FOR WHOM THE FROG JUMPS: THE CASE OF A GOOD PROBLEM SOLVER

FULVIA FURINGHETTI, FRANCESCA MORSELLI

> Beauty is the first test: there is no permanent place in the world for ugly mathematics (Hardy, 1989, p. 85)

The ways in which mathematicians come to create have been discussed by psychologists and neurologists for a long-time. Mathematicians themselves have produced interesting documents on this subject in the form of autobiography (Hardy, 1989; Poincaré, 1952) and essay (Hadamard, 1954). More recently, this subject has been treated specifically in the *milieu* of mathematics education (Burton, 1999a, 1999b, 2002; Liljedahl, 2004). This kind of educational research mirrors the interest of mathematics educators in the elements that characterize the mathematical activity of professional mathematicians, with particular reference to the act of creation. This interest relies on the fact that

> it is through mathematical discovery that we see the essence of what it means to 'do' and learn mathematics (Liljedahl, 2004, p. 249)

The previous sentence applies not only to the work of mathematicians, but also to students' mathematical activity. Burton (1995, 1999a) bases her epistemological framework for coming to know mathematics on the practices of mathematicians. Interesting developments from this kind of study concern the extent to which this epistemology is relevant to students' work (Burton, 2002). From a similar perspective, we have carried out a study on the process of proving at university level. The complex nature of the factors influencing students' processes of proving is widely recognized, especially at this level. In his overview of literature on the areas of potential difficulty encountered by students in proving, Moore (1994) concludes that

> the ability to read abstract mathematics and do proofs depends on a complex constellation of beliefs, knowledge, and cognitive skills. (p. 250)

With the aim of disentangling this constellation of beliefs, knowledge, and cognitive skills we have carried out studies on students' performances in proving. In particular, in our paper (Furinghetti and Morselli, 2004), we have focused on the intertwined nature of affect and cognition by studying the negative performance of a university student asked to prove a statement of number theory. That study identified elements that hinder a good solution. Here, we consider the other side of the coin, that is the intertwined nature of affect and cognition in the positive performance of a university student facing the same task in the same external conditions, allowing us to identify elements fostering a good performance.

Background

We see proof as a special case of problem solving and therefore our basic references are to the literature on problem solving (Schoenfeld, 1992, gives an extensive overview). In this article, we filter this view through the lens of affect. First, we discuss the intertwining of affect and cognition, with particular reference to the model of DeBellis and Goldin (1997, 2006). Afterwards, we focus on some components of affect that characterize the good problem solver and we present an interpretation of this intertwining in terms of creativity.

DeBellis and Goldin (1997, 2006) studied the complexity of internal representational systems as human beings engage in mathematical problem solving. According to them, there are five kinds of internal representational systems, constructed over time, that interact continually in symbolic relationships with each other: verbal/syntactic, imagistic, formal notational, planning and executive control and affective. The affective system refers to changing states of feelings during problem solving (local affect) and more stable and longer-term constructs (global affect). Considering affect as a representational system, it may be said that states of feeling interact with other modes of representation, encode important information and influence problem-solving performance. Attitudes and beliefs are aspects of global affect, emotions are part of local affect. According to DeBellis and Goldin (1997, 2006), emotions interact with cognition. They speak of "affective pathways"– sequences of states of feelings that interact with cognitive representational configurations. In addition to the three components of affect mentioned in McLeod (1992), these authors also consider aspects of the solver's values/morals/ethics. These are much than a belief about what mathematics is:

> they refer to the deep, 'personal truths' or commitments cherished by individuals. They help motivate long-term choices and short-term priorities. (DeBellis and Goldin, 2006, p. 135)

For example, students may feel bad when they do not follow instructional procedures, because they are contravening their values. Another example would be mathematical self-acknowledgement, that is, the students' abilities to acknowledge an insufficiency of mathematical understanding. Such an acknowledgement may lead to surface-level adjustment or to efforts for a deeper understanding. The strongest problem solvers show a straightforward recognition of insufficient understanding with productive responses.

In our opinion, the values/morals/ethics component of affect includes aesthetic values. When we use the term

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aesthetics we are not referring to the way intended by professional mathematicians [1], but to classroom aesthetics Sinclair (2003) clarifies this distinction as follows:

> A student's aesthetic capacity is not equivalent to her [*sic*] ability to identify formal qualities such as economy, cleverness, brevity, simplicity, structure, clarity or surprise in mathematical products. Rather, her aesthetic capacity is her ability to combine information and imagination when making purposeful decisions regarding meaning and pleasure [. .]. (p. 200)

Sinclair's (2003) study shows that students'

> aesthetic behaviors have very functional, yet pedagogically desirable, purposes: establishing personal and social value. (p. 204)

A similar interpretation of aesthetic values is discussed by Featherstone and Featherstone (2002), who comment on the work of Hawkins These authors focus on the way this philosopher "connects aesthetic experience to interest and engagement" (p. 24). In the words of Hawkins, reported in the paper,

> [aesthetics] is a mode of behavior in which the distinction between ends and means collapses; it is its own end and it is its own reinforcement (p. 25)

We may argue that aesthetics is one component of the affective domain that promotes a good performance. Other affective factors interact with cognition in fostering a good performance In the following, we refer to the concept of creativity as a catalyst for all these elements.

Creativity has been described in different ways According to Imai (2001), the key aspects of creativity are

> the ability to overcome fixations in mathematical problem-solving and the ability for divergent production within the mathematical situation. (p 187)

The related concept of divergent thinking is characterized in terms of the following features:

> fluency, shown by the production of many ideas in a short time; flexibility, shown by the students varying the approach or suggesting a variety of methods; originality, which is the student trying novel or unusual approaches; elaboration, shown by extending or improving of methods; and sensitivity, shown by the student criticising standard methods constructively. (*ibid*, 2001)

Fluency and flexibility, that is to say the abilities to overcome fixations and to produce creative thinking within mathematical situations, were already acknowledged as important features by Haylock (1987).

There are other descriptions of creativity. For example, Ervynck (1991) gives the following tentative definition:

> Mathematical creativity is the ability to solve problems and/or to develop thinking in structures, taking account of the peculiar logico-deductive nature of the discipline, and of the fitness of the generated concepts to integrate into the core of what is important in mathematics. (p. 47)

This description is internal to mathematics. A point of view (Urban, 1995) that comes from outside mathematics and mathematics education identifies six components of creativity, with different subcomponents:

1. Divergent thinking and acting	**2. General knowledge and thinking base**	**3. Specific knowledge base, area specific skills**
- problem sensitivity - fluency - flexibility - restructuring and redefinition - remote associations - originality - elaboration	- broad perception and information processing - memory network - analyzing and synthesizing - reasoning and logical thinking - metacognition	- increasing acquisition and mastery of specific knowledge and skills for specific areas of creative thinking and acting - expertise
4. Focusing and task commitment	**5. Motivation and motives**	**6. Openness and tolerance of ambiguity**
- topic/object/product focusing - selectivity - concentration - steadfastness and persistence - task commitment	- need of novelty - curiosity - drive for knowledge - communication - devotion and duty - self-actualization - need of control and instrumental profit - external recognition	- openness for experiences - playfulness - (readiness for) risktaking - tolerance of ambiguity - postponing quick solutions - nonconformity and autonomy - defocusing - regression and relaxation - humor

Figure 1: The six components of creativity with subcomponents according to Urban, 1995.

We refer to this model because it evidences the concurrent presence of cognitive factors (such as, knowledge, skills and divergent thinking) and affective factors (such as, motivation, task commitment and openness) as components of creativity, thus helping us to link the intertwining of affect and cognition with creative behavior

Other issues concerning creativity come from Maslow (1962), who distinguishes two degrees of creativity: *primary creativity* is related to spontaneous behaviors and takes place when a person does not fear their own thinking; secondary creativity is related to the ability of putting order in personal or others' ideas. When both degrees are present, Maslow speaks of "integrated creativity". In particular, Maslow deals with *self-actualizing creativity*, which does not come from a particular talent ("genius"), but exactly from *personality* Self-actualizing creativity is revealed by any behavior of the subject, who tends to act creatively in *any* situation. Self-actualizing people are receptive, *i e.*, open to exterior stimulus, spontaneous and expressive. These subjects act in a more natural way than ordinary people; they are less inhibited by self-criticism and by fear of judgments by other people. Maslow says that self-actualizing creative persons have kept something of their childhood. The description given by Maslow stresses the role of affective factors as a source of creativity.

Process

We considered one student attending the final year of the university course in Mathematics. He had attended all basic courses (such as, algebra, geometry and analysis) and advanced courses in mathematics. His curriculum encompassed the course of mathematics education in which our experiment was carried out. In this course, the students were regularly engaged in activities of proving, developed as follows:

- a problem was given
- the students were aware that the problem was in their grasp
- the students were asked to write down their process of solving, recording the written protocol - the thoughts [2] that accompanied their work
- the students worked individually
- the written protocols produced were analyzed by the whole class.

The students were able to use pseudonyms and they were allowed to work as long as they needed in order to avoid the influence of time in the performance (see, Walen and Williams, 2002).

The goal of these activities was not the performance on proof by itself, nor marks given to the performances. Instead, the goal was to make the students reflect on what happened when a proof was carried out and to develop skills in analyzing problem-solving processes.

The statement to be proved was the following:

> Prove that the sum of two numbers that are prime to one another is prime to each of the addends. (Two natural numbers are prime to one another if their only common divisor is 1.)

The definition of numbers that are prime to one another was given to students to prevent the difficulty of remembering it.

This problem is an adaptation of a part of Euclid's proposition VII, 28:

> If two numbers be prime to one another, the sum will also be prime to each of them; and, if the sum of two numbers be prime to any one of them, the original numbers will also be prime to one another. (Heath, 1956, II, p. 329)

Nowadays, numbers "that are prime to one another" are termed coprime numbers.

We have selected our student because he was very cooperative in providing us with information on his thoughts. Moreover, he was one of the students who was able to finish the task. We looked at the written protocol and an *a posteriori* semi-structured interview with the student. The interview was carried out in a relaxed atmosphere since the interviewer (the author Morselli) was not involved as a teacher in the course. It was audiotaped and afterwards transcribed.

For the purpose of our analysis, the whole text of the written protocol was split into excerpts, which we numbered. In our view, each excerpt corresponds to an episode in the student's process of proving. In the following we report on the translation of the written protocol into English.

Analysis

Since our student used names of trees as pseudonyms for labeling his written protocols during the course, we will refer to him as Albero (the Italian word for 'tree').

Already, at first glance, the written protocol of this student appears to be extremely well organized and clear. It also contains a drawing and is sharply divided in two parts. We analyze the first part (excerpts 1-5) step by step; the second part, which contains the algebraic proof of the statement, will be analyzed globally, since it is less relevant to the focus of this article.

> 1. First of all I want to see prime numbers, I want to grasp their secrets.

Here, we catch Albero's need to give the problem a personal sense as a prerequisite for constructing a proof. We stress the use of the terms "see" and "secrets". The language used by Albero is highly metaphorical. The metaphor of 'seeing' evidences his quest for meaning. Moreover, we may note that Albero does not write, "I want to understand", but "I want to grasp their secrets", thus relating himself to numbers seen as living entities. This approach echoes Devlin's (2000) argument that mathematicians "like gossip" and talk of mathematical objects as if they were characters of a story. The use of colloquial expressions such as "to grasp their secrets" will be the *leitmotiv* of Albero's way of communicating.

> 2. The first way in which I think of them is "as jumps", I tell you in this way.
>
> I imagine a straight line with many equidistant stops (the stops are the numbers). Two stops are prime to one another if [3] - I'm roughly speaking - [in considering]
> the frog that jumps from stop to stop
> the frog that jumps every two stops
> the frog that jumps every three stops
> There is not any frog that reaches both stops (except the frog that jumps every stop).

Albero's first strategy is a sort of translation of the problem into "his mathematical world" terms. In this world, the institutional rules are completely accepted, but they are expressed in a non-conventional way. This is clear evidence of creativity (Urban, 1995). He translates the concepts that are involved in the statement of the problem (which, of course, are mathematical) into dynamical images taken from his imagery. We stress that he refers to images that reflect his own way of conceptualizing natural numbers and turn out to be useful for reasoning. The sentences are accompanied by a drawing (see Figure 2):

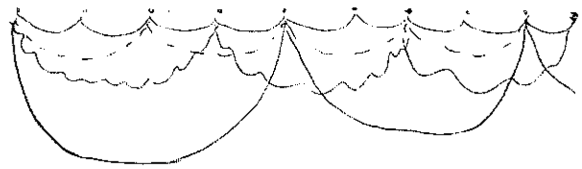

Figure 2: Albero's drawing.

Albero's view of the concepts involved in the problem turns out to be efficient; in particular, his way of representing the relation "being prime to one another" is dynamic and allows him to grasp the mathematical mechanism behind the definition of common divisor. The metaphor is accurately chosen: in an erased row, we see that, initially, the chief character of the metaphor was a travelling man stopping any two stops. The metaphor of the traveller, which may evoke continuity, was discarded in favor of the metaphor of the frog, which is more evocative of discrete aspects. The metaphor shows a high degree of conceptualization and is in line with the argument of Sfard (1994), according to whom metaphors are constitutive of mathematical thinking.

We know that similar ways of conceptualizing multiples were observed elsewhere (Zazkis, Liljedahl and Sinclair, 2003). We are also aware that Albero may have recalled, from his past experiences (maybe his first encounter with natural numbers), a way of looking at natural numbers and the concept of multiples. Anyway, our point is not the originality of the metaphor, but the originality of the choice of using the metaphor in proving. Indeed, the metaphor is a real tool in order to grasp the structure of the problem. The metaphor is empowered by the drawing, which is a support and a guide for the reasoning. We stress that Albero chooses to rely on the metaphor even if, as we'll see in the following, he has at his disposal more sophisticated mathematical tools to solve the problem. In this sense, we feel an aesthetic flavor in his choice: in his proving process, as Hawkins would say, "the distinction between ends and means collapses" (Featherstone and Featherstone, 2002, p. 25). The metaphor is carrier of mathematical meaning and aesthetic values: we see it as potential evidence of the intertwining between affect and cognition.

> 3. Ooh now I see!
> I consider the two stops A and B. I call A + B the stop (do not ask me to be formal, otherwise I lose the good thing [the inspiration]) [...]
> I have to prove that, except the first frog, a frog that stops in A does not stop in A + B,
> a frog that stops in B does not stop in A + B.
> So I would have: if a number divides A or B it does not divide A + B, and this is enough for the thesis, because I would have that if a number divides A + B it does not divide A nor B.

By making the drawing, Albero gets an insight into the relations involved in the problem and when he looks at the drawing that he produced, the solution comes out immediately ("Ooh now I see!").

From now on, Albero explains his solution to the reader, keeping alive the metaphor of the frog. We stress the fact that his reasoning is sharp and formal-like, even within the 'amusing' metaphor [4]. He translates the hypotheses and the thesis into metaphorical language, keeping the isomorphism between mathematics and the pond.

Albero is aware that his way of reasoning is non-conventional, but he does not want to get out of the metaphor so as not to break his stream of thoughts. The sentence "do not ask me to be formal, otherwise I lose the good thing" sheds light on the modalities of his creative act, which strongly relies on being non-conformist.

> 4. It is easy. If a frog stops in A it can not stop in A + B because going from A to A + B is like going from O to B. And, for hypothesis, the frog that, by jumping from O reaches A, can not reach also B. The same argument holds for a frog that stops in B and then it does not stop in A + B.

We note that Albero's proof resonates with Peano's approach to arithmetic (1889). The proof, indeed, is based on the invariance under translation of the sum ("going from A to A + B is like going from O to B"). Once again, we observe that the clever idea was to set the problem in a discrete domain (the frog versus the traveller).

The following quotation from the *a posteriori* interview adds further information to Albero's approach:

> If one wonders [...] I have to work on multiples, then let's look the numbers in the face! [...] I spoke of frogs, but I can imagine this series of numbers, this meter, as bulbs that switch on at the same distance: and you look on them, and once you have looked on them [...]. Perhaps someone does not understand anything and imagines numbers as sacks, this person does not look multiples in the face [...]
> We are talking about numbers, what are they? Numbers are equidistant things, which count equally and never finish. Adding is like going back to zero and starting again to jump.

Albero's use of expressions, such as "look the numbers in the face", stresses, besides his quest for meaning, his peculiar relationship with numbers, that are seen as persons. We underline Albero's awareness of having adopted a discrete approach that is functional to the problem and his skill in analyzing and explaining his way of reasoning. For instance, the invariance under translation of the sum is efficiently illustrated in the chosen metaphor, see the sentence "Adding is like going back to zero and starting again to jump". We stress the accuracy in choosing the metaphors: thinking of numbers as sacks would not allow him to deal with the concept of multiples, which is at the core of the problem ("Perhaps someone does not understand anything and imagines numbers as sacks, this person does not look multiples in the face").

> 5. In this way in intuitive arithmetic and with intuitive methods I have proved (I take on the responsibility for this word, here among friends) the thesis.

We note that at this point Albero reflects on the nature of his proof. Firstly, he focuses on intuitiveness: for him "intuitive"

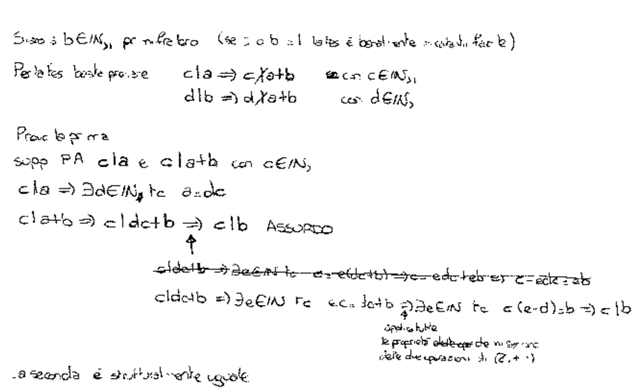

Figure 3: Albero's proof.

means being outside formal mathematics (no formulas, no symbols, no explicit rules of inference) Actually, Albero's reasoning is "isomorphic" to a correct proof of the statement expressed in the conventional mathematical way, only the way of communicating is not conventional. Furthermore, he regards his proof as a real proof ("I have proved"), and feels completely responsible ("I take on the responsibility") of it. This behavior is not common among students: usually they are influenced by the belief that a proof is acceptable only if it is presented in the conventional way, see the ritual proof scheme in Harel and Sowder (1998).

The second part of Albero's written protocol has a completely different flavor in comparison with the first part. He writes:

> 6 Now, since there is time left, we'll make it in the form of an algebraic proof.

His proof is correct, clear and concise as could be found in a textbook (see Figure 3) It appears to us to be an automatic product coming from the brain and not from the heart The natural fluency of Albero's algebraic proof confirms that the previous, non-conventional proof did not originate from a lack of knowledge or skill, but from a natural inclination to metaphorical thinking.

Concluding remarks

In Furinghetti and Morselli (2004) and in this present article we have considered the intertwining of affect and cognition in the performance of an unsuccessful and of a successful solver. Our main conclusion is that, while in the case of an unsuccessful solver affect acts primarily as a burden and a hindrance in the process of proving, in the case of a good solver affect acts as a motivation for mathematical thinking. This article offers insights into the nature of affect in a good performance and clarifies the reasons for such a role.

We see evidence of the intertwining of affect and cognition in Albero's way of approaching the problem and in the peculiarity of his solution. Both facts (approach to the problem and way of solving the problem) are characterized by the presence of metaphors. Albero uses different types of metaphors. We identify *narrative metaphors* that are used to communicate his flow of thoughts (*e.g.*, "I want to grasp their secrets"; "let's look the numbers in the face") and *constitutive metaphors* that ground the reasoning in the mathematical situation (*e.g.*, "the frog that jumps from stop to stop"; "Adding is like going back to zero and starting again to jump"). These metaphors live between affect and cognition.

Other evidence of the intertwining is the strong role of aesthetic values in Albero's proof. In every moment of the proving process, Albero is able "to combine information and imagination when making purposeful decisions regarding meaning and pleasure" (Sinclair, 2003, p. 200). We say that aesthetics permeate Albero's process of proving The mathematical activity and doing mathematics are a pleasure for him. This is in accordance with Polya (1945) who identifies pleasure in solving problems as one of the elements fostering a good solution.

We also note that all through Albero's written protocol there is the distinction between the two types of mathematical activity: the public (the mathematics of the professional mathematicians) and the private (Albero's personal construction of meaning) For him, mathematics is not a construction imported from outside, rather it is a part of his being Mathematics is firstly a private business and there is strong emotional involvement

In the case of Albero, the intertwining encompasses elements that Urban (1995) would ascribe to creativity: knowledge of the mathematics at issue, skills in proving, fluency, originality, playfulness and nonconformity. The work of Maslow (1962) helps to identify other interesting aspects of our student's creativity In particular, Albero is a significant example of *self-actualizing* creativity: he has a powerful intuition, an easy relationship with his thoughts, and manages his ideas through sharp reasoning. Maslow claims that creative subjects do not fear *the unknown and bewildering*: while ordinary people feel uncomfortable with situations of doubt and uncertainty, creative people live in such situations as a pleasant challenge. For this point, the comparison of Albero's written protocol with the written protocol of the unsuccessful solver, analyzed in Furinghetti and Morselli (2004), offers interesting insights The unsuccessful solver makes explicit her anguish about her lack of knowledge and feels unsafe and uncomfortable. On the contrary, Albero feels good and throws himself into the challenge with enjoyment. His behavior fits with Maslow's description of self-actualizing creativity: he is not inhibited by self-criticism and by fear of judgments by other people ("do not ask me to be formal, otherwise I lose the good thing"; "In this way [...] I have proved (I take on the responsibility for this word, here among friends) the thesis.")

In analyzing Albero's protocol we realized how right Maslow is when he claims that being creative means keeping something of one's childhood. Saint-Exupéry (1943) says that all adults have been children [5] – we would add that all adults should remember that and sometimes let themselves go

Notes

[1] Hadamard, 1954; Hardy, 1989; Poincaré, 1952.

[2] Emotions and feelings were not explicitly mentioned as required information, nor were they mentioned in the theoretical lectures delivered during the course of mathematics education.

[3] From the sentence erased by the student, but still readable in the protocol, we know that, initially, the student thought to use a traveler as a chief character, but he shifted to the image of the jumping frogs that fitted better with the metaphor of natural numbers seen as stops in a path

[4] Talking about metaphors and mathematics, it is well known that Hardy used illustrations taken from cricket (one of his favorite sports) for dis-

cussing philosophical or mathematical issues As an example, we recall that his paper (Hardy and Littlewood, 1930) contains the sentences:

> The problem is most easily grasped when stated in the language of cricket [...] Suppose that a batsman plays, in a given season, a given 'stock' of innings [...]. (p. 83)

[5] "Toutes les grandes personnes ont d'abord été des enfants (mais peu d'entre elles s'en souviennent)" (Saint-Exupéry, 1943, in the dedication)

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