the potential to make deeper thoughts about connections. There are many different ways to get the correct result $p = -4$. Here is a small selection:

- You could check whether substituting $x = 6$ leads to the same result and think about why that is.
- If you interpret the left-hand side of the equation as a term of a quadratic function, then the graph of this function has the zeros −2 and 6. The apex $x = -p/2$ is then in the middle of the two, at $x = 2$, and hence, $p = -4$.
- You could use dynamic geometry software and plot the graph of the quadratic function where p can be changed with a slider.
- Vieta's theorem yields $p = -(x_1 + x_2) = -4$.

Seeing this diversity, appreciating it, relating the solution strategies to each other—all of this can trigger joy and amazement, strengthen the conceptual network and thus put the students in the *mode of being*. The focus is on the thing itself, not an external goal.

A tool for reflection

The transfer of the metaphor of *having* and *being* is not about finding a binary association between aspects of learning mathematics and the two modes. Rather, these terms can serve as a lens for reflecting on one's own behavior as a teacher in the classroom. Ultimately, it can be questioned in every situation whether the *mode of being* or the *mode of having* is dominant. Is it about the acquisition of mathematical procedures, about a performance-oriented acquisition of skills that sees the students in a competitive struggle? Or do you focus on doing mathematics yourself, immersing yourself in a mathematical problem and feeling joy solving it, a non-judgmental learning environment, disciplined thinking in the literal sense, establishing relationships within and outside of mathematics?

This reflection should not and cannot have the aim of classifying one's own teaching or attitude on a scale between *having* and *being* using objective criteria. Rather, it is about finding a good way for the learning process between *having* and *being*, creating a humane learning environment and becoming aware of the possibilities that we as teachers can use by adapting our attitude.

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When a few is the right number

SEAN CHORNEY, CANAN GÜNEŞ, NATHALIE SINCLAIR, SHEENA TAN

In his recent article in issue **42**(3), Colin Foster raises the question of how many representations (or models) are required or desirable for learning mathematical concepts. Based in part on the problem of the representational dilemma (Rau, 2017), he argues for prioritising the number line as a singular coherent representation for multiplication. We read this with great interest since we have been working for the past five years on a multi-touch application called *TouchTimes* (Jackiw & Sinclair, 2019), which provides novel and multiple representations of multiplication. We were thus provoked by several aspects of Foster's argument, which we address below. In so doing, we highlight some of the ontological and epistemological assumptions underlying Foster's argument and show how different assumptions about the focus of learning and the nature of concepts would lead to different conclusions. We note, as well, that there is an underlying axiological assumption being made by Foster, which we will make clear at the end of this communication.

Are representations separate from or part of the concept?

As part of Foster's argument for proposing one representation, he raises the issue of the representation dilemma, which refers "to the problem that for students to be able to make use of a representation in order to learn some mathematics they need a certain level of familiarity with that representation" (p. 21). The very formulation of this dilemma assumes that familiarity with a representation *precedes* the learning from the representation. This ontological view is based on the assumption that a representation is separate from a concept. Maffia (2023), in his communication piece in **43**(1) responding to Foster, cites his grounding in a semiotic approach which also assumes that the representation is separate from the object (multiplication in this case). The separation of representation and concept is a view that has been articulated many times with respect to student learning using digital tools, namely, that students first need to learn how to use the tool before they can learn the concept. In a historical, material view of mathematics, the tool and the concept are not ontologically distinct—the very idea of a circle, for example, is intertwined with the strings and compasses and cans that are used to produce circles. From these perspectives, learning to see a representation, to work with it, is part and parcel of developing mathematical conceptual awareness.

We align with Maffia's point that being introduced to many representations is how one learns multiplication; however, where Maffia suggests it is to be able to differentiate between the representation and the object, we see the learning of multiplication as emerging by sewing together experiences from multiple representations. It is Vergnaud (2009) who suggests a concept is not an "explicit object of thought" (p. 93) but comes from a "variety of situations which requires children's meeting and being faced with contrasting situations" (p. 86).

Going back to Rau (2017)'s notion of representation dilemma, four representational competencies (visual understanding, visual fluency, connectional understanding and connectional fluency) were proposed to be important for students to develop in order to learn mathematics with visual representations and to navigate the representation dilemma.

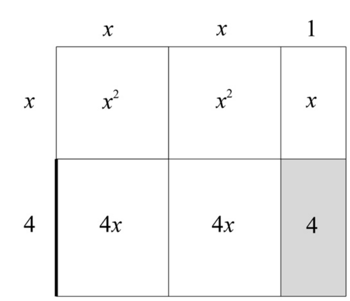

Figure 1: The area model that represents (2x + 1) × (x + 4)