WHO WOULD I SHOW IT TO

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> *And, as imagination bodies forth*
> *The forms of things unknown, the poet's pen*
> *Turns them to shapes, and gives to airy nothings*
> *A local habitation and a name* [1]

Like Laurinda Brown, the editor, I have been reading Barry Mazur's *Imagining numbers*, one of a number of recent general books about mathematics. [2] This one, typically, is said in the preface to be for "people who have no training in mathematics". It does, perhaps untypically, invoke algebraic symbolism and solve quadratic and cubic equations. More intriguingly, it specifically addresses "people at home in the imaginative life of poetry", inviting them to consider how an experience of mathematical imagining might compare with "the imaginative work involved in reading and understanding a phrase in a poem". I was drawn into reading it, and found it raised a number of interesting questions - about imagination, as well as about the nature, history, and teaching of mathematics.

I often wonder who it is who buys such books Are all that many people interested in finding out more about things they turned away from long ago? Is it perhaps the case that it is really people like me, with some adult experience of mathematics, who read them? The author offers analogies between poetry and mathematics, and these trigger questions that I cannot fully answer Would the poets (to use a shorthand) be drawn into some mathematics? Or is it that the mathematicians are drawn into some deeper appreciation and understanding of their subject?

The book's play with analogies and associations starts with the title. Those who already know about complex numbers will understand that the book is going to be about what Descartes called imaginary numbers, to distinguish them from what he saw as real numbers. Those who know some history of mathematics will recall that Cardano took the step of operating formally with the known general method of solving quadratics to solve $t^2 + 40 = 10t$ and so derive the two 'fictitious' solutions 5p:Rm:15 and 5m:Rm:15 Mazur quotes Cardano's remark that his reader would then have to imagine Rm:15, and this is where we understand the book's sub-title: "particularly the square root of minus fifteen" The history of *imaginary* numbers is then presented as a joint enterprise over the centuries of trying to *imagine* them - in effect, in terms of a suitable *image*.

Cardano's innovation was to give the as-yet-unknown a name and then state how you could then manipulate it. For instance, you could always square Rm:15 and get m:15. This is an algebraic imagining, and it seems to be part of Mazur's thesis that this would be felt to be inadequate until a corresponding geometric imagining is found - in this case some three and a half centuries later. This begs a few questions. One is that it assumes that people investigating the same situation over the centuries are thinking about the same things, in the same terms

Mazur discusses early accounts of 'numbers' involving square roots of negative quantities, though these roots would not have been conceived of as numbers at the time. He also refers to Plato's *Meno*, where Socrates gets the slave to say that the square on the diagonal is double the original square, which is not, of course, the same as saying that the length of the diagonal is $\sqrt{2}$ This suggests that a geometric imagination was already at work before, in fact, the displacement of concept that conceived the length of the diagonal being measured by a number

"Why do we call something a number?" asked Wittgenstein

> Well, perhaps because it has a - direct - relationship with several things that have hitherto been called numbers; and this can be said to give it an indirect relationship to other things we call the same name. And we extend our concept of number as in spinning a thread we twist fibre on fibre. And the strength of the thread does not reside in the fact that some one fibre runs through its whole length, but in the overlapping of many fibres [3]

The issue, which remains unresolved for me, is whether mathematics has the historical continuity that Mazur describes, or whether, with Wittgenstein, it continually re-conceives and re-organises its subject matter.

Mazur writes of "the collective mathematical enterprise of imagining $\sqrt{(-1)}$" (p. 223) and interleaves his account of this with various discussions of poetry, in particular the reading of a phrase from the poet, John Ashbery:

> the yellow of the tulip.

The latter is claimed to be "instantly imaginable", whereas the square root of negative quantities was a concept that took a long time before "a satisfactory geometric understanding of it was discovered". I am not clear what is being gained by the juxtaposition of the square root and the tulip. Mazur intends his mathematical account to be not so much a history as a re-creation in the reader of "the shift of mathematical thought that makes it possible to imagine these numbers" (p. 11). He then adds that poetry also has such shifts of thought: "the "turn" of the poem [...] being celebrated in the word *verse*" I was attracted by the free association, but doubted the etymology. [4]

There are some stimulating sections on analogy, and of course much of the book treats of the classical, Cartesian linking of algebra and geometry. For Mazur,

> *Analogy* is everywhere in mathematics; when it is most fruitful it is most unstable, goading mathematicians into producing larger structures in which the *analogy* becomes an *equality* (p 205)

He then quotes - to disagree with - the mathematician, André Weil:

> Nothing is more fruitful - all mathematicians know it - than those obscure analogies, those disturbing reflections of one theory on another; those furtive caresses, those inexplicable discords; nothing also gives more pleasure to the researcher The day comes when this illusion dissolves: the presentiment turns into certainty; the yoked theories reveal their common source before disappearing. As the *Gita* teaches, one achieves knowledge and indifference at the same time. (p. 205)

I was stirred by Weil's last remark, reminding me of those moments, at a more mundane level, where I solve a mathematical problem to my satisfaction - and then lose all interest. I was also moved by the connection with more serious matters, though was again a bit uncomfortable about the analogy being drawn there That mathematicians lost interest, it is said, in invariant theory once it was well worked out, does not really seem to be like the detachment favoured by eastern mystics

An earlier writer who wanted to link poetry and mathematics, Scott Buchanan, once commented on the undercurrent of worry about "the luxurious use of mathematical analogy" in the writings of the early nineteenth-century Cambridge mathematicians.

> Boole had made this worry very explicit in his less known writings, but all of them tried to present the cold rational side of their thought as if to disown the romantic thread which guided them into the great mathematical developments of the nineteenth century. [5]

Mazur, of course, exults in the 'romantic thread'.

> Oh, but how grim education would be if knowledge and indifference were ineluctably conjoined. (p. 206)

He is a compelling and exciting expositor, his lectures must be an inspiration to many students, his book is an outstanding example of the genre. But . Well, wherein lies the reservation? I suggest that there is always something ambiguous about exposition, which stems, inevitably, from a particular individual experience and point of view - the more effective the exposition, the less likely a learner's personal construction, the less robust the grasp.

For example, there are a number of persuasive metaphors that are offered at various expository times to help students 'understand' negative numbers But physical facts about distance, time, speed, temperature, bank balance, and so on *ad nauseam*, are available as metaphors precisely because we have previously created negative numbers, rather than the converse Mazur explains very clearly, how the way that we multiply negative numbers is forced upon us by the distributive law of multiplication. Perhaps he does not stress enough that wanting these new 'numbers' to satisfy the law is a choice - that, in fact, we could decide otherwise Thus, it is always effective to ask students to explore the implications of choosing 'minus times minus is minus' Meanwhile, even Mazur slips a little when he suggests that "arithmeticians agree with grammarians in claiming that a double negative is a positive" (p 48). This is argued on the grounds that a forgiven debt is a newly acquired asset. Like many a new teacher, I did once try that one - and you do not get away with it with any class that is not totally passive. In mathematics teaching, the analogy - be it presented by image or metaphor - that carries you over one hurdle does not always suit the next

In Mazur's capable hands, a negative is at one moment a debt, then a position on a number line, and then a rotation through two right angles - and it is, of course, the latter that leads neatly into an image of the square root of a negative. Such an account may smooth over some history. It certainly ignores the fact that even in the nineteenth century some mathematicians still considered negative numbers, let alone imaginary ones, somewhat suspect Moreover, it seems to emphasise visual, over any other, imaging. An alternative account could move on from the models offered by Wessel, Argand, and others, to the algebraic characterisations preferred by many later mathematicians: Hamilton's number-pairs, their representations as two-by-two matrices, or Kronecker's residues of polynomials modulo $t^2 + 1$.

It is clear that many mathematicians seek some geometric manifestation for the 'objects' they study For such people, a conic might be a section of a cone or a geometrically described locus But for others, it *is* a quadratic equation, and this may be, for them, more satisfying and more developable. The early algebraists' revolutionary slogan was that every problem could be put in the form of an equation, and so then solved automatically (this seemed to Rousseau like playing a tune by turning a handle!) There may be some relevant and interesting psychological explanations for such opposing views, but because there are such well-attested personal differences, it is hardly justifiable to have an epistemological preference

Mazur discusses the age-old bridge between geometry and algebra He echoes Weil's use (mentioned above) of the word *yoked*, finding it "an appropriate word, with its implication that we have two separate, but somehow connected, intuitions" (p. 207). Thus, a point is identified with a number-pair - *like* becomes *same*

> It is a useful, perfectly benign strategy in the teaching of math to utter such statements as "*X* is nothing other than *Y*": these statements are often thoroughly graspable, often logically immediate, and yet often effect some needed change of perspective. (pp 182-183)

This seems to me usefully provocative. I could not find anything in my own teaching experience that could corroborate it, but I will keep on testing it.

Typically, and somewhat unnervingly, there is then an immediate switch into a section on prose and poetry, which are conceived as being on different sides of a mountain. Mazur offers the image of burrowing through the mountain. He quotes from Virginia Woolf seeking poetry in her prose, while he quotes Baudelaire wanting to do the opposite This short section is then immediately followed by a mathematical discussion, opening with the statement:

The recipe for multiplication of complex numbers has two equivalent descriptions, one *algebraic* and the other *geometric* (p 184)

This switch is like many similar juxtapositions throughout the book, which I feel create serendipitous coincidence rather than significant correspondence One is bound to feel that Mazur is forcing an association (perhaps, in Coleridge's terms, with Fancy rather than Imagination). But where does it leave us? Do we think algebra is the prose to geometry's poetry? Or the other way round? Does our experience of algebra and geometry help us capture the distinction between prose and poetry? Alternatively, does the non-mathematically-minded readers' supposed experience of prose and poetry help them to appreciate the different nuances of algebra and geometry? Present readers may have their own answers to these questions

There are various historical examples of the way in which people can take passionate positions on either side of that mountain. People tend to have different preferences for 'analytic' or 'synthetic' solutions to geometrical problems and in the past such preferences have often been expressed with some hostility. There were no 'burrowings' in early nineteenth-century Naples, where the university's synthetic geometers were Bourbon sympathisers, bitterly opposed to the analytic geometers from the institute of civil engineers, who were Jacobin supporters. For the former, mathematics was a spiritual science, a powerful resource against atheism and materialism. Their mentor, Nicola Fergola, was a staunch Catholic, who - it is said - saw God behind the circle and the triangle, and who claimed that his opponents saw only the nothingness behind their formulas [6]

In our time, such controversies recur in different guises. We may now ask whether signs gain significance from what they are supposed to represent, or rather from their relationship with other signs. It is interesting to note that Simone Weil (the sister of André Weil) linked algebra with money and mechanisation as the three monsters of contemporary civilisation. For her,

> the relation of the sign to the things signified is being destroyed, the game of exchange between signs is being multiplied of itself and for itself [7]

A contrasting example might be that of the so-called Copenhagen discussions of quantum mechanics, which led to earlier visual models being replaced by Heisenberg's matrices Further developments like Feynman diagrams suggest that it is no longer the case that an appropriate mathematics is derived from some physical model, but rather that a physical model is derived from the mathematical 'exchange of signs'.

Mazur has some stimulating things to say about different 'imaginings'

> The act of visualization is, to be sure, only one possible act in the repertoire of the imagination To visualize, we play the image on our already existing internal screen But the more difficult leaps of the imagination force us to establish larger screens and, perhaps, new theaters of the mind. (p 142)

One example offered is that of imagining imaginary numbers, which is said to be not an immediate, simple act of visualisation. The first step is to re-interpret the concept of number as a transformation (a 'stretch' of the number line) Then we have to "work at visualising these transformations". The sudden switch (one soon gets used to these) at this stage is to the idea of imagining the inventors of writing Ashberry writes of the "tissues and tracings the genetic process has laid down between us and them". These may briefly invoke some images, which will, however, fade and become, in Ashbery's words, "as useless as all subtracted memories" (p. 143). Mazur's moving comment on this includes a reference to those subtracted memories, which is then (with another switch!) immediately followed by a return to the mathematical discussion. I am torn between impatience at this seemingly arbitrary disturbance of my linear reading, and delight at the invitation to play with unconscious associations and resonances.

Let me then close my highly selective account by playing a little .. Subtracted memories often re-surface as elements of *condensations*, such as symbols or images. And this may be one of the most obvious links between mathematics and poetry. In both cases, any unpacking of meanings demands attention. Mazur emphasises this in his opening chapter He quotes Rilke: "we are the bees of the invisible", and then comments:

> our gathering of the honey of the imaginative world is not immediate; *it takes work* [ ] *It is who we bees are* (p. 4, *emphasis added*)

In this spirit, the non-mathematically-minded could perhaps be invited to work on the condensation of meanings in the statement 'point nine recurring equals one'. Instead, the present reader might be asked to work on the one-line poem, quoted by Mazur:

> who would I show it to

I recommend delaying consultation of the endnote [8] until this poem has had some attention

It is we who bees are ....

Who would I show it to? *How* would I show it? *What* would I show? Ah! If I knew the answers, teaching mathematics would be so easy

Notes and references

[1] *A Midsummer Night's Dream*, 5 (i), lines 14-17

[2] B Mazur (2003) *Imagining numbers (particularly the square root of minus fifteen)*, St. Ives, Allen Lane, the Penguin Press The author is a well-known mathematician who has described (elsewhere) how reading some essays about imagination in literature led him to think about how you imagine a mathematical idea and then to become involved in teaching a graduate seminar in literature

[3] L Wittgenstein (1953) *Philosophical investigations*, Blackwell, p. 32e The passage occurs in a discussion of various things we call 'games' These are held to have 'family resemblances' rather than some supposed common element: "And for instance the kinds of number form a family in the same way"

[4] The word 'verse' does indeed (according to the *Oxford English Dictionary*) derive from a turn, but this is that of the eye or pen when starting a new line. I am not clear whether this is what the 'turn' of a poem is supposed to be This usage feels more like the expression 'the turn of a phrase'.

[5] S Buchanan (1962) *Poetry and mathematics*, Lippincott, p 25. Various analogies are explored in this unusual book, first published in 1929. A

typical example (p. 64) links the counting of objects with the recounting of stories. If this seems far-fetched, we may recall that in ancient times the one who tallied was also the one who told tales.

[6] *Cf.*M Mazotti (1998) 'The geometers of God', *Isis* **89**, pp. 674-701 According to the synthetic geometers, the 'cold algebraists' would reduce mathematics to a practical instrument devoid of meaning. There are some echoes of this controversy in current discussions of computer proofs, or – at another level - of the use of geometrical programs like *Cabri*.

[7] S. Weil (1963) *Gravity and grace*, Routledge, p 139. Weil frequently used analogies from mathematics and science in her writings. Some went beyond just a poetic comparison; for example, the title of this book was meant to be taken very seriously. The opening page includes:

> All the *natural* movements of the soul are controlled by laws analogous to those of physical gravity [...] what we expect of others depends on the effect of gravity upon ourselves, what we receive from them depends on the effect of gravity upon them.

[8] The poem is by W. S Merwin Its title, *Elegy*, triggers some moving – and for the bereaved, almost unbearable - meanings. Mazur quotes it (p. 76) as an example of poetic "conciseness of expression", and after a technical analysis refers to its "desperate constriction" and its "concentration of meaning" I hoped readers would freely associate (from what I prefer to call a condensation) without the trigger of the title. Showing can be a display or a bringing to attention. And, as Wittgenstein observed, what *can* be shown *cannot* be said.

I did show the first chapter to two friends with English degrees, and asked for their comments. L wrote:

> I enjoyed trying to keep up with the maths [...] The idea of an area less than zero belongs in my imagination to an underworld of minus – where invisible activity whizzes about slightly out of control, but maybe resolves itself into something satisfying The literary gobbits didn't sing at all. I wanted them to have a parallel with the maths.

G added:

> The complexity of the imaginative process in the creative arts is simplified to the idea of imaging and some speculations about the reading process. Who is he aiming at? The maths is acceptable to the non-mathematician, but the snippets of poetry and psychology do not give us much to hold onto

Species and numbers.

In the phrase "the yellow of the tulip," there is the usual delicate ambiguity hidden in the repeated definite article, "the." Are we to be imagining the *general tulip* or else some yet unspecified, nevertheless particular tulip? Or both at the same time? If we were watching a *Nature* documentary and heard that authoritative voice-over intoning, "The ring-necked pheasant ....," there would be no ambiguity about that definite article: we are in the presence of the *generic* ring-necked pheasant, whatever that is.

In algebra, however, a fruitful ambiguity surrounds the way in which one thinks of the unknown *X*. Are we thinking of *X* as a placeholder (and nothing more) for yet unspecified, nevertheless particular values? Are we thinking of *X* as a *universal value*, whatever that means? Are we thinking of *X* as a freestanding object to be treated in its own right, yet capable of being substituted (legal tender) for specific values?

(Mazur, 2003, pp. 128-129)