Generalization of cross product to higher dimensions using geometric approach

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Multivariable Calculus students often complain that the formula for computing the cross product of vectors is not revealing enough: the geometric nature of the formula is not readily apparent from its algebraic representation. We present a way to overcome this impediment and deepen students' understanding of the cross product. Our approach also enables students to understand how the cross product can be generalized to higher dimensions. In particular, this can inspire students to learn such concepts as the wedge product of exterior algebra without any mathematical knowledge above a Multivariable Calculus course.

One of the central ideas in a Multivariable Calculus course is the generalization from two to three dimensions. Every semester that Lia Vaš, Associate Professor at the University of the Sciences in Philadelphia, teaches a Multivariable Calculus course, she makes this idea the unifying theme of the course. Her experience is that students can better grasp the three dimensional concepts of Multivariable Calculus if they can see how the concepts originate by adding an extra dimension to the familiar two dimensional concepts in earlier calculus courses. As a consequence, Lia's Multivariable Calculus students are often intrigued by the generalization from two or three to higher dimensions. In Spring 2008, one of Lia's students, Timothy P. Enright, a chemistry major, was especially attracted to the idea of higher dimensions, in particular, the generalization of the cross product.

It all started with the formula for the cross product of two three-dimensional vectors. Being a highly visual learner, Timothy was looking for a way to "see" why the formula for the cross product of two three-dimensional vectors $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$ works:

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$$

Timothy learned in class that the formula produces a vector of magnitude equal to the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$. To "see" why this is true, he began to investigate projections of $\vec{a}$ and $\vec{b}$ onto the different coordinate planes and obtained images as those in Figures 1 and 2. He denoted the projections of $\vec{a}$ onto the unit orthogonal basis vectors in coordinate planes by a_{hor} and a_{ver} and used similar notation for projections of $\vec{b}$. While examining Figure 1, Timothy noticed that the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$ can be obtained as

$$(a_{hor} + b_{hor})(a_{ver} + b_{ver}) - 2 \cdot \tfrac{1}{2}\, a_{hor}a_{ver} - 2 \cdot \tfrac{1}{2}\, b_{hor}b_{ver} - 2\, a_{hor}b_{ver} =$$

$$= a_{ver}b_{hor} - a_{hor}b_{ver}$$

In Figure 2, Timothy noticed that $(a_{hor} + b_{hor})(a_{ver} - b_{ver}) - 2 \cdot \frac{1}{2} a_{hor}a_{ver} + 2 \cdot \frac{1}{2} b_{hor}b_{ver} = a_{ver}\, b_{hor} - a_{hor}\, b_{ver}$. Similarly, Timothy checked all the possible cases depending on position of the two vectors and, in each case, obtained $a_{ver}b_{hor} - a_{hor}b_{ver}$, the coordinate of the cross product of $\vec{a}$ and $\vec{b}$. This convinced Timothy that the formula for the vector product is indeed true.

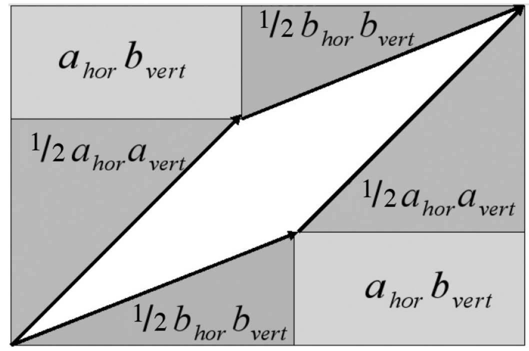

Figure 1.

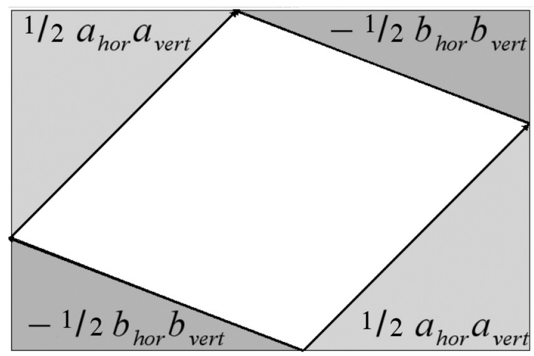

Figure 2.

Then, Timothy started wondering if there is a similar connection between the volume and the cross product in four dimensions. Namely, in the coordinates of the cross product in three dimensions, Timothy recognized the formula for 2 × 2 determinants:

$$\vec{a} \times \vec{b} = \langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \rangle$$

In class, he also learned that the formula for the volume spanned by three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

To direct him further, Lia suggested considering the following fact:

Cross product in	three dimensions	four dimensions
projections	parallelograms	parallelepipeds
i-th coordinate computed by	area of parallelogram 2 × 2 determinant	volume of parallelepiped 3 × 3 determinant

and trying to find the formula for the product of three four-dimensional vectors. With further input from Lia about the signs of determinants, Timothy developed the following formula for the cross product of $\vec{a} = \langle a_1, a_2, a_3, a_4 \rangle$, $\vec{b} = \langle b_1, b_2, b_3, b_4 \rangle$ and $\vec{c} = \langle c_1, c_2, c_3, c_4 \rangle$.

$$\left\langle \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix}, - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right\rangle$$

It was evident to Timothy that this formula would generalize the three-dimensional case in the sense that the coordinates would compute the volume of the projections. However, Lia wanted Timothy to see that this formula retains some further properties of the cross product:

1. It produces a vector orthogonal to the initial vectors. Note that the dot product and the concept of orthogonality are much easier to generalize to higher dimensions than the cross product. It was clear to Timothy that Lia meant the following: the two four-dimensional vectors $\langle x_1, x_2, x_3, x_4 \rangle$ and $\langle y_1, y_2, y_3, y_4 \rangle$ are orthogonal if their dot product $x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$ is zero.

2. The cross product is zero if and only if any of the initial vectors are collinear (for 3 dimensions) or coplanar (for 4 dimensions). Note that this fact easily follows from the geometric representation.

Elated by his findings, Timothy wondered if this could be generalized to higher dimensions. Without knowledge of linear algebra and how to evaluate an $n \times n$ determinant when $n > 3$, Timothy could not understand every detail of Lia's explanation but he could get a very good sense of the basic idea of a wedge product in an exterior algebra. [1] Lia mentioned that in higher dimensions wedge ^ is used instead of cross × and led Timothy to the formula for the product of $n - 1$ n-dimensional vectors $\vec{a_i} = \langle a_{i1}, a_{i2}, \ldots a_{in} \rangle$, $i = 1, \ldots, n - 1$ to be $a_1 \wedge a_2 \wedge \ldots a_{n-1} = \langle A_1, A_2, \ldots, A_n \rangle$ where A_i, $i = 1, \ldots, n - 1$, is defined as:

delete

$$A_i = (-1)^{1+i} \begin{vmatrix} a_{11} & \ldots & a_{1i-1} & a_{1i} & a_{1i+1} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2i-1} & a_{2i} & a_{2i+1} & \ldots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n-11} & \ldots & a_{n-1i-1} & a_{n-1i} & a_{n-1i+1} & \ldots & a_{n-1n} \end{vmatrix}$$

Lia pointed out that this product would retain two basic properties of the two and three dimensional products:

1. The wedge product produces a vector orthogonal to initial vectors.

2. If vectors are such that they all lie in a "plane" of smaller dimension than the space itself (Lia did not use the term "linearly dependent" since Timothy did not take Linear Algebra yet), then the wedge product is zero.

Timothy and Lia decided to share some of these findings with Timothy's classmates. They had a poster presentation on Research Day at the University of the Sciences in Philadelphia, held in the spring semester each year. A good number of Timothy's classmates, non-mathematics majors just like Timothy himself, attended the poster presentation and followed Timothy's arguments leading them to the definition of a wedge product in an exterior algebra. Timothy and Lia presented the poster without using any, in the words of students attending the presentation, "fancy math language". Instead, mathematical concepts were presented using simple language combined with plenty of illustrations, enabling some students to more easily adopt advanced mathematical symbols and concepts.

Following Bruner's (1986, 1990) approach, this Multivariable Calculus topic was extended and adapted for non-mathematics majors. This group of students demonstrated they could appreciate the visual approach to abstract concepts in the curriculum. Using "reflective practices" (see Driscoll, 2005), students can create their own perception of more advanced mathematical theories that are usually in the realm of experts only. This can be applied to various other topics in standard calculus textbooks such as Ellis and Gulick (1994) or Stewart (1999).

Note

[1] Students with more background than a basic Multivariable Calculus course can be directed to Bourbaki (1989) or MacLane and Birkhoff (1999). Wikipedia also has a good introduction to the wedge product and exterior algebras.

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