WHEN PROOFS REFLECT MORE ON ASSUMPTIONS THAN CONCLUSIONS

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"Mathematics is growing at the bottom as well as at the top" (Young, Denton & Mitchell, 1911, p. 7): this quotation reflects the need to engage advanced students in formalizing and systematizing mathematics, in the sense of identifying assumptions that underlie the mathematics they learn. In the structural metaphor of constructing mathematical theory, the top grows by proving conclusions based on previously established claims. Growth at the bottom, which refers to identifying grounding assumptions, was given the name "foundations", thus extending the metaphor. Most mathematics instructors familiar with the subject commonly called "foundations of mathematics" might prefer to spare students the related debates that ensued at the end of the nineteenth century. However, the questions of foundations provoked fascinating mathematics and the formalization of modern truth-functional logic.

Prior to foundations, most advanced mathematical proof was thought to play a *validating* or *explaining* role: epistemically or psychologically assigning truth-values to previously questionable statements to expand the catalog of mathematical facts. In contrast, Hilbert's axioms of geometry, published in 1899, were not motivated by a need to verify theorems' truths, but rather by a desire to show his axioms were sufficient to render those theorems *provable*. Rather than assigning truth-values to statements, the process of axiomatizing emphasized the links between the truth-values of axioms and theorems, which suggests why logic (truth-preserving arguments) was formalized around that time.

In the mathematics classroom, recognizing such relations between hypotheses and conclusions relates closely to what Freudenthal (1973) and Human and Nel (1984) called students' *systematizing* mathematical understandings. In what follows, I explore the entailments of an instructional project intended to foster students' systematizing activity. I demonstrate how investigating questions of *provability* can help students to assess possible definitions or axioms and to comprehend key ideas in the axiomatic game. In the same way that modern systematization arose in response to non-Euclidean geometry, the current investigation features students' learning about neutral axiomatic geometry in an upper-level undergraduate course.

Provability

Given the task of teaching students to produce and understand proofs, a volume of mathematics education literature has clarified the nature and purpose of mathematical proof. As early as 1976, Bell suggested that proofs are used for verification (what has also been called convincing), illumination (explaining or instructing), and systematization. Verification relates primarily to the truth of the claim and the logical validity of the argument intended to prove it. Illumination relates primarily to the experiential cognitive value of the claim and argument. Systematization situates a statement and proof within a theoretical context.

Because proofs of claims are contextual in the sense that they rely on systems of assumptions, definitions, and warrants, Mariotti *et al.* (1997) suggested that mathematical theorems are triads that include 1) the *statement* to be proven, 2) the *argument* said to prove the theorem, and 3) the *body of theory* supporting the warrants and methods within the proof. Like Bell's notion of systematization, Weber (2002) points out that for certain proofs, the central purpose is to highlight relationships with the third piece of the theorem: "proofs that justify the use of a definition or axiomatic structure" (p. 14). It is assumed within formal mathematical practice that proofs depend upon the statement of the claim, the definitions of the terms within that claim, and previously established warrants. However, it is also true that a definition or axiom that does not afford the formulation of precise claims and the subsequent development of valid proofs thereof is useless. This means that specific definitions and axioms are in some sense "justified" by the proofs they afford. Stated another way, proofs demonstrate the sufficiency of the assumptions and definitions on which they are built, which I call the relation of *provability*. Necessarily, provability is not a property of an axiom, definition, theorem, or proof, but rather a collective property within Mariotti *et al.*'s (1997) triad.

Systematizing in the classroom

Freudenthal (1973) points out that though mathematics instructors treat axioms as "starting points" in the *presentation* of formal mathematics, they are the final touches in the *organization* of mathematical theory. Modern, meta-theoretically oriented axioms appeared relatively recently within the historical span of mathematics. As such, Freudenthal deems the presentation of pre-fabricated axioms an anti-didactical inversion of little use to students. Many others similarly perceive the premature introduction of such formalisms as counter-productive (Freudenthal, 1958). Freudenthal (1973) instead recommends that students experience mathematics as a human activity, often through the instructional design heuristic of guided reinvention. He suggests transforming the mathematical elements being taught from presentable *products* into generative *activities* such that axioms, definitions, theorems, and proofs are produced through axiomatizing, defining, conjecturing, and proving. While a growing body of undergraduate mathematics education research describes the influence of such instruction on student defining (Dawkins, 2012; Zandieh & Rasmussen, 2010) and proving (Larsen & Zandieh, 2008), there is also growing evidence supporting the efficacy of guided axiomatizing (Cook, 2012; De Villiers, 1986; Yannotta, 2013).

Human and Nel (1984) explore the possibilities of engaging K-12 students in systematizing mathematics through 1) proving, 2) defining, and 3) axiomatizing geometry. They argue that this ordering is psychologically more appropriate than the traditional, logical order of mathematical presentation, which it inverts. They trace the history of Euclid's *Elements* as a didactical framework, pointing out that because Euclid and his predecessors were preoccupied with logical ordering and constructability, it took almost 2000 years for educators to reorganize the *Elements* for instruction.

Many educators objected to Euclid's order of theorems because the earliest are the most obvious. Proofs of obvious theorems lack illuminative purpose, which may be a barrier to students' acculturation to proving. Human and Nel (1984) argue that students benefit from proving non-obvious statements first (with plenty of assumptions) such that proof maintains both illuminating and convincing purposes. Only after proof has become a normative practice do they suggest that students should begin to reduce and organize the set of definitions and assumptions that supported their proving activity. This argument for systematizing after validating and illuminating helps address Harel's (2008) "necessity principle" for mathematics education, which states, "For students to learn the mathematics we intend to teach them, they must see a need for it, where 'need' refers to intellectual need, not social or economic need" (p. 285).

A natural question facing most proof-oriented mathematics instructors is how to help students experience a cognitive need for proof, much less systematizing a body of theory. In the teaching experiments from which this investigation grew, I adopted a different approach to systematizing than that described by Human and Nel. Part of the difference relates to my undergraduate context, but I shall highlight how focusing on *provability* can help students experience a cognitive need for systematization, even when the conclusions are obvious.

Experimenting with systematizing

In ongoing, iterative teaching experiments, I am exploring how to engage students in systematizing neutral axiomatic geometry by formalizing their spatial intuitions. I alternate between whole-class (14-17 students) and small group (2-4 students) teaching experiments to balance the insight and focus of small-group data against the constraints and affordances of larger-group interactions. The study participants are upper-level mathematics majors at a US university who are in most cases taking their first predominately proof-oriented course, and their first geometry course since learning Euclidean geometry in secondary school. In three representative episodes from my fourth iteration of the experiment (4 students), I highlight how *provability* provides a criterion for modifying the body of theory they create. I led this teaching experiment (1-2 meetings weekly) outside of regular lectures. Their course was taught by an effective, experienced mathematics professor.

In the participants' regular lectures, the professor first introduced the Euclidean, spherical, hyperbolic, and Minkowski planes analytically before presenting basic units on logic and set theory. These students had not taken an introduction-to-proof course, and their introduction to logic consisted of making syntactic transformations on primarily non-mathematical statements (*e.g.*, the negation or contrapositive of "If there is a major earthquake, then the baseball game is cancelled."). Students in the teaching experiment reinvented axioms in advance of their instructor's presentation thereof. The instructor provided a brief history of the parallel postulates and hyperbolic geometry, but neither he nor I directly lectured about meta-mathematical topics like independence and consistency or meta-mathematical terms like undefined terms, axioms, definitions, and theorems. The goal of the teaching experiment was to see how I could guide students to apprehend these meta-mathematical concepts within their axiomatizing, defining, and proving activity.

Guiding reinvention of a particular body of theory

I engineered the reinvention tasks toward the body of theory in the text our institution uses in this course. The textbook (Blau, 2008) defines a *plane* as a collection (**P**, **L**, d, [ω], μ) where **P** is the set of points, **L** is a set of subsets of **P** called lines, d: $\mathbf{P} \times \mathbf{P} \rightarrow \mathbf{R}$ is a real-valued distance function, ($AB = d(A, B)$) and μ an angular distance function. The distance function of each plane induces a "diameter" parameter ω, which is either the supremum of the distances or ∞. After observing some commonalities among the examples provided in the textbook, I invite students to propose "rules" for how any geometric plane should "behave". For instance, when asked about rules (or "axioms") for distance functions, students usually propose three axioms that articulate their intuitions of "distance" which they interpret as "the space between points": positive ($AB \geq 0$), non-zero for distinct points ($AB = 0$ iff $A = B$), and symmetric ($AB = BA$).

Students in my experiments often propose two other distance axioms: collinear distances are additive ($AB + BC = AC$) and the triangle inequality ($AB + BC \geq AC$). Both of these suggestions are essential to the way the body of theory is subsequently formulated. The textbook represents spatial location via the distance function such that a point B is between points A and C whenever the distances $AB + BC = AC$. This "betweenness relation" is denoted $A - B - C$. From this relation, segments and rays are defined as sets:

$$\overline{AB} = \{A, B\} \cup \{X | A - X - B\}$$

and

$$\overrightarrow{AB} = \{A, B\} \cup \{X | A - X - B\} \cup \{X | A - B - X\}$$

In the spirit of reinvention, I present very few of these elements of theory ready made. Instead, I invite students to articulate and use their own axioms, definitions, and conjectures. In the three episodes that follow, I demonstrate how students' used provability relations to make or confirm their various choices. Episode 1 portrays how students used *provability* to decide how to define these concepts. The students' intuition of additivity corresponds to the Betweenness of Points Axiom (BP), portrayed in Table 1. BP intuitively describes Euclidean and hyperbolic lines where two points (A, B) divide the line into three sections, each uniquely characterized by a betweenness relation. In Episode 2, the students considered how one could prove that BP is not provable from the previous axioms, introducing axiomatic independence. Episode 3 presents how the group responded when they proved a subcase of the Triangle Inequality Axiom using Axiom BP. In this case, they used the *provability* criterion to interpret the difference and relationship between axioms and theorems. They further elaborated their understandings while discussing the Quadrichotomy Axiom for points (Table 1) that organizes lines as the union of two rays.

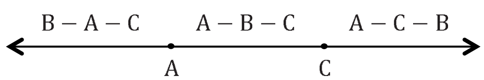
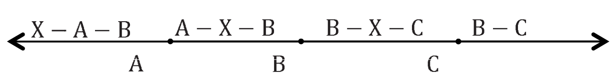
Betweenness of Points Axiom (BP)
If A, B, and C are different, collinear points and if AB + BC ≤ ω, then there exists a betweenness relation among A, B and C.

Quadrichotomy Axiom for Points (QP)
If A, B, C, and X are different, collinear points and A – B – C, then at least one of the following must hold: X – A – B, A – X – B, B – X – C, or B – C – X.


Table 1. Betweenness of Points Axiom (BP) and Quadrichotomy Axiom for Points (QP).

In the textbook, 21 axioms of planar geometry appear over the course of 9 chapters culminating in a characterization theorem of all absolute planes as Euclidean, spherical, or hyperbolic. The introduction of new axioms allows for proof of new claims adding more "rules" or structure to the system. Prior to introducing many new axioms, the text guides students to examine or create example geometries that satisfy particular subsets of the axioms. These idiosyncratic planes motivate the need for the next axiom by demonstrating what is still possible within the current set of assumptions. From an intuitive standpoint, these planes appear as "monsters" (Lakatos, 1976) that students often want to remove. From a systematization standpoint, these idiosyncratic planes serve as counterexamples to possible theorems of the form "Axioms 1-*n* ⇒ Axiom (*n*+1)". Such proofs of *non-provability* foster discussions of the *independence* of axioms.

Episode 1: creating definitions

The matter of defining "between" arose in the teaching experiment in conjunction with the Triangle Inequality Axiom (Axiom T). During a previous session, Benjamin suggested Axiom T in response to the idiosyncratic Gap plane (where T fails). The group discussed exactly which points guarantee strict inequality or equality between AB + BC and AC. Their two observations were that equality occurred when B was "between" A and C or whenever B was on the "segment" $\overline{AC}$. The students' reasoning inverted the defining hierarchy in the text (distance → between → segment). The students likely perceived that distance relations depended upon spatial arrangement because they were analyzing geometric diagrams. When viewing diagrams, students often treat segments as unitary objects rather than collections of points.

I pushed the group to define either *between* or *segment* to clarify their observation. Benjamin especially found himself in a circular loop of defining "segments" as the points between the endpoints and "between" as being on the segment. Acknowledging the limitations of his defining language, Benjamin observed, "Between's a funny math word. It's different in English than it is in math."

Other members of the group wanted to use interval-type representations to define "between." Anthony tried to define "between" saying, "An element of the set bound by AC. Umm, because in order for it to be in the set it has to fall within the two bounds." While this reflected the familiar number line structure (or Cartesian coordinates) with less than and greater than relations, I challenged this approach asking him to explain what it meant for one city to be "greater than" another city. Matthias modified their approach by comparing distances, which are numbers unlike points. He proposed that B is between A and C whenever AB < AC and BC < AC. Benjamin contested this definition because it was satisfied by points not on $\overleftrightarrow{AC}$, prompting the group to add the condition $B \in \overleftrightarrow{AC}$. The group agreed to this definition of A – B – C because it described only those points they perceived as spatially between.

During the discussion, Anthony's suggestion first introduced set structure. The group observed that the segment should be a subset of the line (also a set), but they struggled to formulate a definition precisely. Benjamin attempted saying, "[The segment's] contained or it's an element of the line through A and C. But it has to be a straight line, you can't have bumps and waves." Here, Benjamin made clear recourse to spatial arrangement, which is especially hard to formalize into the set structure used by the text. Benjamin then revisited the analogy between the points in segments and the interval "[A, C]" which Matthias elaborated as "The set of all points between A and C, inclusive." When asked to translate this into set notation, they denoted the points between by A < B < C, showing they did not initially connect "between" to their newly-formed definition. After re-encountering the incompatibility of points and "<", Benjamin suggested "Oh then we take [Matthias's] inequality."

Episode 1, continued: using provability to choose a definition

To transition the activity, I verbally summarized the group's progress recalling that they 1) conjectured that the equality

$AB + BC = AC$ holds when B is between A and C, 2) defined between in terms of distance inequalities and 3) defined segment $\overleftrightarrow{AC}$ using their definition of between. I asked them to prove their conjecture using their definition, hoping it would help them see that their axioms did not yet require distances to sum as they anticipated. They spent several minutes trying to prove the claim without questioning its provability.

Their interpretation of the definition probably still referred to the Euclidean diagrams from which it grew (Gravemeijer, 1994) and those diagrams did not foster skepticism toward the claim. Neither their algebra nor diagrams afforded consideration of non-additive distances ($AB + BC > AC$). I intervened by recalling such an instance in a finite geometry they had created in a previous session:

I: Okay, so what does that mean for us?

M: Bad definition.

I: Why do you say that?

B: Well because we can fall, we can hold our definition true, but logic is false. I mean $AB + BC = AC$ is logic to me. That just should hold, that's just logic. [...] And because of this definition missing probably one little thing it isn't true.

Benjamin then conjectured that, though the triangle equality was not provable from Matthias's definition, Matthias's definition could be proven from the equality. The group produced such a proof using their previous axioms. After some reflection, Matthias asked, "So we can prove that [pointing to "$AB < AC$ and $BC < AC$"] from [$AB + BC = AC$], does [$AB + BC = AC$] become our definition of betweenness?" The other students agreed to ratify this new definition (with $B \in \overline{AC}$). Thus Benjamin's insistent *monster-barring* of planes where $AB + BC \neq AC$ led to productive refinement of the underlying definitions (as described in Larsen & Zandieh, 2008) helping the students' body of theory approach the text's.

Observations from Episode 1

More like classic Euclidean axiomatics (De Villiers, 1986), the students began forming definitions as *descriptions* of their understanding of geometric objects in an almost empirical sense. They identified relationships among "between", segments, and the equation $AB + BC = AC$, but they did not attend to mathematical structure (numbers, sets, relations) or to hierarchies among definitions. Their visualizations made spatial arrangement and objects like segments self-evident (even as Hilbert called "between" an undefined term). Distances appear to depend upon arrangements. Furthermore, students struggled to work outside of the Cartesian framing for Euclidean geometry. They progressed by attending to mathematical structure. When I suggested they could not apply < to points, Matthias introduced distances as viable numbers to compare. Anthony's suggestion to use set structure refocused the group's attention away from segments as locations or unitary objects in space. Thus, a focus on mathematical structure led the group to unpack these concepts and embed them in a formalizable structure.

Next, analyzing implications among properties led the students to impose hierarchies among these properties. Intuitively, $AB + BC = AC$ seemed a strange definition of between. However, once students recognized that it rendered other concepts *definable* and properties *provable*, they adopted the definition. They had not been taught about "stronger" definitions, but their proving activity afforded the meta-theoretical concept. The group followed a very similar pattern later regarding the definition of four-point betweenness relations ($A - B - C - D$): they proposed two definitions, found that one implied the other but not *vice versa*, and ratified the stronger condition. The group thereby proved their first theorems about betweennesses before choosing a definition. Their attention shifted from describing familiar planes (like classical axiomatics) to reasoning about meta-theoretical relations (like modern axiomatics; De Villiers, 1986). This shift also occurred in the following episode.

Episode 2: conceptions of key mathematical tasks

Several of the tasks presented in the textbook invite students to create finite geometries that satisfy certain axioms and not others. Finite geometries are represented in the text by a set of points (letters), a set of sets of points (lines), and a table for the distance function. This allows students to easily create and modify planes. In previous semesters when instructing the course, I assigned follow-up questions asking students to explain the importance of such exercises. Students provided three types of explanations:

- The activities are pedagogical (a psychological frame): by creating planes that satisfy some axioms and violate others, students better understand what the axioms say.
- The activities reveal how axioms remove counterintuitive possibilities and add structure (a classical, descriptive axiomatic frame). It shows how new axioms make the formal system more compatible with intuition. Unfortunately, responses in this category sometimes show that students found finite planes in the set-and-table representation completely non-geometric.
- The activities prove independence (a modern, meta-theoretical axiomatic frame): an uncommon response prior to guided reflection.

During later experiments, I sought opportunities to help students discover the idea of independence. Such an opportunity arose when Benjamin proposed Axiom T. I began the session describing an activity like that used in episode 1. The students pointed out that their professor had recently assigned such a task. I asked, "What is the point of [this] activity? Why do we ask you these questions?" Their initial explanations were pedagogical. Benjamin said, "To show a full understanding of what the axioms actually mean." Ashley suggested it was an exercise on the logic of statements, saying, "Manipulating them too. It's false, now can you make it true? Or *vice versa*." This suggested they gained no specifically mathematical information from the task.

I invited the group to create a finite geometry that satisfied the first 7 axioms, but not Axiom T. They produced a 3-point geometry satisfying these conditions, calling their example the "Waffle Cone plane" after a diagram they drew (Figure 1). I then asked whether they could prove Axiom T from the previous seven. Benjamin, who proposed the axiom, said he thought it would hold true as long as lines in the plane were straight. Then, when I asked again about proof, Benjamin suggested that they would be able to with later axioms, but not with these seven. Matthias added that the Waffle Cone was a counterexample, "Because if it's possible to create something that that satisfies all of our 7 axioms, but doesn't satisfy that [Axiom T], then those 7 axioms alone cannot be enough to satisfy that." I asked:

I: So what if now instead of the first 7 axioms we started talking about the first 8 axioms, including Triangle Inequality? [...]

B: This example would fail that. So it wouldn't follow our 8 axioms so this would not be a legitimate plane based on our first 8 axioms [...] We keep saying [in class] that eventually some of these planes are gonna fail our axioms and they are not going to be considered useful planes anymore.

A: It's getting a lot more specific [...] There are not a lot of planes left.

Benjamin added later that the odd planes indicated that, "We need more axioms [...] Our 8 axioms aren't specific enough."

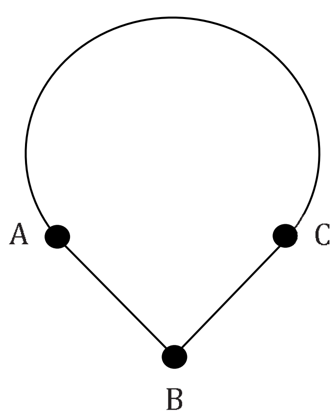


Figure 1. The Waffle Cone finite plane.

Observations on the participants' explanations

The group's explanations of the game of creating planes showed a marked shift as they reflected on Axiom T. Benjamin clearly thought the claim was intuitive, but the group seemed certain that they had not assumed enough to prove it. Furthermore, they articulated that the Waffle Cone was a counterexample to the proof they were asked to produce (Axioms 1-7 ⇒ Axiom T). Though they did not yet know the language of axiomatic *independence*, they recognized their example as evidence that Axiom T was *non-provable* from their current assumptions. Such meta-theoretical relations are hallmarks of modern axiomatics.

The group identified several other aspects of adding axioms. First, their axiomatic system became "more specific" by "ruling out" idiosyncratic planes. This explanation acknowledges that axiomatic theory is quantified over a set of example planes (rather than descriptive of prototypes), and adding axioms narrows the set of exemplars. Such a quantified view of mathematical axioms parallels the common mathematical parlance of "spaces" of abstract structures (continuous functions, abelian groups, *etc*.). Prototypes maintained a guiding role as axioms narrow toward more "useful planes." Such interpretations of axiomatizing are fitting in a course that culminates in a characterization theorem. It also reflects the group's reinvention process, which began with observing geometric patterns in the prototype planes. Much modern mathematics mimics this blueprint of constructing hypothetical-deductive theory from familiar mathematical paradigms.

Clearly articulating proof of non-provability

To focus their meta-theoretic reflection, I asked them again how they could ensure that no proof of Axiom T was possible. Benjamin recalled from their unit on logic that, "if it's an 'all' statement, you only need one [counterexample] to prove it false." I asked then where the "for all" was in the theorem "Axiom 1-Axiom 7 ⇒ Axiom T". The group initially suggested "for all" seven axioms, but not every proof required every axiom. I asked for which planes the proof should hold, and Ashley acknowledged it should be all planes. Benjamin elaborated, "All abstract planes that follow those rules." He specified that the proof should work for the Waffle Cone plane, showing why the proof could not exist. So, the theorem should be stated: "For all geometries that satisfy Axiom 1-Axiom 7, Axiom T holds also."

Statements of theorems in mathematics texts rarely include all antecedent assumptions. Usually the order of presentation entails any logical or epistemic dependencies (proofs may reference warrants on previous pages, but not later ones). However, some students in my experiments struggled to understand the epistemic shifts that occur when axioms are installed (old examples are rejected and new theorems are provable). The question of provability helped Benjamin's group perceive the importance of both the conclusions of a theorem and the assumptions that render them provable. This coincides with Durand-Guerrier *et al.*'s (2012) suggestion that students should understand how "the 'truth' of statements depends on definitions and postulates of a reference theory" (p. 358). In episode 2 the students recognized axiomatic *independence*; in the next they considered axiomatic *duplication*.

Episode 3: making an axiom a theorem

As mentioned above, Benjamin assumed the additivity of distances on a line. Once the group defined betweenness as AB + BC = AC, they proposed an axiom to guarantee this property: "For all three collinear points A, B, C, one must be between the other two." The book's caveat that AB + BC ≤ ω, which affords special cases on the sphere, is less than obvious to students. So, I presented the motivating examples and this condition to the group. I then asked them to prove "If X, Y, Z are distinct, collinear points, then XY + YZ ≥ XZ", which they did using the text's Axiom BP.

I asked the group to reconsider the status of the Triangle Inequality now that they had proved (a restricted form of) it. They explained:

> M: Now it doesn't have to be an axiom, it can be a theorem based off of Betweenness Points Axiom.
>
> B: Right, BP implies the Triangle Inequality.
>
> M: Because axioms are assumptions [...] but theorems are things you can prove from your assumptions so they are no longer just, 'Hey, we are going to say this' [...] Nope, I can prove this.

These reflections revealed the students' emerging view that provability distinguished theorems from axioms. Also, Matthias displayed the mathematical preference for validating claims over assuming them (*i.e.,* reducing assumptions).

During the next research session, I introduced Axiom QP to the group. Matthias immediately asked, "But now, why is that an axiom and not a theorem? [...] I feel like somehow we would be able to, if we could prove it, it would be a theorem." Because it appeared intuitive, Matthias felt the claim should be provable just as many ancient mathematicians felt toward Euclid's Fifth Postulate. I acknowledged that we should either a) find a plane that satisfies eight axioms and not QP or b) prove it. I acknowledged the text calls QP an axiom, so Matthias anticipated there must exist a "wonky plane that we don't care about" that satisfies the former condition. Benjamin added that the plane would then get kicked out. The students thus displayed an emergent picture of the axiomatic game:

1. New axioms should be established *non-provable* before installation.
2. "Wonky" example planes proved *non-provability*.
3. Adding that axiom removed the idiosyncratic plane narrowing the set of exemplars.

Some results from assessing meta-mathematical learning

In a later session, the students completed an assessment survey on which they were asked how to test whether the Triangle Inequality Theorem held true on an example plane without directly testing every triplet of collinear points. Only Matthias recognized that verifying the first 8 axioms would guarantee the theorem by implication. The students' responses on this survey reiterated their previous observations about the game of systematizing axiomatic theory. They tended to focus on conforming their geometric theory to the paradigm examples and kicking out strange planes (descriptive axiomatic frame), while also attending to establishing non-provability relations (meta-theoretical axiomatic frame). Benjamin, for instance, explained that the idiosyncratic planes helped "to show that all the axioms do not imply the next. That in fact each axiom has its own unique way of describing a given plane." They also suggested that odd examples helped them learn by portraying how new axioms modified their axiomatic system (pedagogical frame). So, the students' meta-mathematical conceptions of the axiomatic game continued to reflect these three common interpretations simultaneously, each of which helped them explain certain aspects of axiomatizing.

Conclusions

These learning episodes exemplify how guided reinvention experiences can help students develop conscious meta-mathematical understandings of formal mathematical games like systematizing. Clearly, students like Benjamin struggled to coordinate their visual geometric reasoning with the abstract, hypothetical-deductive method. However, students' abstraction and generalization activity can be formative. I observe that the students' reflection on their defining, axiomatizing, and proving activity fostered their engagement in rich conversations about meta-theoretical relations. I claim such apprehension of theoretical relations represents one of the most valuable dimensions of advanced mathematical learning because it gives students access to the rules, assumptions, and practices that underlie the modern mathematical game.

The questions of provability are essential for understanding systematizing and proving activity. In geometry, where students have rich intuitions, formulating the definitions and assumptions carefully can be more important than the specific conclusions to be proven. It is not remarkable that the conclusions are true, but that they are rigorously *provable*. I claim attention to provability provides a fruitful antidote for the common complaint that geometry inhibits proof because the claims are too obvious. Though many claims were intuitive for the students, they had to make less obvious choices among various assumptions and definitions.

The ready presence of non-standard geometries was very important for the reinvention experiment. My experiments have also identified two pitfalls of this approach: 1) students may ignore "wonky" planes as non-geometric (based on spatial intuition) or 2) they may perceive proof as an impractical, formalist game. As these episodes portrayed, the reinvention process affords a third option in which both odd and familiar examples maintain a purpose. Odd planes produce skepticism about what is possible while prototypes provide *common sense* (as in Freudenthal, 1973) about what to pursue. Students produced axioms descriptively before they assessed them formally.

So, I claim that this mathematical learning environment fostered a sense of intellectual need for systematization and formalization by emphasizing *provability* relations. As Matthias explained when asked why they prove obvious statements, "Though something may seem trivial now, it is helpful to have an understanding of what 'common sense' ideas are provable and therefore usable to examine future behaviors and which are not." This explanation, like others I observed, reveals an emergent, conscious delineation and coordination of "common sense" alongside the claims, norms, and practices of formal mathematics.

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The Greeks did not think much of propositions which they happened to hit upon in the deductive direction without having previously guessed them. They called them *porisms*, corollaries, incidental results springing from the proof of a theorem or the solution of a problem, results not directly sought but appearing, as it were, by chance, without any additional labour, and constituting, as Proclus says, a sort of windfall (*ermaion*) or bonus (*kerdos*) […] We read in the editorial summary to Euler (1756-7) that arithmetical theorems "were discovered long before their truth has been confirmed by rigid demonstrations". Both the Editor and Euler use for this process of discovery the modern term "*induction*" instead of the ancient "*analysis*" (ibid.). The heuristic precedence of the result over the argument, of the theorem over the proof, has deep roots in mathematical folklore. Let us quote some variations on a familiar theme: Chrysippus is said to have written to Cleanthe: "Just send me the theorems, then I shall find the proofs" (cf. Diogenes Laertius, c. 200, VII. 179). Gauss is said to have complained: "I have had my results for a long time; but I do not yet know how I am to arrive at them" (cf. Arber, 1945, p. 47), and Riemann: "If only I had the theorems! Then I should find the proofs easily enough." (cf. Holder, 1924, p. 487). Pólya stresses: "You have to guess a mathematical theorem before you prove it" (1954, vol. 1, p. vi).

Lakatos, I. (1976) *Proofs and Refutations: The Logic of Mathematical Discovery* p. 9. Cambridge, UK: Cambridge University Press.