Interacting Reflections on a Young Pupil's Work

JANET DUFFIN, ADRIAN SIMPSON*

The thinking which gave rise to this article came from a piece of work done by a seven year old girl in a school which is taking part in the CAN (Calculator Aware Number) curriculum project. It came about because one of us, the mathematics educator, showed the work to a colleague who is a mathematician. Both of us were impressed by the work, but our initial responses to it were so radically different that it prompted us to think further about the work and the reasons for our different initial reactions.

Our thinking took us through ideas relating to creativity in mathematics and into reflecting as an intrinsic part of mathematical development. The experience was rewarding for both of us and led us to recognize that our own reactions and subsequent reflection, and the fusing and extension of our perceptions which the experience engendered, were things which could be replicated in the classroom. We saw the pupil in the role of the mathematician, the teacher in the role of the mathematics educator and we felt that our experience was one to be shared with workers in the field who might also be interested in the processes and product of the learning of mathematics.

We became aware, as part of this personal growth, that our previous experience and independent thinking on the learning of mathematics had itself contributed to the growth we shared. The mathematician's interests are in undergraduates' mathematical development related to the Perry scheme, the educator's in the implications of the work of the constructivists for classroom learning.

Our different focuses formed a catalyst for our initial reactions; our shared perceptions identified aspects of learning we might neither of us have come to without the interaction with the other.

Janet's reaction

Vicky is a seven year old girl in one of the second wave CAN schools. CAN schools are those which are participating in the Calculator Aware Number project, designed originally as a curriculum development exercise, seeking a number curriculum which would take account of the growing use of calculators in the community.

Initial guidelines to participating schools stipulated the free availability of a calculator in the classroom and the development of mental facility to accompany its use. The most controversial aspect of the project is that teaching of the standard arithmetical algorithms for the four basic operations was to be discontinued. Open-ended activities were introduced into classrooms and investigative work, both with and without calculator use, became the norm.

The changed role of the teacher emerged, perhaps as a result of the ban on the algorithms, and took the project outside its original function of curriculum reform. This changed role freed the children from former constraints and enabled them to demonstrate a hitherto unobserved power to devise their own methods and strategies. Teachers became managers of the children's environment and learned to observe and guide children's progress rather than being the authoritarian prescribers of content and technique.

Vicky's school had entered the project three years after it began, so her teacher's experience of the project's way of working was still at a relatively early stage. He was pleased with what Vicky had done and suggested I talk with her about it.

V: Our teacher asked us to try and make dots for the square numbers and work out how many dots there were.

T: Can you tell me about it?

V: I did the dots and counted them up to there (pointing to the 8 square) and then I decided it would be too hard to add up so I put them down and added two together to make 18 and I did it all the way down. And then I added the two eighteens together to make 36 and the last one was nine, so that made 45. So I had 36 and 45 and that made 81.

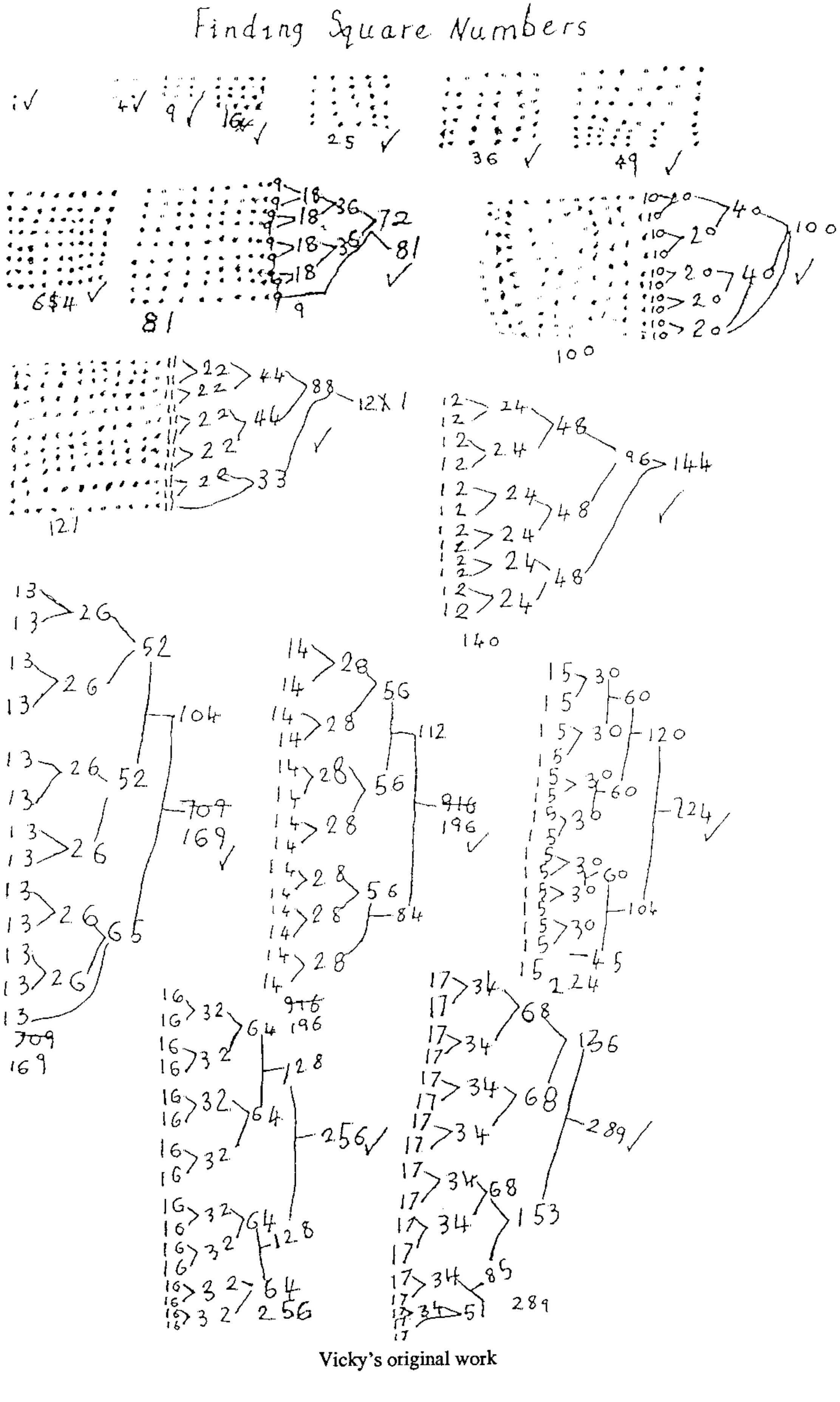
T: How did you add them?

V: I said 30, 40, 50, 60, 70 and then I added the 6 and that made 76 and then I added the 5 and that made 81. And then I did the same for all the others up to 17, but I didn't put down the dots.

(My reason for asking how she had done the addition was that I am constantly checking children's methods in order to discover prevailing strategies.)

Quite suddenly, at the end of our conversation, she said "But this isn't the original. I did it first all on one page, but my teacher said it was too squashed up and asked me to do it again like this." My immediate response to this was to wish that I had seen the original rather than the "fair copy" because I have noticed that teachers are greatly concerned to have a neatened up version of children's work, whereas I like the originals because they are more informative for those interested in the child's thought processes and procedures.

*This author is supported by an S.E.R.C. research studentship.



Vicky's original work

It was when I had seen both versions that the whole exercise erupted into something more significant than first impressions suggested. As Evaluator of the project I was interested in the way that Vicky discarded the dots when she had established a systematic way of arriving at the total for any one square. It also impressed me that she had continued this as far as seventeen squared. Her device of "chevroning" did not surprise me as I had seen it on several occasions amongst CAN children as a device for aiding additions; nor did her idea for grouping in pairs and thereby doubling while halving the number of calculations to be done, for CAN children's experience includes halving and doubling exercises as well as investigating odds and evens. So the kind of strategies she was using seems to come naturally out of the kind of experience she was likely to have had in the project.

This initial response to the work did not at all match that of my co-author, who carries on from here.

Adrian's reaction

If I try to reconstruct my initial reactions to seeing Vicky's work when I was shown a copy of her original version for a couple of minutes, what stands out in my memory is, having returned to my office, wiping clean the blackboard and copying down what she had done for a couple of cases. It was perhaps a typically selfish reaction of a mathematician: I wanted to take Vicky's work and translate it into *my* mathematics, to see what lay at the root of what she did.

I began just to play with her method, notation and all, by using it on larger numbers, 20, 21, 22 and so on. The surprising efficiency of her algorithm, not in its recording but in the number of calculations it took, caught my attention quite quickly. Naïvely adding up the 9s, for example, would have required 8 additions, yet she needed only four ($9 + 9 = 18$, $18 + 18 = 36$, $36 + 36 = 72$, $72 + 9 = 81$).

An emerging pattern seemed to be important—an "odd man out" which appeared or was added to in certain columns. In her work on nines it appears in the second column as a nine after four 18s and then remains until it gets added in at the last stage. Reconstructing her work for myself meant that many of her inconsistencies did not appear on my blackboard and the "odd man out" was handled more clearly.

The oddness or evenness of the lines in the column (not including the possible extra line for the "odd man out" or "leftovers") seemed to determine whether the leftovers were added to or not, so for 33 I had 33 numbers in the first column, 16 and a leftover in the second, then 8 and a leftover, 4 and a leftover, 2 and a leftover, 1 and a leftover and then the final result. Specializing further with 34, 35, 36 and so on, looking only at the *quantity* of numbers in each column led me to see the importance of halving and to examine the process in binary (perhaps an unstated inkling at the back of my mind gave me the push in this fruitful direction).

I was still interested in the number of additions she was using in her method and by examining a binary set-up, it became very clear. For example:

For 37^2

100101	1st column:	10010 entries 1 leftover 1 addition
	2nd column:	1001 entries 1 leftover 1 addition
	3rd column:	100 entries 1 leftover (added to) 2 additions
	4th column:	10 entries 1 leftover 1 addition
	5th column:	1 entry 1 leftover 1 addition
	6th column:	1 entry 1 addition

So in Vicky's method she needed to do one addition to move from each column to the next and an extra addition each time there was an odd number of entries to add on to the leftovers (except for the first time that happens when there are no leftovers to add to, and the last stage where the leftovers are added in to the main additions). That is, she requires $n + k - 2$ additions where n is the length of the binary representation of her number (the number of columns) and k is the number of 1s in that representation (the number of times the leftovers are added to).

But looking at the process in this way did more than just highlight the number of calculations needed, it struck a chord with me—this halving and doubling was precisely the way I recalled being told that a computer multiplied (halving, doubling and adding being cheap processes for a binary machine). A quick visit to the library brought up a book that not only explains the method, but how it can be derived from the naïve way of adding one value to an accumulator the other value's number of times and, in an exercise, shows how the method relates to the classical method of multiplying in base 10. [Berlioux and Bizard, 1986]

This startled me sufficiently to look back at Vicky's work and reflect upon how I had worked on it to get as far as I had. I began to realize that it raised issues which touched upon my own area of research, the development of undergraduates' understanding of the nature of mathematics, as I saw in the work of a seven year old what I felt was missing from the work of undergraduates—discovery, autonomy and creativity.

Ideas on creativity

We found that our initial common focus was the idea that Vicky's work deserved the epithet "creative" and, reading a thesis [Copes, 1974] dealing with aspects of creativity, we came across another episode with features somewhat akin to those shown in the work of Vicky. These two incidents (the work of Vicky and the example quoted by Copes) prompted us to investigate meanings of creativity in order to try to negotiate a meaning which satisfied us both.

Creativity appears to be a concept which allows for a number of different interpretations. We should like to consider those of White [1968] and Copes [1974] in an attempt to arrive at a perception of creativity which could usefully be applied to Vicky's work and that of other children and pupils in an educational setting.

White's approach to the concept seems to be to offer a definition and then to proceed to giving examples which satisfy that definition. Copes, on the other hand, prefers to approach the question by looking at examples of work which appear to him to be creative and then to try to identify a common element or elements which can be used to provide a definition within the context. We prefer Copes's method of approach but we are led to bring an extra dimension into the picture, a dimension which arises from our different reactions to both Vicky's work and the example quoted by Copes.

White postulates what could be called "global" creativity. For him:

> A creative thinker is one whose thinking leads to a result which conforms to criteria of value in one domain or another. "Creative" is a medal we pin on public products.

He cites Newton and Einstein as creative thinkers by virtue of the original work done by them which changed the face of mathematics in their time. His definition would not allow Vicky the "medal" of creativity because there is no way that what she did could be said to have added to the sum of human knowledge. Indeed, it is unlikely that any school pupil could receive this accolade.

His definition has another limitation. What would be the attitude of White to some hypothetical work which might be discovered replicating, but predating, that of Einstein or Newton? Would we have to strip them of their "medal" and postulate them to be no longer the creative workers we have long thought them?

Copes, on the other hand, appears to offer an answer to this and to put the concept within the reach of more ordinary thinkers by virtue of the requirement which he identified as characterizing creative work. He believes that, in creative work, there is an element of "surprise" which comes into the observer's response to it. He cites the case of a third grade boy in an American classroom, described by Davis [1966], when his teacher was introducing the well-established decomposition technique for subtraction. The example was 64 - 28 and he quotes:

> She said "You can't subtract 8 from 4, so you take 10 from the 60..." A third year boy, Kye, interrupted. "Oh yes you can! 4 - 8 is -4, and 60 - 20 is 40, 40 - 4 is 36 so 64 - 28 is 36."

Although this method was new to the teacher and to Davis at the time, it was not original, but, according to Smith [1954] is attributable to a man called Colson in the eighteenth century. Kye's use of it would, therefore, not be seen as creative under the White definition, though it would be under that of Copes. "Creativity," he says, "involves an unexpected way of looking at things.... it must be unexpected, unpredicted in exact content."

This concept of creativity therefore extends the White definition to enable it to be applicable to workers other than those in the White domain. It does not exclude White's domain but rather extends it. It could perhaps be said to create a "local" creativity, the surprise and unexpectedness in the classroom context being the feature that makes it creative.

What was interesting to us was the way in which our initial reactions to both the Kye and Vicky episodes differed: work done by children of different sexes, in different countries and different mathematical contexts but with strong features in common. Both children surprised their teacher; both children surprised one of us but not the other, because the latter had often met the Kye method before amongst CAN children and because the Vicky work could easily be seen to be derived from the kind of activities current in the CAN classrooms. What Vicky had done was to use her experience of those activities to enable her to find a procedure which helped her in this new situation.

We should like to return to the perceptions of the observer in such situations as the Kye and Vicky episodes describe because we feel they have an important bearing on classroom practice. But at this point what we want to do is to extend the Copes definition beyond that of surprise and unexpectedness to one which suggests that creativity comes from taking past experience and building on it: in Kye's case, to be able to see another way of thinking about subtraction than that of his teacher; in the case of Vicky, being able to develop a procedure which enabled her to calculate the squares of numbers by calling on her previous experience. In each case the procedure was useful for the occasion and for future occasions in their learning.

By this means we can come to an extension of both the White and Copes definitions of creativity, moving from the "global" (with its requirement of universal originality) and from the "local" (with its requirement of something unexpected and surprising) to the "personal" which requires that the procedure shall be "useful and new" to the person creating it.

For this to be possible in the classroom, children have to be permitted to create their own procedures, have to be encouraged to take an active part in their own learning instead of being in a situation in which, at worst, they are told what to do by their teacher and at best, they are "helped" toward the teacher's way of proceeding by questioning and guidance. In both of these cases success is measured by the closeness of the children's output to what the teacher is looking for.

In contrast, children in the CAN project, and indeed all participants in it, are encouraged to be autonomous and responsible for their own development within it. The children develop their own strategies which are accepted by the teacher, whose new responsibility is to help them to this autonomy by valuing instead of judging children's contributions.

It is at this juncture that the place of perception of the observer as we experienced it in connection with Vicky's and Kye's work again comes into prominence. For we were obliged to reflect upon our initial responses and it is to that reflection that we should now like to turn.

Reflecting on reflection

Though we were both impressed by what Vicky had produced, we responded to the work in different ways. One of us was intrigued by the mathematics and spent time extending and identifying it as a legitimate algorithm capable of generalization; the other was much more concerned with the way in which Vicky came to a systematic way of working, after which she was able to discard the dots which had been her starting point, in order to develop an arithmetical strategy for generating square numbers to a point beyond what she was likely to have been able to reach by straight multiplication.

When we had shared our responses to the work and reflected upon their differences, we became aware that we had approached the work with a different focus: either on the mathematics or on the child's processes. The act of sharing those approaches had both extended our personal viewpoint and made us conscious of the effect of our own professional positions and experiences on our initial responses. It was not until the mathematician had divulged his response, which uncovered the general algorithm hidden within the work, that the educator began to focus on the mathematics and it was not until the educator had divulged her response that the mathematician became aware of the significant processes involved in the development of the method. The act of sharing our responses focused our attention on the other's perception and our interest in the work was furthered. We were conscious of the way in which a realization of the other's perception increased our satisfaction in the work besides extending our own individual appreciation of it. We began to reflect on these reflections in order to identify what it had done for us. Because we began to perceive that what had come about for us as we viewed this piece of work might have sources of useful information about what may be achieved in the classroom, we tried to articulate the processes we had used and the outcomes reached.

The initial reactions to Vicky's work took us no more than a few hours to form and articulate. Considerably more time was taken up after that reflecting on those reactions and, indeed, it is the reflection which we feel was the more significant and valuable activity. This reflection took place in two noticeable ways as we came to try to understand how and why our reactions differed and how we felt about the work that we had done in decoding Vicky's work from our own perspectives. We worked apart, trying to articulate our feelings, to distinguish personal key points and to check our understanding of our own and each other's reflective statements. We also worked together, discussing our individual work, matching significant points and justifying and explaining our accounts. In doing this, we both felt that we had come to extend our understanding of the issues raised for us by Vicky's work, we came to clarify and justify our feelings about some of the notions involved that had previously been hazy or ill-formed. With this came a great sense of personal satisfaction and achievement. The interchange of perspectives that came from attempting to reconcile two distinct initial foci, highlighted and improved our sensitivity to the problems faced by us as teachers, learners and observers in mathematically active environments.

It would certainly be trite and clichéd to suggest that we are now "better people", but we feel that we have been educated by the experience of trying to understand Vicky's work, of trying to understand our own reactions and of trying to understand our own reactions and of trying to understand each other's viewpoints, and that that education has come most significantly from the act of reflecting.

Applying these ideas back to the scenario from which we built so much, the pupil in the classroom, we may explore the role of reflection there and ask what reflection is and how people in different roles could go about reflecting. Both the pupil and the teacher can use reflection as a tool for clarification. They can use it to ensure that they feel happy about and can articulate to the satisfaction of others their construction of meaning of an incident. In the child's case, they can make clear what they mean by certain mathematical concepts and methods; in the teacher's case, they can clarify their understanding of what the child seems to have come to know.

In the classroom, the pupils may use reflection to construct arguments that justify their methods and which may be used to convince others. The teacher may use it to consider what led the pupils to their understanding and how their approach relates to the struggles or successes of others.

The classroom can provide for three types of reflection to allow the growth of understanding along these lines:

- self reflection:

  —on the part of the teacher (though, given the fierce pressure of time and attention on the teacher in an active classroom, this must necessarily take place away from the class).

  —on the part of the pupil.

- interaction between the pupil and the teacher which allows them to form a mutual account of the mathematics learning that has taken place.

- peer group reflection:

  —by the teacher with observers and fellow-teachers with whom incidents can be shared and discussed.

  —by the pupil with fellow pupils, where the clarity of an argument can be checked, and discussion of solutions or solution methods take place in an atmosphere more of co-exploration and co-discovery than can usually be expected in teacher-pupil discussions.

We have been led from an interesting, but by no means astonishing, piece of work to consider how we reacted to it, to discuss what we meant when we were led to describe it as creative, to reflect on how and why our reactions differed and on how that reflection had helped us gain from the experience and finally to reflect upon the role of similar sorts of reflective processes in the classroom. There, the child's interest is on creating and discovering mathematics whilst the teacher's is on creating an environment in which the children may further their ownership of their mathematics. So the focus of the children is on the mathematics whilst the focus of the teacher is on the learning of mathematics, which, if one looks back, are just our two initial reactions to Vicky's stimulating work.

References

Berlioux, P. and Bizard, P. [1986] *Algorithms: the construction, proof and analysis of programs.* John Wiley and Sons, Chichester

Copes, L.E [1974] *Teaching models for college mathematics.* Unpublished doctoral dissertation, Syracuse University

Davis, R.B. [1966] Discovery in the Teaching of Mathematics. In Shulman, L.S and Keislar, E.R [Eds.] *Learning by discovery: a critical appraisal.* Rand McNally & Co., Chicago

Smith, C.A.B. [1954] *Biomathematics: the principles of mathematics for students of biology and science* Hafner

White, F.P. [1968] Creativity and Education: a Philosophical Analysis, *British Journal of Educational Studies,* Volume 16, pp. 123-137

Bibliography

Cobb, P. [1988] Multiperspectives. Paper presented at ICME-6, Budapest

Copes, L.E. [1982] The Perry Development Scheme: a Metaphor for Learning and Teaching Mathematics, *For the Learning of Mathematics,* Vol. 3, No. 1, pp 38-44

Davis, P.J. and Mason, J.H. [1990] Notes on a Radical Constructivist Epistemology Applied to Didactic Situations. Preprint

Ernest, P [1991] *The philosophy of mathematics education* Falmer Press, Basingstoke

Glaserfeld, E. von [1987] Learning as a Constructive Activity. In Janvier, C (ed) *Problems of representation in the teaching and learning of mathematics.* Erlbaum, Hillsdale

Kamii, C. [1985] *Young children re-invent arithmetic.* Teachers College Press, New York

Mason, J. [1989] Teaching (Pupils to Make Sense) and Assessing (the Sense That They Make). In Ernest, P. (ed) *Mathematics teaching: the state of the art.* Falmer Press, Basingstoke

Perry, W.G [1970] *Forms of intellectual and ethical development in the college years: a scheme.* Holt, Rinehart and Winston, New York

Perry, W.G. [1981] Cognitive and Ethical Growth: The Making of Meaning. In Chickering, A.W. (ed) *The modern American college.* Jossey-Bass, San Francisco

Sinclair, H. [1988] The Interactive Re-creation of Knowledge. Paper presented at ICME-6, Budapest.

The move I am making beyond Kant can be summed up in the following way: I am suggesting that Kant s greatest contribution to our understanding of meaning and rationality was his work on imagination which ironically his system forces him to separate sharply from reason and understanding I am thus led to deny that the metaphysical and epistemological dichotomies presupposed by his system are rigid and absolute I regard them rather as poles on a continuum of cognitive structure By taking imagination as central I see its structures as a massive, embodied complex of meaning upon which conceptualization and propositional judgment depend Meaning is broader and deeper than the mere surface of this experiential complex — a surface that we peel off (cognitively) as concepts and propositional contents We also see that meaning is not always, or even usually univocal as Kant seems to think when he defines concepts as rules specifying lists of features At least where human conceptualization is metaphorical there is not a core underlying set of literal propositions into which the metaphor can be translated Finally rationality resides in *all* of these structures taken together. *each* with *their* own special constraints

Mark Johnson