PARADOX IN MATHEMATICS EDUCATION: PSYCHO-ANALYSIS, METONYMY AND PRECARIOUS TIMES

ALF COLES

One oddity in trends of thought has been the limited treatment of ideas from psycho-analysis in English language mathematics education journals. I say an oddity, because psycho-analytic theory has articulated sophisticated ideas around learning from experience and group dynamics, potentially relevant to classrooms and meetings of teachers. This journal has been a notable exception, for instance, a special issue edited by Dick Tahta (including Pimm, 1993a) and, some years later, writing by Berdot, Blanchard-Laville and Bronner (2001), this latter piece translated by David Pimm and Nathalie Sinclair. A theme across such writing is the presence of unconscious processes in the teaching and learning of mathematics, often linked to symbolism and the act of symbolising, and the potential for tension and conflict to arise. The psycho-analytic strands of Pimm's writing are less well known than his work on language (although highly inter-connected) and one purpose of this article is to articulate threads of that thinking, which I argue are particularly relevant now.

A significant linguistic distinction Pimm (1988) draws is between metaphor and the less well-known idea of metonymy. He defines metonymy as "a *substitution* of names, X *for* Y, rather than a metaphor, seeing X *as* Y" (p. 32) and offers four forms of metonymy: "part for whole"; "object used for user"; "place for event"; and, "producer for produced". Particularly relevant to learning mathematics is the part-for-whole form of metonymy. Tahta (1998) wrote about ways in which number names can be invoked, metonymically, as a substitute for number concepts, in allowing children access to patterns in our number system. For instance, the linguistic transformation from "six" to "six-ty" and "seven" to "seven-ty" potentially offers insight into number structure and an entry into working with number (name) relationships.

Invoking ideas of metonymy, Pimm (1995) pointed to how there are times in the doing of any mathematics in which symbols might be invoked with relatively little connection to their meaning. In solving a problem, I might write $x^2 + 2x + 1 = 0$ and re-write this as $(x + 1)^2 = 0$, with attention on little else than the symbols and their rules of combination. The transformations I perform are metonymical in the sense that any context or relation the variable has to a wider problem is substituted for the symbol itself. Having arrived at a solution of $x = -1$, there may then be a more metaphorical searching for what this means in the context of the wider problem (if there is one). One of the difficulties for learners of mathematics is to know just which properties or associations (which parts) are relevant in any particular concept at any particular time, since metonymical substitutions are often invoked invisibly (p. 113).

Perhaps deeper sources of difficulty for learners might be in play, as Pimm (1993a) suggests, when he asks what must be suppressed, in order to gain the kind of metonymical fluency with mathematical symbols pointed to above; what meanings and associations must be denied and ignored, and, at what cost? (p. 35). Mathematics can be valued for its seeming other-wordly precision, a precision we might be cautious about; Brown (1993) urged a questioning of anywhere we spot an aesthetics of clarity being viewed as a natural or assumed way mathematics should be done, or learnt, or taught.

And when it comes to questions of imposing clarity, and concomitant suppressions, psycho-analytic thought potentially has something to offer. Wilfred Bion was a psycho-analyst who was one of the first to develop approaches to group analysis, from having to deal with shell-shocked soldiers during the First World War. He viewed the development of thinking as intimately linked to our capacity to tolerate frustration, a frustration being experienced in the time between a moment when a "want" is felt, and the moment (which of course may never come) when that want is satisfied. The primary frustration invoked by Bion is the absence of the mother, when hungry. He took "thought" to be a combination of a preconception (*e.g.,* an unconscious expectation of food) and a frustration. So, thought is provoked by absence and the intimate linking of thought and emotion is present from the outset.

Bion (1967) suggested there are two parallel mental developments in humans, the development of thoughts and the development of the capacity to cope with those thoughts. It is this latter capacity which Bion called "thinking" and, he claimed, it is the pressure of thoughts that provokes the need to learn a way of coping with them (*i.e.,* thoughts produce thinking, not the other way around). However, things do not always run smoothly. If conditions are overwhelming, then rather than thought leading to thinking, which makes frustration tolerable, the psyche resorts to evading frustration. In such cases, Bion suggested: "What should be a thought […] becomes a bad object, indistinguishable from a thing-in-itself" (p. 180). Rather than substituting X *for* Y, or seeing X *as* Y, in order to think about it, X *is* Y. Instead of tolerating frustration through thinking, the "want" becomes the "bad object" itself (not just a thought about that object), which must be evacuated or suppressed. Words are experienced as actions or objects (Segal, 1991, p. 34). There is no learning from experience and there are well recognised processes where suppression takes place through projecting danger onto someone or something else (such as mathematics, or, conversely, everything apart from mathematics).

Pimm has pointed to the role of metonymy in becoming successful at mathematics. In working mathematically, there are times when we treat the symbol *as* the object or concept. Bion points to a different form of substitution that inhibits the capacity to learn, the symbol *is* the object. What could have been experienced as a thought about, or awareness of, an absence, becomes the thing feared or the absent-object itself and, as a consequence, must be projected, suppressed or annihilated. The psyche intervenes before thinking is able to hold a thought at all.

Paradox in precarious times

At a time of potential ecological collapse there are pressing thoughts about our relationship to the world and each other that are no doubt often evaded, but also (to phrase the issue in Bion's language) each time we give ourselves to contemplating our precarity, we may be developing our capacity for thinking. A question it seems an increasing number of mathematics educators (*e.g.,* Skovsmose, 2019) have been asking, is, how and when will global crises impact on teaching and learning mathematics? Psycho-analysis, seen as a form of rehabilitation, perhaps has renewed relevance at a time when the world is experiencing such a concatenation of crises that the word crisis (which has a sense of a temporary hiatus) may not be appropriate. Instead of *crisis*, the word *precarity* perhaps better captures the sense that instability is (and always has been) our permanent condition. The skilfulness with which so many of us, in affluent parts of the world, seem to avoid acting on our knowledge of precarity suggests some kind of pyscho-analytic process of repression may be in play; the alternative may be intolerable. When we do contemplate the paradox of how we seem to know, and yet not know, what we are collectively doing to the world, a common and unhelpful response seems to be guilt. Stengers (2013) challenges us to action on precisely this point:

> This conviction situates me as part of a generation that may turn out to be the most hated in human memory. We knew, and just felt guilty. (p. 156)

The question implied by Stengers is, how might our knowing lead to new actions? Or, to put it differently, is there an alternative to using our capacity to split our knowledge of the implications of current ways of living in the global North, and our acting, for instance in mathematics education, as though we can continue with business-as-usual? If the knowing of relevance here is about precarity and the messiness of the world, how might we *stay with the trouble* (Haraway, 2016) long enough to reach new insight, before annihilating unwanted thoughts?

This article is an attempt at staying with some of the troubles of mathematics education, through working deliberately with paradox. Winnicott (1971), following Bion (1967), viewed the tolerance of ambiguity, frustration and paradox as essential to the birth of play and thence creativity and culture. Winnicott (1971) suggested there is a *potential space* in which contradictions can exist simultaneously and unresolved. The aim of resolving paradoxes suggests a dialectic or synthesis, whereas the space Winnicott describes is one where contradictions are legitimised. My hope is that such potential spaces might offer possibilities about a curriculum and ways of working relevant to a mathematics education better suited to precarious times.

If a paradox is to be held in conjunction it seems clear that our usual logics will offer little help. I align with Stengers (2009) in believing that: "The point is to try and learn to feel affinities and divergences, which are not psychological ones, but are related to the very exercise of thought" (p. 30). A form of logic that might be appropriate is described in the next section.

A recursive logic

Recursiveness and paradox become linked in the phenomena where distinctions "re-enter" themselves. For instance, "All truths are relative" is a statement that invites a distinction between absolute and relative truth and, in proposing all truths are relative, the statement seems to eliminate the very distinction that gives it meaning. Wolfe (2010) suggests that such recursiveness is not just an arcane aspect of analytic philosophy but rather points towards profound paradoxes of observation and self-reference that he sees as central to existence as a living being. He proposes the appropriateness of such a recursive logic in our current time, a logic that shifts between different levels or orders of observations, a logic that can loop between taking a distinction as a given and bringing that distinction into question, or between what is a part and what is a whole.

A recursive logic is in play in any moment of observation. An observation presupposes an observer and something observed, yet a recursive logic undercuts distinctions between inside/outside or system/environment by drawing attention to the temporality and self-reference of all observation (human or non-human), in that an observation can never include the fact of its own observation. An observation can never escape the historical, material and contingent perspective from which it is made. Any observation creates a momentary subject, from whose viewpoint that observation is made. Such a subject is a contingent, evolving being that is as much affected by an observation as affecting. The apparent continuity of our (human) experience can perhaps lead us into imagining ourselves as stable and relatively fixed and possessing certain elements of knowledge or skill. A recursive logic disrupts such easy assumptions; paradoxical contradictions can combine, I can both make a distinction (*e.g.,* between myself and my surroundings) and undercut it (*e.g.,* knowing that I am inseparable from my surroundings).

In the next sections, I look towards paradoxes (a) in mathematics, (b) in teaching mathematics, attempting to bring a recursive logic to bear. Links to David Pimm's thinking are sometimes implicit, but have guided each choice of focus. The theme of precarity is one I return to by way of conclusion.

Paradox in mathematics

The indented text is a thought experiment, which I invite you to follow, slowly.

> These statements are axiomatic:
>
> 1. There is 1 point common to any 2 lines.
> 2. There is 1 line common to any 2 points.
>
> Imagine two lines (*m* and *n*) crossing at point P, with line *m* rotating anti-clockwise around A.
>
> Notice how the point of intersection (P) moves as *m* rotates (Figure 1). What happens to P when *m* is in the same direction as *n*? According to Axiom 1, *m* and *n* must still have a common point. Point P must be somewhere! When offering this task to participants at the symposium in Vancouver, David Pimm stood up and, without words, moved one arm to point up and the other arm to point down. It seems we are forced to propose a "point at infinity" that can be approached from either of the two directions along line *n*, and which is also a point on line *m*. If there is a point at infinity in one direction, there must be a point at infinity in the other direction, but since there is only 1 point common to 2 lines (Axiom 1), these points must be the same point.
>
> Now, keep *m* and *n* fixed in the same direction and introduce a new point Q and a rotating line *l*, which crosses *m* and *n* (Axiom 1). When line *l* swings around to be in the same direction as *m* and *n*, how many points at infinity are there (Figure 2)?
>
> If we retain the label P for the point at infinity common to lines *m* and *n*, then we know there must be a line joining P and Q (Axiom 2). This line cannot cross m and n anywhere else except at P (Axiom 1). Hence the line joining P and Q must be line *l*, and we can conclude that *l*, *m* and *n* all share a point at infinity [1]. Our placement of Q was arbitrary, hence, *there is one point at infinity common to all lines that share a common direction.*
>
> Is the same point at infinity common to lines in different directions?
>
> If lines *n* and *m* (in Figure 3) share a common direction, they meet at a point at infinity (Axiom 1); and, similarly for lines *r* and *s*. Lines *r* and *m* meet at point P, so they cannot share another common point (Axiom 1). Hence, *lines not sharing the same direction have different points at infinity.*
>
> We can now conclude that, in Figure 3, the point at infinity of lines *m* and *n* and the point at infinity of lines *r* and *s* must be joined by a line (Axiom 2). This is the (unique) line at infinity.

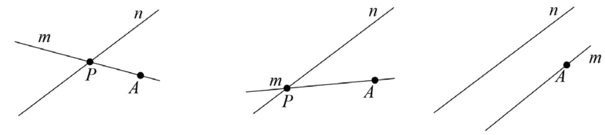

Figure 1. Two lines crossing.

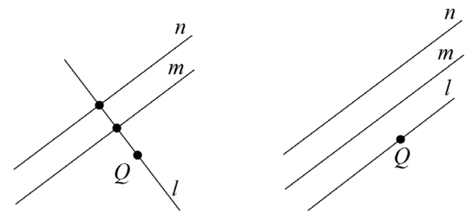

Figure 2. Three lines.

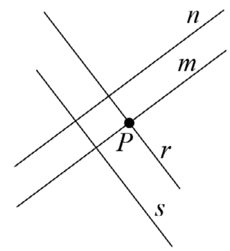

Figure 3. Lines in different directions.

The exercise above, if unfamiliar, requires a suspension of typical ways of making sense. The seeming ambiguity or even senselessness of new results needs to be tolerated, if there is to be an expansion of thinking (in Bion's terms). The experiment is one within projective geometry; the expansion of mind needed to contemplate points and lines at infinity is something I know I can feel physically as a loosening or softening. A point at infinity is reached in one direction and in the opposite direction, and in both directions; the part is the whole. A single point at infinity can be reached from an infinite number of different lines. The line at infinity somehow encircles the plane and requires a holding open of possibilities not acceptable within Euclidean thought.

Projective geometry leads to results that are false, from a Euclidean perspective, while at the same time Euclidean geometry is a special case of the projective. Parallel looking lines can both meet and never meet, depending on our assumptions. Invoking Pimm, we might ask what must be suppressed in becoming fluent with such thoughts? Bion's expansion of thinking, to cope with difficult thoughts, implies growth and maturation; Pimm suggests there is an inevitable cost. Perhaps one potential cost is the temptation to insulate paradoxes within the relatively safe systems of mathematics. We are led to the, perhaps familiar, distinction between truth seen as an absolute and truth relative to assumptions. And that distinction is one that, if we allow it to, brings into question the ground from which it is made.

Thinking from different sets of axioms gives some control and power to the mathematical concepts themselves and their materiality. The meeting of parallel lines can be invoked by the gesture of a tilt of the head (looking along a pair of lines drawn on paper), in contrast to the apparent "view from nowhere" that keeps their mutual distance constant [2]. Projective geometry invokes the perspective of the observer, to make sense of results and, in doing so, shows up the relativity, choice and, ultimately, arbitrariness of our point of view. Such shifting ground can be disturbing, and if we have made simplifications as adults (evacuating certain thoughts), in order to cope with life (perhaps through recourse to mathematics), it is the right of others to explore complexity in their learning, with all the discomforts this may provoke.

Paradox in teaching

Brousseau (1997) put it succinctly:

> Everything that she [the teacher] undertakes in order to make the student produce the behaviours that she expects tends to deprive this student of the necessary conditions for the understanding and the learning of the target notion; if the teacher says what it is that she wants, she can no longer obtain it. (p. 41)

Or, as interpreted by Mason (2002, p. 17), the issue for a teacher of mathematics is that the more explicit you are about the behaviours you want to see from students, the less likely it is those behaviours will emanate from students' awareness. And, the less explicit you are, the more likely it is students will not see the point of what you are offering. Pimm (1993b) refers to this as the didactic tension: "Whenever trying to teach someone something, there is the difficulty of indicating what the teacher is attending to and what is seen as being available to be learnt" (p. 28). A recursive logic is in play here, in that, attempting to offer a class, or a student, a distinction (attend to this/not anything else) the distinction undermines itself, as there is inevitably the ambiguity of attention to the act of distinction (*e.g.,* we attend to the teacher in the act of pointing, as well as, or instead of, what they are pointing towards).

Gattegno (1987) wrote about attending to families of concepts in mathematics and (as a teacher) discovering the invariant element of transformation. Such transformations can then be offered as a starting point for students, the transformation allowing a focus of attention on a "part" of the family of concepts on which students can act, and from which "part" a fuller concept can develop over time:

> The whole of my technique designed to lead to a mastery of the abstract concept can be reduced to […] beginning with a dynamic family of active concepts and to discovering, in the dynamic image, the invariant element of displacement or of transformation, which creates a dynamic and abstract concept arising from the sensory-motor concept, embracing it, but irrevocably larger. (p. 47)

This is a compact and complex quotation and one I have returned to consider, time and again, over the last 25 years. The current sense I make of it is exemplified through the case of learning number structure. Gattegno's Tens Chart offers an image of number-naming structure (there is a row of numerals 1 to 9; directly above them 10, 20, 30 …, 90; above that a hundreds row; more rows extend as far above, or below into decimals, as you want). The chart allows attention on transformations, such as the ones already mentioned, from "six" to "six-ty", "seven" to "seven-ty". Within what Sinclair and I have labelled a "symbolically structured environment" (Coles & Sinclair, 2019), I might work with children to develop a sensory-motor concept, in the sense of a patterning of gestures as they point to the units row and then the tens row in saying names. Precise constraints on communication and action are required initially (*e.g.,* provided by a teacher and by the environment) in order to focus attention onto the manner in which transformations link together (*e.g.,* moving from one row to the one above, and back again) occasioning students' awarenesses about their own actions (not the teacher's actions) that can be used creatively. Students can begin working with a narrow aspect of a group of mathematical concepts *as if* they are a full mathematical structure, for example, working with relations between number names, *as if* they hold the entire structure of the number system. Such an approach does not attempt to get around or solve the paradox of teaching but rather uses the tension as an affective entry point into mathematics for students themselves.

In other words, one perspective on the teacher's paradox is that the impossibility for a teacher to be precise about what she wants from students, points to the need for students to have the opportunity to express and act and to gain feedback from the environment. A game-like structure can offer a sense of intrigue about what this feedback will be and a desire to find out more. There is a sense in which the teacher can initially take responsibility for distinctions being used and then erase that role, if students take on the creative possibilities of a potential space. In common with conclusions from the previous two sections, in the process of supporting students' expressions of mathematics, different perspectives will need to be heard, tolerated and allowed as legitimate, while also negotiating spaces of conjunction. There are demands on the teacher's capacity to listen and hear, and a need to stay alert to our own blind spots, intolerances and suppressions.

Reflections

The possibilities envisioned in the preceding two sections have not related explicitly to the theme of precarity mentioned at the outset, although I hope they may have done so obliquely. I now turn to this, by way of conclusion. The UN's Sustainable Development Goals include "Quality Education" and UNESCO *et al.*'s (2016) "Education 2030 Framework for Action", states:

> Quality education fosters creativity and knowledge, and ensures the acquisition of the foundational skills of literacy and numeracy as well as analytical, problem-solving and other high-level cognitive, interpersonal and social skills. (p. 8)

It would be possible, as a mathematics educator, to take succour in the recognition of the need for numeracy and analytical problem-solving skills, in preparing students for the future. However, Winnicott's (1971) sense of the importance of tolerating paradox and frustration might make us wary of too quick a ducking of complex issues.

If, however, my teaching can help students experience staying with difference, difficulty, confusion, without an emotional closing down, perhaps it can serve as part of an education for precarious living. Suggestions that have emerged from the considerations of the earlier sections include: moves towards a curriculum that deliberately works with contradictory systems; and, teaching mathematics through working with dynamic families of transformation and invariance, in symbolically structured environments. These proposals might be interpreted as mechanisms for a dramatisation of mathematics learning, or for feeling our world with mathematics (Stengers, 2011). Mathematical concepts emerge with some agency, if offered to students within structures that allow a butting up against their constraints.

One aspect of our current time, related to precarity, is a recognition (*e.g.,* in many universities around the world) of the need for serious reflection and action in relation to historical and contemporary processes of colonisation and racial privilege. There is a pressing need for the capacity to consider viewpoints and ways of knowing other than our own. The recursive logic used in this article is present again in the distinction implicit in the notion of "others' ways of knowing", which can only be contemplated from our own point of view, under-cutting the very distinction. Yet the distinction is vital to hold and contemplate, while at the same time recognising it erases itself. We *can* learn from others, and from perspectives contradictory to our own. If we rarely do this, Pimm challenges us to consider: "How are we implicated in making ourselves (and others) unaware of certain connections?" (1993a, p. 35).

If we cannot escape paradoxes and the attempt to resolve them becomes self-defeating (the equivalent of evacuating thoughts as a way of coping with frustration) then Bion and Winnicott encourage us that it is through the tolerating of contradiction that we develop our thinking, our capacity for creativity and for new possibilities. And, to return to the Stengers quotation from that start of this article, the test of any new thinking is surely, now, whether it provokes us to do more than know and feel guilty.

Notes

[1] With thanks to John Mason for this reasoning.

[2] With thanks to Ricardo Nemirovsky for this thought experiment.

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This is perhaps where Winnicott's notion of 'third area' comes into play. […] This third area, both mine and not-mine, potentially provides an interesting means to pass beyond the seemingly perpetual desire to distinguish concrete from abstract (and possibly tactile from digital) dichotomously within mathematics education. And this is perhaps a core reason why students seem perennially directed to stop using their hands in order to access more formal (abstract) mathematics. However, TT provides a clear instance of the significance of the interaction between the hands and the eyes on the one hand, and the hands/eyes and the device on the other, when engaging in and thinking about multiplication.

— Sandy Bakos and David Pimm, from p. 163 of 'Beginning to multiply (with) dynamic digits: fingers as physical-digital hybrids'in *Digital Experiences in Mathematics Education*, **6**(2).