## References

Boaler, J. (1997) *Experiencing School Mathematics. Teaching Styles, Sex and Setting*, Buckingham, Bucks, Open University Press

Burton, L. (1995) 'Moving towards a feminist epistemology of mathematics', *Educational Studies in Mathematics* 28(3), 275-291

Burton, L. (1998) 'Thinking about mathematical thinking – heterogeneity and its social justice implications', plenary address to the Mathematics Education and Society Conference, Nottingham, September, 1998

Burton, L. (1999a, in press) 'The practices of mathematicians: what do they tell us about coming to know mathematics?', *Educational Studies in Mathematics* 37(2), 121-143.

Burton, L. (1999b, in press) 'Fables: the tortoise? the hare? the mathematically under-achieving male?', *Gender and Education*

Fischbein, E (1987) *Intuition in Science and Mathematics: an Educational Approach*, Dordrecht, D. Reidel

Hersh, R. (1998) *What Is Mathematics, Really?*, London, Vintage Books

Mason, J., with Burton, L and Stacey, K. (1982) *Thinking Mathematically*, Wokingham, Surrey, Addison-Wesley.

Resnick, L. B, Levine, J. M. and Teasley, S D (eds) (1991) *Perspectives on Socially Shared Cognition*, Washington, DC, American Psychological Association

Seymour, E and Hewitt, N M. (1997) *Talking about Leaving: Why Undergraduates Leave the Sciences*. Oxford, Westview Press

Smith, J. P. and Hungwe, K. (1998) 'Conjecture and verification in research and teaching: conversations with young mathematicians', *For the Learning of Mathematics* 18(3), 40-46

Steffe, L. and Gale, J (eds) (1995) *Constructivism in Education*, Hillsdale, NJ, Lawrence Erlbaum Associates.

von Glasersfeld, E. (ed) (1991) *Radical Constructivism in Mathematics Education*, Dordrecht, Kluwer Academic Publishers

Whitehead, A N (1962) *The Aims of Education*, London, Ernest Benn Limited


# A Four-Sided View of 'Function'


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The term 'function' means the mutual connection and dependence of things, in any way, from a dynamic/operational perspective. Therefore, function must be viewed both from the aspect of things that are in mutual interdependence, that is structurally, spatially, statically, geometrically, and from the aspect of procedures, that is in operation, dynamically and algebraically.

The concept of the function is one of the two basic notions in modern mathematics, the other being that of set. The first has had an interesting evolution and gives us an example of the trend in mathematics to extend and generalize its concepts Function is greatly involved in mathematics, which is why famous mathematicians (e.g. Mac Lane, 1986) support its use as a unifying and centralizing principle in organizing and teaching of the respective lesson. This concept seems to be the natural and main guide in selecting, organizing and developing mathematical subjects in general and particularly in mathematical texts A very good example here is the *Conceptual Mathematics: a First Introduction to Categories* (Lawvere and Schanuel, 1993)

There are also problems, that this concept brings along with it, either when we want to conduct research or to teach it to students of mathematics The realization of those difficulties out of the entire mathematical community has as a consequence the tremendous increase of the related research and publications, during recent years. Researchers such as Sierpinska (1991), Sfard (1989, 1991), Harel and Dubinsky (1992), Vinner (1983), Tall (1991) have expressed their views about the subject In all these works, as well as in Barneveld and Verstappen (1982) and Even (1990), the subject of function is extensively discussed

The approach of this author differs from those mentioned above, mainly in two respects:

(i) by drawing on Jung's views and the corresponding ones of Rucker;

(ii) trying to classify all views of the function in a holistic schema, which will allow us to face up to and respect the individual differences of the students.

The necessary elements of Jung's theory that are used here are included in this essay when needed. For a further study of Jung's work the reader has recourse to Jung (1957-1977). Except for the well-known and widespread elements of his work, there is the aspect, which is based on his so-called *hypothesis of archetypes* According to this hypothesis, the concept of the function may very well be of an archetypal nature. That means that human beings have built in the archetype/model of a 'good' relation between things, which projects onto the external world In this way, they indirectly realise the existence of relations into the system of the universe, according to the Greek view *hen to pan* (everything is a unity). Consequently, we could consider the concept of function as the archetype of the healthy and regular relation to the psychophysical continuity of the one World. This view seems to be close to the idea of the arrow between objects in Category Theory (Mac Lane, 1986)

## A four-sided view of function

Decisions, positive or negative, and questions regarding the teaching of function are known and considered as the common acceptable aspect which predominates today. But, it is natural to be aware of and curious about other aspects less well known. Nevertheless, as more aspects are expressed about a topic, the more hope we have of looking at it as a whole and not only from one side

Jung claims (1957-1977, vol. 6: Psychological Types) that in order to live in space-time and in the relations of the things of the world, in order to communicate and to exchange information with the environment, we need psychological functionings, of which here are four basic types:

(i) *sensation*, a psychological functioning which informs us that 'something' exists;

(ii) *thinking*, which defines what exactly is and what this 'something' that exists does;

(iii) *feeling*, which determines if this 'something' that exists has any value for us, if we like it or not; this psychological functioning, generally, binds us to the things of the world;



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(iv) *intuition*, which tells us where this 'something' comes from and where it goes to; knowledge 'via the unconscious'.

These four psychological functions correlate to four ways with which we orientate in relation to experience, communication, information. The basic character of these four psychological functions makes them proper criteria for consciousness and classification. So, this four-sided view is the minimum precondition which is demanded for a minimum fulfillment of judgment, perception and balance. If into this set of four basic psychological functions, we add one more set of four - will, temperament, imagination, memory - then we reach a maximum fulfillment of judgement and perception. The psychological type of human beings is created depending on which psychological functioning(s) is used as central and which as secondary. Jung places counter-diametrically the four basic psychological functionings as follows.

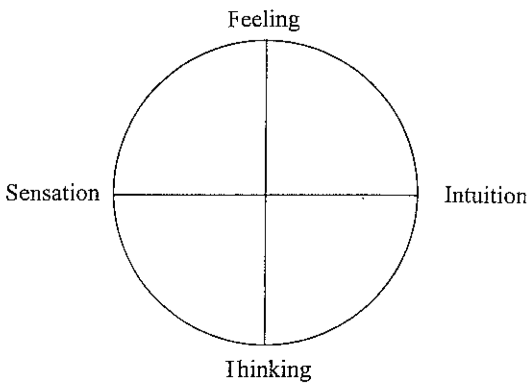


*Figure 1*

Thinking and feeling (the vertical axis) are called *rational* psychological functionings, that is reasonable deduction and formation of a judgment, which reach their ultimate meaning when they comply with the laws of logic. On the contrary, sensation and intuition (the horizontal axis) are called *irrational* psychological functionings, that is non-rational data processing, which are forced to overlook logic in order to have a better conception of the whole flow of facts.

The harmonious co-functioning of all partial basic psychological functions eventually leads us to a full and balanced, a rounded, entire and multidimensional view of this 'something' that arrives in our consciousness. For this reason, the centre of the circle, where all psychological functions meet, I label with the Greek classical word *pemptusia* (quintessence); on the one hand because it is fifth after the four psychological functions, and on the other because it contains the full meaning of this 'something', the main element, the most important content, the final extract. From this point of view, we can transform the above figure as follows:

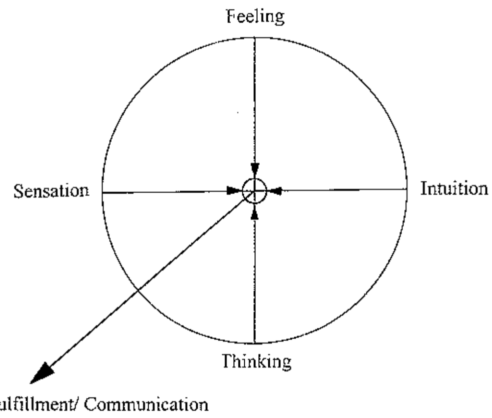


*Figure 2*

Rucker (1988, pp. 14-35) draws on Jung's views to explain why mathematics has developed in five ways: that is, number, logic, space, infinity and information. These five ways are related to the five above-mentioned psychological functionings, where here he regards information as quintessence. Then he makes the following correlation with mathematics:

(i) *Sensation ≈ Number*

This 'something', to which the sensation is referred, is the result of a distinction, and this development of distinction is the one that leads us to number. The world of sensation and that of number is the world of the discrete.

(ii) *Thinking ≈ Logic*

Thinking defines the abstract structures with which sensation and intuition are feeding us. In a thinking procedure we look for underlying patterns, we link them together and this leads us to logic.

(iii) *Feeling ≈ Space*

Feelings attach us to things. Therefore, in order to have a feeling about something, and in order to evaluate it or to be attached to it, we must go out and find ourselves in continuous space.

(iv) *Intuition ≈ Infinity*

Intuition means to have a deep sense of reality as a totality. By taking the whole world as unity, one can understand the concept of infinity.

(v) *Quintessence/Fulfillment/Communication ≈ Information*

The conception of the world is based on the exchange of information and on communication with it in all possible ways

We can say that information unifies all the branches of mathematics. Considering mathematics as a set of methods, of rules and practices which helps us gain access to and explore all kind of worlds (material, conceptual, ideal), in fact transforms one species of information into another one and gives us a tool for decoding messages sent by these worlds. This tool operates the algorithms of mathematics.

Likewise, if we examine through time the basic concept of mathematics in each era, then we observe that this is number according to the Middle Ages, space according to Renaissance, logic according to Industrial Revolution and infinity according to modern times. Nevertheless, with computers' progress, we are already following the new era in mathematics, the fifth in a row, in which the dominant concept is one of information.

Finally, from a biological point of view, the left hemisphere of our brain is responsible for the digital operations, while the right hemisphere for the analog ones. That means that the left hemisphere thinks in terms of discrete number, while the right hemisphere in terms of continuous space. The related figure is as follows

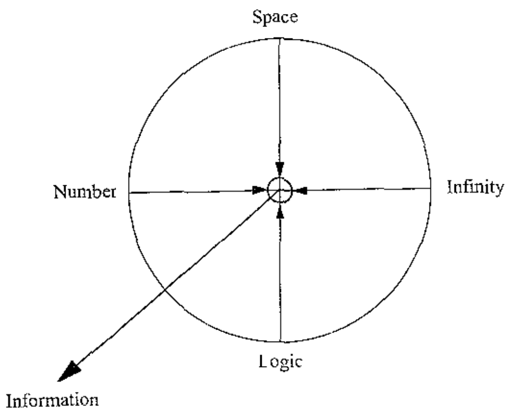


*Figure 3*

Lawvere (1970) says that in the evolution of mathematics dominates the dialectic relation between geometry and logic, in which geometry and the geometrical manner play the leading role. One can consider, as Drossos (1995) does, that analytic-elementary study of this dialectic relation gives analysis, while the holistic-structural gives algebra and the mathematical structures. Summing up, we have the following basic dialectic quaternity, where mathematics is the quintessence and in which all dialectic components are re-constituted

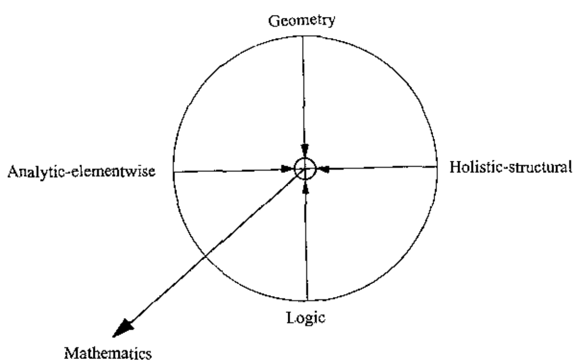


*Figure 4*

If we now combine these three aspects, we have Figure 5.

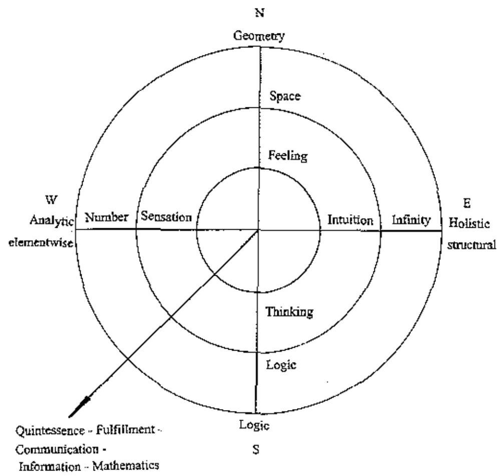


*Figure 5*

What interests me most is to apply this synthetic dynamic schema to the specific case of the function. Then we have the following:

(W) gives us the information that there is a procedure, a co-change of quantities, a relation, a correspondence, etc., that is a particular situation which is clearly distinctive among others; a case where we realize the function in a first, pre-typical shape. We can also analyze this function to its elements and to get its table of values.

(S) specifies what this function is and what it does, using logic and putting it into a mathematical format.

(N) connects us with the function and evaluates it as 'good'/'bad' and further represents it as a graph.

(E) tries to conceive where the origin of the function is and to where it goes, that is to understand the behavior of the procedure in a holistic way

If we interpret these four points as mathematically relevant, we have the following:

(W) Existence of the function, its distinctiveness from others, table of values, distinctive co-relation of elements of the domain with these of the co-domain, function machine, etc.

(S) Formalization, that is definition as a set of ordered pairs, function's formula, properties and mathematical study.

(N) Graph, evaluation of the function as 'good'/'bad' from the point of view of its mathematical study, interest for the function in accordance to this attitude

(E) Behavior of the function as procedure, study of the function when $x$ tends to infinity, holistic conception

Accordingly, to have the quintessence of a function requires this minimum four-sided view

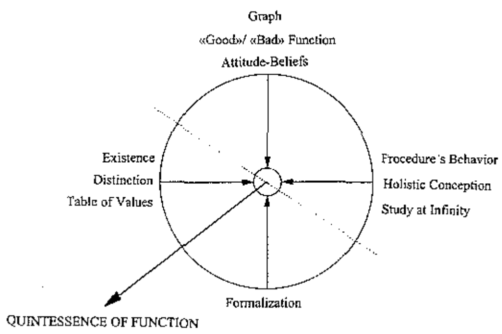


*Figure 6*

This schema can give us a help not only in the research on functions, but also in teaching them, no matter by which way this is done, i e traditionally or by the use of new technology. The above analysis shows us that if we have this four-sided view of a function, we have the total view of this function in all its width (extension) and depth (intention). It is also possible that the relevant four-sided views of other basic mathematical objects, for example number (Franz, 1974), set, etc., may lead to the correct research and teaching and ultimately to their quintessence. In this way, we can shortly systematize many mathematical objects and facilitate conducted research as well as the methods of teaching them.

Notice on the right of the segmented line in Figure 6 the spatial, continuous and analog view of the function; consequently this one corresponds to the right hemisphere of the brain. On the left of the segmented line, there is the arithmetic (numerical), discrete and digital view, corresponding to the left hemisphere of the brain. This information makes things easier not only in research but mainly in the teaching of function, by personalizing it in respect of individuals, who are unilateral on the right or left hemisphere, respectively

It is a fact that left hemisphere controls the time conception of a procedure, the verbal thought, the analytic-serial thought and words which come from sensation, the sequential succession of logic, the notation, the formalism, etc. On the other hand, the right hemisphere controls the optical view of space, the non-verbal thought, the holistic combining of thought with pictures, the intuitive correlation with random succession, sentimental evaluation, etc. It seems that for modern human beings, the left hemisphere looks closer to the conscious side of mind while the right one to the unconscious. As an epigram, as to what is of interest to us here, the left hemisphere analyzes within time while the right one synthesizes into space.

The educational system mainly targets to the development of logic and verbal abilities of analytic and serial type and to the objective realism of sensation. In other words, it is addressed to the left hemisphere. On the contrary, the irrational and non-verbal abilities of the synthetic and holistic type are degraded as well as the intuitive and sentimental side, that is the right hemisphere. Therefore, it is obvious that education does not offer a harmonic and balanced development, in other words does not contribute to the contest, the synthesis and the completion of all human functions. On the contrary, it leads to competition, separation and partiality. The result is the creation of persons with unbalanced development and everything that is possibly entailed from that fact in general and especially in mathematical knowledge. The complete person uses all kinds of psychological functionings as well as the whole brain, utilising logic as a tool and intuition as a source of inspiration.

## A didactic proposal

After the mathematical study of a function, the suggested method can operate as metacognition (Schoenfeld, 1987). Students are asked to order the whole procedure of the mathematical study according to Figure 6. He or she can then see:

(i) if his/her initial conjecture/intuition about the function was either confirmed or not;

(ii) which psychological functions she or he mobilised, in each phase of the mathematical study.

As a natural consequence, the student can become more aware of the procedure of his or her psychological functionings and improve his or her self-realisation and self-regulation.

## Epilogue

As a result, a four-sided view of every object, and in particular of the function as described above, which covers both hemispheres and the four basic psychological functions, is the minimum precondition required for a complete knowledge of this object. In other words, when we teach a function, this function must be presented to students from all four views. Only then we can say that they have fully been taught the mathematical object of function. In this way, we also succeed in the synthesis of those two different positions, languages and codes, those of the left and right hemisphere, to a hypercode which corresponds to totality, unity and fulfillment, a state which I have already termed *quintessence*. This is that holistic understanding which can help students to understand the concept of mathematical function

## References

Barneveld, G van and Verstappen, P. (eds) (1982) *Proceedings of the Conference on Functions*. Enschede, The Netherlands, N.I.C.D

Drossos, C. A (1995) *Introduction to Mathematical Thinking: Mathematical Tours*, Patras, Greece, Patras University Press.

Even, R. (1990) 'Subject matter knowledge for teaching the case of functions', *Educational Studies in Mathematics* **21**(6), 521-544

Franz, M.-L. von (1974) *Number and Time*, London, Rider and Co.

Harel, G. and Dubinsky, E. (eds) (1992) *The Concept of Function: Aspects of Epistemology and Pedagogy*, Washington, DC, M.A.A Notes vol. 25.

Jung, C. G. (1957-1977) *The Collected Works*, Princeton, NJ, Princeton University Press (Bollingen Series XX), 20 vols.

Lawvere, F. W (1970) 'Quantifiers and sheaves', in *Actes du Congrès International des Mathématiques, IMU*, Nice, pp 329-334.

Lawvere, F. W. and Schanuel, S. H. (1993) *Conceptual Mathematics: a First Introduction to Categories*, Buffalo, NY, Buffalo Workshop Press.

Mac Lane, S (1986) *Mathematics. Form and Function*, New York, NY, Springer-Verlag.

Rucker, R. (1988) *Mind Tools: the Mathematics of Information*, New York, NY, Penguin Books.

Sfard, A. (1989) 'Transition from operational to structural conceptions: the notion of function revisited', *Proceedings of PME 13*, Oaxtapec, Mexico, pp. 151-158.

Sfard, A (1991) 'On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin', *Educational Studies in Mathematics* **22**(1), 1-36.

Schoenfeld, A H (1987) 'What's all the fuss about metacognition?', in Schoenfeld, A H (ed.), *Cognitive Science and Mathematics Education*, Hillsdale, NJ, Lawrence Erlbaum, pp. 189-215.

Sierpinska, A. (1991) 'Some remarks on understanding in mathematics', *For the Learning of Mathematics* **10**(3), 24-36.

Tall, D. (ed.) (1991) *Advanced Mathematical Thinking*, Dordrecht, Kluwer Academic Publishers.

Vinner, S. (1983) 'Concept definition, concept image and the notion of function', *International Journal of Mathematics Education for Science and Technology* **14**(3), 293-305.