Mathematical Fluency: the Nature of Practice and the Role of Subordination

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There has been considerable debate recently in the UK about claims that students are arriving on mathematics degree courses without that same fluency in calculus and algebraic skills as students had many years ago A joint report from the London Mathematical Society, the Institute of Mathematics and its Applications, and the Royal Mathematical Society [LMS *et al*, 1995] says that there is *a serious lack of essential technical fluency—the ability to undertake numerical and algebraic calculation with fluency and accuracy* [p 2] as one of three problems they highlight. This follows a series of articles in the national press [for example, Barnard and Saunders, 1994; Ernest, 1995], and a similar debate where there were claims that the introduction of GCSE, replacing "O" level and CSE, meant students weren't so fluent in some algebraic skills taken to be pre-requisites for an "A" level pure mathematics course. Tahta [1985] has commented, with reference to notation, *We do not pay enough attention to the actual techniques involved in helping people gain facility in the handling of mathematical symbols* [p 49]. The joint report from LMS *et al* [1995] calls for .. *an urgent and serious examination of what levels of "traditional" numerical and algebraic fluency are needed as a foundation for students' subsequent mathematical progress, and how such levels of fluency can be reliably attained* [p 14, their emphasis] I consider traditional ways in which attempts have been made to help students become fluent, and offer a model for ways in which fluency can be achieved with a more economic use of students' time and effort than through the traditional model of exercises based on repetition Examples of impressive learning from everyday life can offer insight into possible ways forward inside a mathematics classroom and I begin with an example of impressive learning that we have all achieved (unless we have suffered an accident, illness or a disability which has prevented us. In which case an equally impressive alternative can be substituted).

Impressive and not so impressive learning

> If you want to practise walking ..
> then start learning to run.

I am fairly good at walking, but this was not always the case. I cannot remember now, but I observe babies who cannot walk and conclude that there was a time when I could not walk either. Walking is something that young children learn Up to one point in time, a child has never walked. Then they walk. Having walked once does not mean that they can always walk; the next time they try, they may fall down after the first step Practice is required, and it is effective practice since walking is an example of impressive learning—it is retained for the rest of a child's life (unless an unfortunate accident or illness determines otherwise).

Children, having learned to walk, are not content to continue just walking. They want to walk on walls, walk on kerbs, walk missing the cracks on the paving stones, walk up and down stairs, they want to run The practice of walking is not just done by continually walking along a plain, flat area. The practice of walking is done by *subordinating* walking to some other task Such a task as walking up and down stairs can be understood by a child even though that child may not be a proficient walker The child may be carried up and down stairs in the course of a day, or have already engaged in the different task of crawling up and down stairs. So, that child can consider the combination of two things—getting up and down stairs (which they already know about)—with the new activity of walking. This means that walking does not have to be mastered before such tasks can be considered and engaged upon. Walking is not a pre-requisite for *comprehending* the task. Yet walking is subordinate to carrying out the task, since the task is *walking* up and down stairs. In this way, a child may engage in the task of walking up and down stairs before they are particularly successful at walking along a flat floor. Likewise, children can comprehend the task of trying to run before they are proficient walkers. Children have already got experience of the dynamic of *speed* within such activities as making noises, or moving an arm. So, they can consider the combination of speed (which they already know about) with the new activity of walking. In fact, children, in engaging with the practising of walking, can combine walking with any other dynamic of which they already have experience, even though they may not be proficient at walking on a flat floor. Many parents of young children report noticing their child trying to do things before that child has gained the skills required to achieve those things Pimm [1995, p29] quotes Robert Browning as saying *Man's reach should exceed his grasp, or what's a Heaven for?* In fact, I will argue that it is precisely because a child may engage in tasks beyond the skills they currently possess, that they become good at those skills, and become so skilled that they can do them with little or no conscious attention being given to the skills themselves. Pimm [ibid] points out that *Children are very good at practising certain things until they have mastered them—they are willing, it seems, to pay the necessary*

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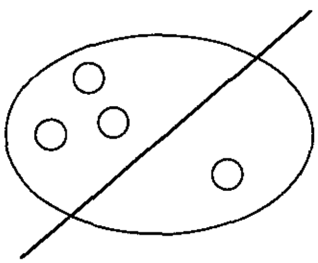
attention [p 177]. Yes, children practise, and children do pay the necessary attention, but what is important is *where* children place their attention. I suggest that children become good walkers precisely because they do not spend their time attending to the act of walking along flat floors and nothing else. It is because they place their attention in tasks which subordinate walking: they try to walk up stairs; they try to walk along walls; they try to run.

In contrast to such effective practice, I once asked a class of 14-15 year olds what they wanted me to teach them. After much discussion, a clear majority of the class wanted me to teach them how to add up fractions. I asked them whether they had ever been taught how to add up fractions before. *Yes* was the answer. I found out that they had all been taught to add up fractions in their primary schools and at least another twice in their current secondary school. Yet here I was going to waste their time for a fourth time, when more than likely they would learn it for a while and forget it later. Many of these students said that they had understood at the time of being taught, but could no longer remember now. There is so much that is taught apparently successfully at one time—success being measured, for example, by the fact that students can correctly answer an exercise of fractions to be added together. Yet, this learning is temporary and, unlike the skill of walking, is forgotten and needs to be re-taught at another time. Desforges [1987] gives the following example:

> Time was often spent practising a procedure until the teacher was satisfied that the children understood the underlying idea. In her attempt to get children to understand subtraction Mrs. D. had them read out subtraction "number stories" from diagrams [like the one below].
>
> 
>
> After some initial hiccoughs all the children were able to say, "four take away one leaves three" in response to this diagram. Mrs. D. concluded that they, "had got a good grasp at last". In fact post-task interviews showed that just fifteen minutes later and without Mrs. D.'s conducting they could not reproduce their behaviour. [p 117].

If the skill of understanding subtraction from such diagrams, or adding fractions, needs to be taught again, then the time and effort taken up during the first attempt to learn has been wasted with regard to the learning of that particular skill. It is not an economic [Gattegno 1971, 1986; Hewitt, 1994] use of the students' time if they are being asked to give up their time to learn something only to need to give up more time in the future to learn it again.

My students had succeeded in remembering how to add fractions long enough to be successful at some exercises. Then they equally successfully forgot. *Remembering for now* leaves the learners mortgaged to the teacher; they will have to come back for more at a later date. *Learning for life* enables progression. It breaks the need for the learners to return to the teacher and enables them to be free to seek new, more demanding tasks, secure in the knowledge that what has been learned will always be available.

The nature of practice

Practice is clearly required for something new to become something which is known so well that it can be used when little or no conscious attention is given to it. However, there are many times when the carrying out of repetitive tasks through a series of questions in a traditional exercise does not succeed in helping that skill be retained beyond a relatively short period of time, the fraction example being indicative of this. So, what is significant about effective practice which repetition does not offer? Consider a traditional exercise within a textbook currently used by some schools in the UK. Although taken from a particular book, this is an example of a type of task which might be in many other text books, verbally introduced or written on a board, etc.

In *ST(P) Mathematics 2B* [Bostock, L; Chandler, S; Shepherd, A; Smith, E, 1991], there is an exercise on place value which begins with the following example:

> In the number 627, find the value of
> a) the 6 b) the 2
>
> a) The 6 has the value 6 hundreds, i.e. 600
> b) The 2 has the value 2 tens, i.e. 20.
> [p 20]

This is followed by four questions:

> 1. Underline the hundreds figure in 524. What is its value?
> 2. What is the value of the 5 in 745?
> 3. What is the value of the 9 in 497?
> 4. In 361, find the difference between the value of the 6, and the value of the 3.
> [p 20]

There is little help here for a student to learn about place value, so there is an assumption that a student already knows about place value to some extent. There is little attempt to teach, rather this exercise offers an opportunity to state what each digit represents within a number. A notion behind such an exercise is that *practice makes perfect*, or, perhaps more appropriately, *repetition makes perfect*. The more someone repeats saying what a digit represents within a number, the more they will become "perfect" at this. Yet this kind of exercise is what my students reported having had over many years in their mathematics classrooms with the addition and subtraction of fractions. At the time, most students had reported being successful in the questions they did. However, the repetition they did at the time was not sufficient for this success to last for an extended period of time

One improvement in this situation could be for more repetition to take place. Perhaps an exercise of 100 questions rather than four And perhaps this could be done more frequently so that, as well as practising 100 questions over one week, this would be followed by a lesson once a fortnight with similar types of questions. Although this argument may offer the possibility of helping students to become fluent in adding and subtracting fractions, or knowing the value of digits in a number, there are several fairly obvious problems Even ignoring many issues such as potential boredom and the carrying out of mechanistic procedures rather than working on understanding, there is simply not enough time for students to be continually carrying out work of this type If every taught skill requires regular sets of questions, then there will come a time when there is no time left for students to engage with anything new. Furthermore, repetition is designed to help students to stand still—to keep something they have learned and not to go backwards and forget This seems quite different to the practising that young children do when learning to walk. With walking, yes, there is lots of practice, and there is regular practice—every day However, it is not mere repetition, the practice is practice where the children also go forward—they are progressing (walking on walls, running, etc.). It is not repetition in order to stay still but practice whilst moving forward and progressing in their learning This is what I call *practice through progress*—practising something whilst progressing with something else. For example, consider a different task which is still concerned with knowing the value of digits in a number, and is based on the idea of the old computer game of *Space invaders*:

> Enter the following number in a calculator:
>
> 52846173
>
> The task is to *zap* the digit "1" (turn it into a zero), whilst keeping all the other digits as they are. The only operation allowed is a single subtraction Next, the digit "2" has to be *zapped*, then the digit "3", etc., until all the digits have disappeared. (In some cases, such as with "5", the digit may disappear rather than being turned into a zero.)

Like the previous task, this provides opportunity for work on place value. However, the practice within this task has several different properties to the repetition of the previous exercise Firstly, the main task presented is to do with *zapping* digits—to make each digit turn into a zero (or disappear) in numerical order, until all the digits have disappeared. This is a focus which can be understandable to someone, even if they are not particularly good at place value, as long as they know about certain other things, such as the digits "0" to "9" and the subtraction button on a calculator. So, one does not have to possess a good understanding of place value to be able to engage in the task of practising place value. After all, isn't practice needed precisely at those times when someone is unsure of something? Secondly, the focus of attention is not on what is going to be practised, but on the consequences, or results, of the practising So the practising of place value is subordinate to the task of "zapping" digits in numerical order. And thirdly, when I am engaged in this task I can see whether I am correct or not, by looking at the consequence of any subtraction I make. I do not need a teacher to tell me whether I am correct or not I can see the consequence of my decisions and the consequences can help me learn about place value For example, 52846173 may turn into 52846163 after my first attempt at zapping the 1. This provides me with information about what happens to the number when I made my attempt of subtracting 10, and can help me consider what to try next time

A model of subordination

One holiday, I took a sailing lesson I knew very little about sailing and this was my first lesson. I was using the rudder to steer in a particular direction. There was an unusual element for me in using the rudder, in that I seemed to need to move my arm in the opposite direction to the one in which I wanted the boat to go. Then, I was introduced to a rope which affected the plane of the sail with respect to the rest of the boat Lastly, I began moving my body by leaning over the side of the boat. This affected the angle of both the boat and the sail with respect to the vertical There seemed to me to be too many possibilities to consider at once If I concentrated on any one of the rudder, rope, or body position, I forgot about the other two. My attention was darting from one to another. On the arrival of my attention to any one of the three, I found that I needed to make large corrective adjustments before turning my attention to the next, which, in turn, needed large adjustments also I felt frantic and unable to cope. I soon found myself in a situation where the boat was becalmed. My teacher then told me to keep my attention only on the sail, and to ensure that the sail was always *just* not flapping in the wind. On doing this, I found that I was able to make increasingly finer adjustments with the rudder, rope, and my body position such that the sail just did not flap. I became more relaxed and felt in control of not only the boat but also the possibilities of moving the rudder, rope, and body position

The rudder, rope, and body position all affect the plane of the sail relative to the wind. And, to change the angle of the sail with respect to the wind, at least one of these three needs to be moved, and so I say that *they are subordinate* to the angle of the sail relative to the wind.

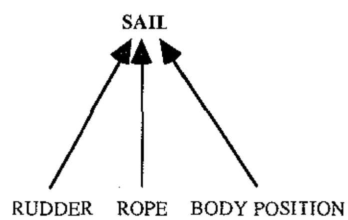


Figure 1
Diagram representing some dynamics of subordination within sailing

Also, I say that the angle of the sail is at a *higher subordinate level* than the rudder, rope, or body position, since these three are subordinate to the task of changing the angle of the sail. This is not an absolute judgment of levels, but is dependent upon the particular situation. There might be a device created which would subordinate the angle of the sail to the movement of the rudder, in which case the rudder would be at a higher subordinate level than the sail when that device is in use.

During the sailing lesson, my skill at moving the rudder, rope, and body position improved partly as a consequence of my attention being taken away from these and put onto the sail. Each of the three had an effect on the sail. I attended to the *consequences* of any movements in these three, rather than attending to the movements themselves. Thus, my attention was away from where my learning was taking place. This was not just a withdrawal of attention from an area in which I was learning, but a placement of attention on a task which subordinated the skills I was trying to learn.

I use *subordination* in a particular way. The movement of rope, rudder, and body position have a number of features with regard to the task of keeping the sail "just not flapping". Firstly, when sailing, I do not stand up in the boat and physically get hold of the sail and move it. I could—it is a possibility—but, there are issues of stability, etc., which make it better that this is not done. Thus, affecting the angle of the sail *requires* the use of rope, rudder, and body. Secondly, I am able to *see the consequences of my actions* with the rope, rudder, and my body position, on the flapping of the sail *at the time I am making those actions*. Lastly, the task given to me is one which I understand without having to know about the ropes, etc. I can see the sail and know when it is flapping and when it is not. I say that a skill, A, is subordinate to a task, B, only if the situation has the following features:

(a) I *require* A in order to do B. (This may be an existing necessity or can be created through the "rules" of a task);
(b) I can *see the consequences of my actions* of A on B, *at the same time as making those actions*.
(c) I do not need to be knowledgeable about, or be able to do, A in order to understand the task, B.

The requirement, within (b), of simultaneity is important. As I was sailing I began to feel *as if* I was directly altering the flapping of the sail, yet in reality I was moving a combination of rudder, rope, and body position. I began to see the flapping of the sail *through* the movements I made. Thus, although in reality there were times when my attention was with the rudder, etc., I was not so much attending to the rudder but was attending to the flapping of the sail *through* the rudder. Simultaneity helps this link between rudder and sail to be established.

Solving linear equations: what is being practised?

I will explore the notion of subordination further, by comparing a fairly traditional exercise of repetition within another textbook currently used in some UK schools, and a whole class activity which also offers practice. Both are concerned with the solving of linear equations. However, lying behind this surface of similar content, is a completely different idea of what is being practised, and learned, for a student engaged in each of the two tasks.

In Holderness [1994], a section on solving equations begins with the following:

> You can add equal numbers to both sides.
> You can subtract equal numbers from both sides.
> You can multiply both sides by the same number.
> You can divide both sides by the same number, (not 0).
> [p 162]

Then, there follow seven examples, a sample of which are:

> Example 1: $x + 10 = 17$
> Subtract 10 from both sides
> $x = 7$
>
> Example 5: $13x - 20 = 6x + 8$
> Subtract 6x from both sides
> $7x - 20 = 8$
> Add 20 to both sides.
> $7x = 28$
> Divide both sides by 7
> $x = 4$
> (Instructions for checking this example followed)
> [p 162-163]

An exercise with several equations to solve followed these examples. The first 25 questions were under the heading of "Solve these". Three of these questions are offered here as a flavour of the exercise:

> 1 1 $a - 7 = 14$
> 2 7 $\frac{1}{4}(g - 2) = 6$
> 3 3 $2(5 + x) - 3(6 - x) = 42$
> [p 164]

Unquestionably, these tasks offer the opportunity to repeat the processes of solving linear equations as described in the given examples. There are assumptions within this exercise that a student will already be able to work with a letter appearing within an equation (and not be put off with all the examples involving an *x*, whilst the first twenty questions involve other letters). Also, that a student can look at an equation and know the order of operations and the syntax of formal algebraic notation. And that a student can carry out other algebraic procedures such as multiplying out brackets. The rules which are offered at the beginning are only a part of what a student will need to know in order to carry out successfully the solving of these equations. A high level of mathematics is already assumed to be available to a student. If someone does not feel confident with letters, or multiplying out brackets, or the order of operations, then that student may be in a position where even attempting some or all of the questions is difficult, and so would get no practice at all. This is the classic Catch 22 [Heller, 1962] of traditional exercises—if someone doesn't understand already, then they won't be able to begin; if someone does understand already, then there is little need for them to begin. The perceived task, of carrying out the stated procedures to solve an equation, is only understandable if someone can already understand those stated procedures. So, to understand the task, one already has to be able to carry it out.

As a student works through these tasks of solving equations, his or her attention has been drawn to the four operations, given at the beginning, which can be carried out on an equation. This is what is emphasised in the given examples, and this appears to be what the authors intend to be repeated in the exercise. Thus, attention is being placed on carrying out the repetition. Those things which are subordinated during this repetition (use of letters, syntax of formal algebraic notation, multiplying out brackets, order of operations, etc.) are assumed to be already known and understood.

A different activity is based on a "think of a number" idea, where I say that *I am thinking of a number and then add three to it, multiply by two, and get 14. What is my number?* Through working with students in particular ways for only about 15 minutes, outlined elsewhere [Hewitt, 1994], but which are carried out purely verbally and not making use of pen and paper, the students become aware of the inverse procedures required to find my number, and can articulate those procedures. Once the students are in a situation where, if I tell them what I do to my number, they will know what to do to get back to my number, I deliberately give a long list of operations to do to my number. (The following transcripts come from a lesson with a middle set of 13-14 years olds, excerpts of which appear in the video *Working Mathematically on Symbols in Key Stage 3*, Open University, 1991.)

DH: *I'm thinking of a number. Oh dear, what am I going to do with this one? Oh yes, I'm going to add three, times by two, take away five ... divide by three ... add 72 ... Got a problem with this? Do you want me to write down what I am doing?*

Class: Yes

The students are now in a situation where, if only they could remember the list of operations, they would be able to work out my number. However, I have deliberately made the list too long for them to remember without some written record. At this stage, I offer to write on the board what I have done to my number. As I say again what I do to my number, I gradually write the following on the board, being careful to write the symbols associated with the words I am saying, at the same time as saying them:

$$6\left(\frac{2(x+3)-5}{3}+72\right)=100$$

DH: *OK. So, let me see I am thinking of a number* (writes x on the board) .. *I add three* (writes +3) .. *then I'm going to . multiply by* (writes brackets round the expression so far) .. *two* (writes 2 in front of the brackets) . *then I am going to take away five* (writes −5) ... *then I am going to divide by* (writes a line underneath the expression so far) . . *three* (writes 3 below the line) ... *then I'm going to* .. (makes a noise whilst going along the division line from left to right, writes + following on from the division line and makes a different double noise whilst the addition sign is being written) *add... 72* (writes 72 after the addition sign) ... *then I'm going to multiply by* (writes brackets round the expression so far) .. *ummm six* (writes 6 in front of the brackets) .. *and I get* (writes = to the right of the expression so far) ... *umm 100* (writes 100 to the right of the equals sign) . *So you are going to?*

Shona: Think of a number, add three, times by two, take ...

Girl5: Five.

Shona: .. five, divided by three, times six, add 72, equals 100.

DH: *I think I said add 72 and then times by six.*

Shona: Oh yeah ... add 72, times by six, equals 100.

For the students, this may be the first time they have met standard algebraic notation, and the first time they have seen a letter being used to represent a number. Here Shona ignores my invitation to work on finding my number but decides to publicly rehearse what I have just said by trying to do so through looking at the notation. Sometimes a student reads the notation in a way different to the convention. In which case I either repeat what I had said, as above, or offer how I would write what they had just said:

$$6\left(\frac{2(x+3)-5}{3}\right)+72$$

This keeps us communicating through the algebraic notation, which ensures that the notation is being subordinated. During this lesson I attempt to keep the focus of attention on the task of what they have to do in order to find my number. The notation is the only thing available which is helpful to remind them of what I did to my number, since I had not allowed them to write anything down themselves. Thus, the notation, which has been met for the first time by many students, becomes immediately subordinate to the task of finding my number. Although the notation may be new, the task of finding my number is not, and so the *task* is understandable without the need to be able to interpret the notation. Rousseau [1986] said that:

> Before you can practise an art you must first get your tools. [p 90]

Relating the algebraic work to this statement might suggest that there needs to be some lessons on learning about notation before notation-as-a-tool can be used in the art of solving equations. However, I suggest that:

> As you practise an art you will acquire your tools. [p 90]

The practising of the art of solving an equation (which may be understood by the students at this stage as "finding my number") will help the acquisition of the tool of algebraic notation. The desire to practise your art subordinates the tools required to carry out that art.

> A painter is disciplined in his art in the degree in which he can manage and use effectively all the elements that enter into his art—externally, canvas colors, and brush; internally, his power of vision and imagination. Practice, exercise, are involved in the acquisition of power, but they do not take the form of meaningless drill, but of practising the *art*. They occur as part of the operation of attaining a desired end, and they are not mere repetition
> Dewey [1933, p86]

Not only is it through making progress in your art that your tools will be practised, but that when the tools are met for the first time, they can be immediately subordinated and practised within your progress in art. In fact, I argue that the tools will be acquired sooner if their use is immediately practised through their subordination to a task, than if they are met separately and practised in isolation with attention remaining purely on the practising *per se*. For example, I learn the facilities of a word-processor through using those facilities when I have something to write and want to present my writing in particular ways.

As the lesson continued, the following equation was gradually written on the board below the original one:

$$\frac{3\left(\frac{100}{6} - 72\right) + 5}{2} - 3 = x$$

DH: *OK. So, how am I going to work out my number? Gemma.*
Gemma: 100 divided by six
DH: (Writes 100) *100* (Points to 100 in the original equation)
Gemma: Divide by six
DH: *Aha. And you're dividing by six because ..* (writes a line underneath the 100 and then 6 underneath that line)
Gemma: Because it was times six. So you ... (unclear) .. divide by six
DH: *So that's that done.* (Points to 6 in original equation)
Gemma: Yeah. Then take 72
DH: *Aha* (Writes −72 to the right of the division line) *OK. And after that, I've done that* (Covers 72 in original equation)
Gemma: Times three.
DH: *Because I ..*
Girl8: Divided ..
Gemma: Divided three
DH: *So you're going to*
Gemma: ... so it's times three.
DH: *Times by ..* (Writes brackets round the expression so far and writes 3 to the left of the brackets) *Right.* (Covers 3 in the original equation) *We are left with this.* (Hugs $2(x + 3) - 5$ in the original equation)
Gemma: Divide it by two ..... oh no ...
Girl9: That's what I thought.
Gemma: Add five.
DH: *Go on*
Gemma: Add five. Divide it by two.
DH: *Sorry .... add ...* (Writes + to the right of the brackets)
Girl9: Five.
Gemma: Five. (DH writes 5 to the right of the addition sign)
DH: *Right. So that's that done.*
Gemma: Divide by two.
DH: *Aha.* (Makes a noise whilst drawing a line underneath the expression so far) *Divide by two.* (Writes 2 underneath the division line)
Gemma: Take three.
DH: *Right and ....* (Makes noise whilst going along the division line from left to right, and a different noise whilst drawing –) *take .*
Gemma: Three.
DH: *Three* (Writes 3 to the right of the subtraction sign) *And I end up with .*
Girl10: A number.
DH: (writes = to the right of the expression so far) *What do I end up with?*
Several: Four.
DH: *Gosh, you worked that out carefully. I hadn't worked that out . Well, I end up with the number I was thinking of, whatever that is. And when I said "I'm thinking of a number", what did I write down? When I first started off saying "I'm thinking of a number", what did I write down?*
Student2: x.
DH: *Right* (Writes x to the right of the equals sign) *So I end up with whatever that number is.* (Bangs on the x in the equation just written) *And we could work it out and find out what it is*

Students can successfully carry on with their task of finding my number when the equations have become quite complex and include several other symbols (β, *k*, *d*, α, ...) They subordinate the notation to this task (since I have constructed the situation so that they have no other choice), and as a consequence become fluent in interpreting and writing formal algebraic notation. Gattegno, when talking about the use of Cuisenaire rods for work on introducing the Arabic numerals to young children (Cuisenaire, G; Gattegno, C [1959]), said, *It is now that we hope to introduce a notation .. but here we must wait for a need to arise or we must create one* [p 5]. Here, through my role as a teacher, I have created a need for algebraic notation, through using a task which requires use of notation. So, the notation is not taught as a separate lesson, with attention placed on the memorising of arbitrary conventions, but is subordinated to a task where the task can be understood and seen through the notation which has been introduced. Mason [in press] talks about this "looking through" in connection with awareness of generality:

> One way to work at developing awareness of generality is to be sensitized by the distinction between *looking through* rather than *looking at,* which leads to the primal abstraction and concretization experiences, namely *seeing a generality through the particular,*

and *seeing the particular in the general*. [in press]

I want to use Mason's differentiation between *looking at* and *looking through*, to suggest that subordination is concerned with *looking through* that which is being subordinated (notation, in this case) onto a different focus of attention (finding my number). A focus of attention is required in order for what is new to be subordinated. However, the particularity of that focus of attention is not so important, a variety of foci would succeed in subordinating the notation. The desirability of immediately subordinating something which is to be learned, is that practice can take place without the need for what is to be practised to become the focus of attention. As Mason [ibid] has observed, *Practice tends to focus attention on precisely those aspects of a technique which have to be done without attention when the technique is mastered* [in press] What I offer here, is a way to practise something, helping it to become known and used with little conscious attention—something which is an important aspect of mastery.

I have found that those things which have been learned through being subordinated to another task are retained longer than those things which have been the focus of attention. For example, when algebraic notation has been met through subordination to the task of solving equations, it is the familiarity of the notation which is often retained, whereas the processes of solving a simple linear equation remain a little hazy. The use of subordination within learning can mean that those things which are given conscious attention may not be learned as well as those things which have conscious attention taken away from them through their subordination to a task. One year, I taught a mixed ability class of 12-13 year olds For about two weeks we spent time working on solving linear equations. The first lesson had been similar to the one detailed above The next academic year, the year group were split into sets and one boy, Paul, had gone into set three out of four and was being taught by another member of the department In the summer term of that year I met Paul when he had been sent out of his class. He had no work to do, so I asked him whether he had done any algebra this year. He said that he hadn't. So I thought I would find out how many of the ideas from the previous year had remained with him. I gave Paul the following information written on a piece of paper, asked him to solve it for me, and left the room.

$$\frac{3\left(2\left(\frac{3x+6}{7}\right)-6\right)}{7}+8=y \qquad y=20.8$$

On returning I found that Paul had written:

$$7\left(\frac{\left(\frac{\left(7\left(\frac{y}{3}\right)6\right)}{2}\right)+6}{3}\right)-8=x$$

The following conversation took place:

DH: *If I double and add one, what are you doing?*

Paul: Halving and taking one

DH: *I've got a number I times it by two, add one and get five. Do you know what my number is?*

Paul: I take one and divide by two.

DH: *That's not what you said before.*

Paul: Ah ... I know ..

Paul began writing again and I left the room. When I returned I found that Paul had now written:

$$\left(\frac{7\left(\frac{\left(\frac{7y-8}{3}\right)+6}{2}\right)-6}{3}\right)=x$$

One year on from doing such work, Paul was having partial success at the perceived task of solving the equation, but what he was quite confident with was interpreting and writing what he wanted within formal notation. The notation, which had always been subordinated to various tasks had remained with Paul. He was not so sure of the solving of equations, which had not been subordinated to the same extent in the lessons he had the previous year

With the traditional exercise on solving equations mentioned above, attention was placed on four procedures which were to be repeated during many questions requiring an equation to be solved. There is no mechanism to help these procedures become something which is carried out *without* conscious attention, other then repetition And repetition takes up considerable time, leads to boredom and lack of progress, and is mainly concerned with short term "success". Furthermore, what I have not dwelt on in this article is that the processes offered within the traditional exercise are offered as if they have to be told by a teacher and memorised by students. The "think of a number" activity involves the teacher only "telling" those things which must be told—such as those things which are conventions Everything else comes from the awareness of the students, including the processes involved in solving an equation

Although it appears that the traditional exercise is about practising the use of those stated procedures, there are a number of other "knowings" which are required to be practised if someone is to carry out the exercise successfully I listed some of these, including the use of a letter in an equation, the order of operations, and the syntax of formal algebraic notation. It is actually these "knowings" which are being subordinated to the task of carrying out the exercise, and are thus benefiting from being practised whilst attention is elsewhere. This can lead to mastery of those "knowings" rather than the solving of equations. However, these knowings are assumed to be in place already. This is a risky assumption, since, if they had been learned in a similar way to the procedures introduced in this exercise, they may well have been memorised at the time, and forgotten by the time a student meets this exercise. The "think of a number" activity does not assume someone has memorised these knowings, but introduces and immediately subordinates them to the focus of finding my number. Thus, a mechanism is offered to help these become mastered, rather than temporarily memorised, so that they remain available for the future.

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The more I have examined the workings of my own mind, [...] the less respect I feel for the part played by consciousness. I begin with others to doubt its use altogether as a helpful supervisor and to think that my best brain work is wholly independent of it. The unconscious operations of the mind frequently far transcend the conscious ones in intellectual importance. Sudden inspirations and those flashings out of results which cost a great deal of effort to ordinary people, but are the natural outcome of what is known as genius, are undoubted products of unconscious cerebration. Conscious actions are motived, and motives can make themselves attended to, whether consciousness be present or not. Consciousness seems to do little more than attest the fact that the various organs of the brain do not work with perfect ease or co-operation. Its position appears to be that of a helpless spectator of but a minute fraction of a huge amount of automatic brain work

Francis Galton
