From the Cognitive to the Epistemological Programme in the Didactics of Mathematics: Two Incommensurable Scientific Research Programmes?

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In the ICMI study *Mathematics Education as a Research Domain: a Search for Identity* (Sierpinska and Kilpatrick, 1998), an eclectic point of view is adopted in terms of both the object of study in mathematics education and the objectives of research, the types of problems that it is to take care of and the nature of mathematics education as a discipline

> Is there a specific object of study in mathematics education? This question is even more difficult to answer than it first appears, because the term 'mathematics education' is ambiguous. It signifies both a practice (or rather a set of practices) and a field of knowledge. [...] An analysis of the nature of research in mathematics education raises many issues, but again and again the issue of the complexity and multifaceted nature of the field keeps emerging. There are various objects of enquiry, research approaches and paradigms, traditions, institutional locations, practices, evaluation criteria, adjacent fields of knowledge and a growing body of theories, results, and publications that deserve a full and thorough consideration. (Ernest, 1998, p. 72, p. 84)

In this context, it seems strange that some of the contributions to the ICMI study propose taking the comparison of theoretical foci in mathematics education as a problem of research in itself:

> we also need theoretical tools of the 'theoretical perspective' type concerning comparison of different types and methodologies of research in mathematics education; [...] Classifications, and models - if possible - are necessary. (Boero and Szendrei, 1998, p. 208)

One of the objectives of this article is to contribute to the above-quoted discussion. To do this, I first present the preconceptions which I disagree with and basic hypotheses that guide my point of view on the nature of didactics in mathematics and its relation to mathematics education.

(i) I believe that the conglomerate of 'foci', 'paradigms', 'theories', 'traditions' and 'professional practices' of diverse types that constitute what is commonly called 'mathematics education' cannot be considered globally as a scientific discipline, but instead should be seen more as a professional area and a social and cultural project which involves, among other things, different disciplines

(ii) I postulate that it is possible to select, although not in a unique manner, a group of theoretical perspectives which, starting from the study of the *problématiques* of teaching and learning mathematics, enable a sketch to be made of the 'evolution' of 'the didactics of mathematics', as a scientific-experimental discipline (in its foundational phase)

(iii) *Rational reconstruction* (in the sense of Lakatos, 1971) of the evolution in the didactics of mathematics, described in Gascón (1998), is assumed. This reconstruction does not intend to be an objective and neutral description of historical facts. Neither is it intended to exhaust the theoretical perspectives completely nor, even less so, to encompass the enormous richness and variety of investigations that tend to be included in the area of 'mathematics education' as described, for example, in Kilpatrick (1992) or the ICMI study mentioned above.

(iv) The reconstruction of the evolution of the didactics of mathematics that I draw on here has been elaborated by selecting those perspectives which successively increase the *problématique* and that, at the same time, can be situated in the direction of the development that culminates in the anthropological theory of didactics.

(v) The reconstruction essentially envisages two successive enlargements of the object of study of the didactics of mathematics which give rise, respectively, to two research programmes in the sense of Lakatos (1978): the *cognitive programme* and the *epistemological programme*.

(vi) Both of these research programmes, although arising from different theoretical perspectives, are characterised by the nature of their 'hard cores' (to use Lakatos' terminology, which describes the central set of tenets of a theoretical programme tenaciously protected from refutation), by the type of problems that can be formulated within them and by the mechanism of production of such problems. While in the hard core of all of the cognitive perspectives is a model of cognitive activity, in the epistemological perspectives this is replaced by an epistemological model of mathematical activity. [1]

A decision about programme *incommensurability* (Kuhn, 1962/1970) will depend on answers to the following questions

- Is it possible to present in one of the programmes (some of) the problems that are presented in another?

- In each case, which are the primitive notions? How do they relate to each other? And how do they interact with the corresponding empirical bases?

- Do the phenomena which are studied in such programmes have the same nature or are they deemed to be irrefutable phenomena?

- Is there agreement between the nature of the discipline and the basic norms of work in the respective scientific communities?

To start answering some of these questions, and due to the fact that the limits of this project do not permit a complete study of the theme, I focus on some of the perspectives (which I call *proceptualist*) that tend to integrate what is called *Advanced Mathematical Thinking,* in order to study the question of commensurability or otherwise. [2] I take some representatives of the most recent advances in the *cognitive programme* and try to relate them to recent developments in the *anthropological theory* of Chevallard and colleagues [3], which I choose to represent the *epistemological programme*

1. The cognitive programme in the didactics of mathematics

Once past the pre-scientific stages, which are centred in the spontaneous *problématique* of the teaching and learning of mathematics (see Gascón, 1999b), the cognitive programme in didactics emerges with great strength. This programme considers learning in general, but especially the learning of mathematics, to be a psycho-cognitive process which is strongly influenced by motivational, affective and social factors. It represents the first systematic analysis of didactical facts and has the honour of having broken with a pre-scientific way of thinking concerning the analysis of the didactic for the first time.

Brousseau (1986) characterised this programme, which he termed "a classical approach" (p 39), remarking that, in the explanation of didactical facts:

> [the] cognitive activity of a subject is central, taking as given that such activity can be described and explained in a relatively independent manner from the other aspects of the didactical relation (p 39; *my translation*)

Historically, the initial evolution of the cognitive programme was explicitly determined by the manifest insufficiency of the general notion of 'human learning' and by the means that were available to describe the mathematical knowledge of a student. [4] At this point, I want to underline the decisive importance that the epistemological model (of mathematics) has had, which goes hand in hand with successive perspectives of the cognitive programme. I will show how, throughout the evolution of such a research programme, although it has always been postulated that its primary object of research is formed by the cognitive activity of the subject, such activity has been interpreted as a function of the epistemological model of mathematics which implicitly supports it. Taking such an epistemological model as my criterion, I distinguish three types of theoretical perspectives within the cognitive programme.

(1) *Primitive conceptualists,* who implicitly identify the mathematical knowledge with a system of concepts. In these perspectives, the 'concepts' approximate to 'black boxes' without any operative internal structure.

(2) *Psycholinguists,* who expand purely conceptualist perspectives to include in the epistemological model the dimension of the *language* it possesses for mathematical knowledge.

(3) *Proceptualists,* who attempt to model cognitive processes related to the construction and development of mathematical concepts This leads them to make mathematical knowledge, in a more or less implicit way, a problem in itself. The epistemological model that backs up these perspectives is, however, more explicit than that of the conceptualists and psycholinguists It is also the 'finest', in the sense that, for example, the concepts - such as mathematical objects and not just as psychological objects - stop being black boxes. These perspectives tend to frame themselves under the label of AMT (Advanced Mathematical Thinking) or APOS (Actions, Processes, Objects and Schemes) theory.

Early conceptualist perspectives

Within the frame of the cognitive programme, the initial *problématique* revolved around the notion of *significant knowledge* in the sense of Ausubel (1968), namely making the mathematical knowledge of the student and its evolution the primary object of investigation According to Novak (1977), a disciple and subsequent collaborator of Ausubel, it is essential to take a theory of human learning as a base for constructing a model of *curriculum* (that is, a structured series of previewed results of learning) and to design an adequate teaching programme.

The research work realised within the primitive conceptualist perspectives attempted to study the laws that constitute cognitive phenomena relative to the acquisition and the development of the concepts. Some of the prototypical problems that are proposed in the study of such phenomena are the following

PCP1 What are the spontaneous or primitive conceptions of the students concerning determined mathematical concepts? How do such conceptions, difficulties and mistakes influence the acquisition of knowledge in which such concepts intervene or have an influence?

PCP2 How could the similarities and differences between the conceptual structures of the students and the corresponding structures of the mathematical concept system be utilised in order for students to reach their potential in significant learning?

These first cognitive perspectives take the processes of construction and the acquisition of concepts as the primary object of investigation. The adoption of this object of study derives from an assumption about the interpretation of *school mathematics* in terms of a system or net of mathematical concepts (and it is for this reason that I label such perspectives *conceptualist*) At the same time, it also involves a learning model based on the acquisition or construction of concepts that the student must carry out throughout his or her experience in the classroom.

One could say that the *early conceptualist* perspectives predominated during the 1970s and, for the most part, the 1980s (Rojano, 1994). Since then, a complex evolution, one which is evidently still happening, has taken place. As mentioned before, I do not attempt to describe this evolution 'historically'. But going back to the rational reconstruction I have been using as a guide (Gascón, 1998), I can say that in the successive enlargements and completions of these first conceptualist perspectives that can be considered to be within the cognitive programme (essentially carried out by *psycholinguists* and *proceptualists*), the following postulates are conserved in a more or less open way

A. Its primary object of research is conformed by specific cognitive processes of the subject which are enriched by other aspects (linguistic, logical, epistemological and social) of the didactic relationship

B. Teaching activity has a decisive influence over the processes of construction of the mathematical knowledge of the student. In fact, one of the primary characteristics of the cognitive programme consists of identifying the teacher with the teaching system and, also, taking him or her as a direct and immediate mediator between the mathematics to be taught and the cognitive processes of the students.

Psycholinguistic perspectives

In the development of didactic research, there was a period in which purely conceptualist accounts appeared as clearly insufficient and non-operative. In the 1980s, interest in the study of *semantic and syntactic aspects* of mathematics emerged, in order to explain empirical observations about the interpretations that students give to mathematical symbols (Rojano, 1994) Psycholinguistic perspectives enlarge the epistemological model of purely conceptualist perspectives, because the semiotic dimension of mathematical activity [5] is taken into account. In these perspectives, the type of problems treated by conceptualist perspectives is significantly transformed. The main reason for this change is that school mathematics alters from being considered a net of concepts that a student has to construct to a language that can be taught.

It could be said that the main phenomenon attempting to be studied within the frame of psycholinguistic perspectives is the phenomenon of the disagreement between significations It is postulated that the notion of 'meaning' is a key factor in the didactics of mathematics, because it supposedly permits the description, and even the explanation, of certain facts relating to the teaching and learning of mathematics. It starts from the disagreement between the 'meaning' that the teacher (or the scholastic institution) attributes to mathematical objects and the ones that the students give back to him or her. Among the problems that are presented within this perspective, with the object of explaining such phenomenon and its didactic consequences, is the following.

PSP1 How can we construct a semantics for the symbols and the mathematical operations related to the situations that are present in the phrasing of the problems? (Kaput, 1987)

When considering mathematics as a *language,* this language can be considered as the result of conceptual activity (Laborde, 1990) but, also, as a problem-solving instrument. This way, as psycholinguistic perspectives take form, the conviction of the insufficiency of investigating the possibilities that the student has of "giving sense to the concepts" he or she is expected to acquire also takes form Studying the nature of the *methods of utilisation* in problem solving also becomes more necessary

When emphasising the role of mathematics as a language, the existence of many mathematical objects which are not 'concepts' [6] becomes evident Therefore, new types of problems of didactic investigation emerge which are still centered around the *discordance of meaning,* but with a certain ambiguity with respect to the 'objects' which, within mathematical activity, are susceptible to 'having a meaning', and with respect to the nature of such meanings. From these new types of problems, I will address the following.

PSP2 What is the role that symbols play in the appropriation of the basic algebraic ideas by the students, in the context of the solution of verbal problems? What are the ideas which are represented by the symbols of school algebra? How should we direct the activities designed to the stimulation of interrelationships between symbols and ideas? (Radford and Grenier, 1996)

PSP3 How do we make the students "active searchers of the sense"? How do we help the students interpret the situations, interpret their own actions and assign the right meaning, in each case, to the symbols and the operations? (Sfard and Linchevski, 1994)

These formulations put forward that there is nothing clear concerning the types of objects – as well as 'concepts', the 'methods', the 'means of solution' and the 'well-formed expressions' – that the student should make sense of the 'symbols', the 'situations', the 'actions or operations' that he or she uses There is also ambiguity with respect to the nature of the presumed 'meanings' of such objects: are they 'ideas', are they mathematical objects?

Psycholinguistic perspectives are still maintained next to primitive conceptualist perspectives, in the frame of the cognitive programme, since the 'hard core' (the group of basic presumptions that are not open to discussion as the result of a methodological decision) continues to come from a model of cognitive activity which is more or less explicit Due to the enlargement of the epistemological model of mathematical knowledge, this does not end the production of an important change in the nature of the phenomena that are studied in the didactic problems which are presented and even in the teaching models that are proposed.

Proceptualist perspectives and advanced mathematical thinking

The limitations of the early conceptualists not only favoured the extension of their theory to include language, but also made more apparent the need for a model of a concept as well as of the cognitive processes that intervened in its construction. A new model, one finer and more functional than that presented by conceptual maps, was needed. New perspectives consequently emerged within the cognitive programme to overcome the many limitations and deficiencies of the previously-mentioned perspectives [7] Here, I focus on perspectives that work around the relationship between processes and concepts. I call them *proceptualists* or *APOS theorists.* Historically, their appearance dates back to 1985 when a Working Group of the *Psychology of Mathematics Education* conference was interested in what is usually called *Advanced Mathematical Thinking*

The origin of this new *problématique* is situated in the ratification of the great difficulty which teachers encounter when teaching (and the students when learning) the basic concepts of calculus. For example, 'limits' and 'functions', in the context of high-school teaching and the first year of university (see Schwarzenberger and Tall, 1978). This *problématique* rapidly grew to encompass the analysis of the difficulties, contradictions, confusions, cognitive obstacles and, in general, the (cognitive) phenomena that appear in the transition from *Elementary Mathematical Thinking* (EMT) to *Advanced Mathematical Thinking* (AMT).

According to David Tall, who could be considered the main author within these perspectives, many of the activities that are utilised in AMT are also utilised in EMT; the distinction essentially resides in the necessary use of formal definitions and deduction in AMT (Tall, 1991b, p. 3). With more precision, Tall subsequently characterises the differences among both levels of mathematical thought as follows

> The move from elementary to advanced mathematical thinking involves a significant transition: that from *describing* to *defining,* from *convincing* to *proving* in a logical manner based on those definitions. This transition requires a cognitive reconstruction which is seen during the university students' initial struggle with formal abstractions as they tackle the first year of university (p 20; *emphasis in original*)

Cognitive structure associated with a concept

To explore this transition (in what refers to at least a different role that definitions play in both levels of mathematical thought), a first model of the cognitive structure associated with a concept was initially used (Tall and Vinner, 1981; Vinner, 1983; Vinner, 1991). In this model, the existence of two different 'regions' in such structure is assumed. One region is occupied by the definition of a concept *(concept definition)* and the second by the mental images associated with that concept *(concept image),* which naturally includes partial and even mistaken representations of the concept in question. In general, it is postulated that in informal learning of concepts (which is the most common), the concept image is utilised instead of the concept definition and also when the concept definition has been constructed (parting from the terms of the definitions, if these have already been introduced), this will tend to stay inactive in the mind of the person and may even be forgotten

In the case of the learning of mathematical concepts, and before the student confronts the formal definition of the concept, it can be the case that both 'regions' of the cognitive structure are empty Alternatively, it is also possible that the person has previously constructed a concept image associated with the concept in question. When the teacher introduces the formal definition of the concept, the following situations could happen (Espinoza, 1998)

(1) *Interaction:* The concept image, if it already existed, is thereby enlarged, including the mental representations derived from the new definition or, if it did not exist, is thereby formed as mathematical objects that satisfy (examples) and do not satisfy (counter-examples) the new definition. The enlargement (if it already existed) as well as the formation (if it did not exist) of the *concept image* does not have to be in perfect concordance with the concept definition

(2) *Independence:* The concept image, if it already existed, stays untouched after the appearance of the definition of the concept given by the teacher or, if it did not exist, it never forms In this case, both structures develop in an independent manner. If the person is asked directly about the concept, he or she will respond with the definition stored in the concept definition, but if he or she is assigned to do a specific task that involves such a concept, then he or she will use the concept image in order to carry it out.

The pre-eminence of the *concept image* is clear when it is time to act or to solve a concrete problem.

H1 *In the move from EMT to AMT, many new mathematical concepts appear (function, limit, continuity, derivative, etc.), which are extremely complex (they all involve at least the multiple structures of the real numbers). They are also interrelated with each other in a particularly intrinsic manner. It is for this reason that any attempt to simplify them in order to make their learning possible will not do anything other than achieve the creation of inadequate concept images that, by producing 'misconceptions' in students, create great difficulties and cognitive obstacles which will make the formal learning of the concept harder and, therefore, will constitute a new form of contradiction and confusion in the realisation of assignments involving such concepts.*

Therefore, a first hypothesis (H1) emerges explaining some of the cognitive phenomena that appear in the transition from EMT to AMT, at least with reference to the construction of mathematical concepts.

Flexible mathematical thinking: the dual structure of the objects

This first hypothesis of proceptualist perspectives suggests that the transition from EMT to AMT cannot only be explained at the level of difficulties in the learning of formal mathematical concepts, but that special emphasis is necessary to understand the *new type of associated mathematical reasoning*. Tommy Dreyfus (1991) tells us that elementary and junior high students learn, in their mathematics courses, a great number of standardised procedures and a great quantity of concepts, but they acquire almost none of the methodological tools that mathematicians use. In particular, they do not learn to use their mathematical knowledge in a flexible way, in order to solve problems which are new to them.

The notion of *flexible mathematical thinking* can be described starting from more primitive notions which Tall originally takes from Piaget (1972) and from others' work which interprets Piaget's work, such as Dubinsky (1991), Sfard (1989, 1991) or Harel and Kaput (1991). Such notions are those of mental processes (interiorised systems of actions) and concepts produced by the *encapsulation* of processes. Thereby, the concepts which are obtained are objects upon which a system of actions can be applied and this can again be interiorised and give rise to a mental process of a higher level, which is again susceptible to being encapsulated in an object of higher order, and so on, successively.

Gray and Tall (1994) refer to the combination of process and concept produced by the process as a *procept* and it is represented together by the same mathematical symbol, thereby manifesting the dual nature of the mathematical objects and the role mathematical symbolism plays in the encapsulation (of processes into objects) - see Tall (1996). A second hypothesis (H2) of proceptualist perspectives consequently emerges to provide an account of difficulties in the transition from EMT to AMT.

H2 *The three basic notions of calculus: function, derivative and integral (as well as the fundamental notion of the 'limit') are good examples of procepts (Tall, 1996). The study of elementary calculus therefore requires, from the very beginning, enough flexibility to manipulate the same symbol in different ways: as a representative of a process which acts on determined objects or as a single entity to which other processes can be applied to obtain new objects. The strength of AMT comes precisely from the flexible use of the dual structure of the determined objects (and of the ones that are constructed from such objects). This is enabled by the ambiguity of the notation in use. The rigidity of the standardised procedures that characterise EMT therefore becomes an important cognitive obstacle and it also explains many of the extravagant conceptual errors (Dreyfus, 1991) that most of the students display in their first encounter with calculus*

With the goal of proposing a general theory of the development of mathematical thinking which will cover all the stages that come before the introduction of elementary calculus, as well as the stages that will follow (especially mathematical analysis), Tall (1994), inspired by the work of Bruner, proposed a model with three systems of mathematical representation: enactive, iconic and symbolic. This system is very much related to three levels of knowledge of calculus or, in a more general way, to three levels of development of mathematical thinking. These levels are differentiated from each other not only by the specific system of representation each of them uses, but also by the way they treat mathematical objects and by having their own proofs (Artigue, 1998).

The first level belongs to the *enactive* system, that is, one where visuo-spatial representations and experiences are used which relate to action and constitute a first intuitive base in calculus. At this level, the 'proofs' are undertaken by physical experiences upon 'real-world' objects. The second level belongs to the *iconic* representations (numerical, symbolic and graphic). It corresponds to the proceptual treatment of mathematical objects which are part of elementary calculus. The proofs are based on such proceptual structures. The third level belongs to *symbolic* representations. Here, formal definitions are used and not descriptions. The proofs are carried out by following the laws of logic and this level corresponds, according to Tall, to mathematical analysis.

A new enlargement of the didactic-cognitive problem hereby emerges, relative to the difficulties of the transition from 'proceptual' to 'formal' thinking. Tall (1996) mentions, in this case, the insufficiency of the encapsulations of the proceptual level to ensure the transition to the formal level and underlines the chasm between them:

> At the formal end of the spectrum there is a wide conceptual gulf between practical calculations or symbolic manipulations in calculus and the theoretical proof of existence theorems in analysis. I conjecture that this is so wide that it causes a severe schism in courses (particularly in 'college calculus') which attempt to bridge the gap between calculus and analysis during the first encounter with the subject. (p. 296)

Dubinsky's model (1991)

Among the proposed models in the frame of these perspectives, Dubinsky's (1991) model stands out. In a certain way, it completes that established by Tall. Taking Piaget's *reflective abstraction* as the central notion, Dubinsky attempts to elaborate a theory of mathematical knowledge and its acquisition to be applied, especially, to mathematical education at university level (Dubinsky, 1996).

One of the objectives in Dubinsky's general theory is to isolate small coherent portions of mathematical objects' and processes' complex structure that conform to each individual's mathematical knowledge and therefore give explicit descriptions of the schemes and possible relations among them. When this is done to a particular concept, *a genetic decomposition* of the concept arises. Such a decomposition does not have to become *the* valid decomposition for all students; it simply represents a reasonable path that they could use to construct the concept

Following Piaget, Dubinsky considers the existence of three kinds of abstraction: empirical, pseudo-empirical and reflective. Only this last one is able to produce a constructive generalisation. In particular, *reflective abstraction* permits the realisation of the following five types of constructions

(1) *Interiorisation*: moving a succession of material actions (applied to objects) to an interiorised system of action. The result of interiorising a system of actions receives the name of *process*. Numbers, functions, vector spaces, groups, etc. are all examples of *objects*. Among the actions that are considered are: the dual determination, calculus of the derivative function, etc.

(2) *Co-ordination*: using one or more processes to construct new processes.

(3) *Encapsulation*: converting a dynamic process into a static object.

(4) *Generalisation*: extending and enlarging the applicability of a scheme.

(5) *Inversion*: inverting a process that already exists internally, in order to obtain an inverse process.

For Dubinsky, mathematical knowledge can be described in terms of a structural collection of schemes and the existence of a scheme is inseparable from the dynamic action of construction *and* reconstruction. He postulates the necessity of creating a model of the structure and the internal dynamics of schemes that will serve as the foundation for the construction of a theory of cognitive development of subjects. Dubinsky interprets the schemes as necessary cognitive models for the understanding of concepts. Among concepts (and among schemes), he assumes relations of co-ordination and subordination. In this way, within the scheme (of the concept) 'topological space', there can be a scheme for the concept 'continuous function'; within the scheme of 'vector space', there can be a scheme for the concept 'linear function'. By co-ordinating 'continuity' and 'linearity', we get the idea of the 'continuous linear function'.

The availability of an explicit model of the internal structure and the development of concepts results in some problems that could have been enunciated formally in early conceptualist perspectives, but which now involve a completely different approach.

PP1 How could the differences and resemblances between the conceptual structures of students and the corresponding structures of mathematical concepts be utilised in order to maximise student learning?

PP2 How should traditional methods of teaching be modified in order to help students construct the necessary schemes for the understanding of concepts (Dubinsky, 1991, p. 119)? In particular, the lack of emphasis on visual aspects can become a serious impediment that will make the students' learning process more difficult?

In general terms, to create a certain type of problem of didactic investigation, proceptualist perspectives take as a hard core a more or less sophisticated model of the concept's structure (which also presents differences among the different authors), as well as of its own development.

In the works of Dubinsky, such a model together with empirical data (especially the difficulties of the students) is utilised to elaborate a genetic decomposition of a mathematical concept. This decomposition is used to guide instructional design.

2. The epistemological programme: a new research programme

Within the frame of the cognitive programme, the primary object of investigation is formed by cognitive processes relative to the mathematical knowledge of the subjects. This knowledge is strongly conditioned by the educational activity of the teacher, who is considered to be the direct and immediate mediator between the mathematics to be taught and the students. These assume that the mathematical knowledge that comes into play has been given as transparent, unquestionable and, most definitely, not problematic; also, that such knowledge can be assigned a certain 'absolute' character, thereby ignoring institutional relativity.

Under these conditions, it is clear that the implicit epistemological model, which is dominant in an institution, cannot be questioned: therefore, neither is the studying of the didactic-mathematical phenomena that consequently remain invisible [8] In fact, a common characteristic in didactic-cognitive research (that attempts to study the teaching and learning of mathematical knowledge **C** in an institution **I**) consists of assuming the interpretation *C* that, because of whatever reason, has been imposed in **I**. A paradigmatic example of this assumption and the limitations that it carries is given by the dominant epistemological model of 'elementary algebra' in current secondary school teaching: this model identifies 'elementary algebra' with a sort of *generalised arithmetic*. This has lead to locking in many of the investigations to an 'arithmetic frame of reference', relative to the learning and teaching of elementary algebra in high school (Gascón, 1993, 1994).

Among didactic phenomena that have stayed invisible for many decades are:

- the de-algebraicisation of the curriculum of compulsory secondary teaching;
- the mathematical irresponsibility of the students;
- the atomisation of the process of the institutionalised teaching of mathematics;
- the paradox of mathematical creativity;
- the absence of a genuine mathematical discipline in scholastic institutions;
- the algebraicisation of school differential calculus [9]

Such phenomena can only be looked at from a scientific perspective if determined objects (such as 'creative mathematical activity', 'school differential calculus', 'mathematical discipline', etc.) that are taken as transparent (para-didactic objects) in the cognitive programme come to be objects of study in themselves: that is, they are taken as didactic objects (modelled by the didactics of mathematics), which integrate the correctness of the didactic *problématique* (Gascón, 1998, p. 17). This is precisely the originality of the epistemological programme in the didactics of mathematics. It opens a new means of studying didactic phenomena through the explicit modelling of taught mathematical knowledge

One of the essential characteristics of this point of view consists of taking mathematical activity in itself, i e student mathematical activity, as a primary object of investigation. This provides the origin of the term *experimental epistemology* which Brousseau initially gave to the didactics of mathematics. Therefore, it is postulated that all phenomena, relative to the teaching and learning of mathematics, have an essential mathematical component (Brousseau, 1994) and this brings about a new research programme in the didactics of mathematics - the epistemological programme, which also receives the name *fundamental didactics*

The step from the cognitive to the epistemological programme constitutes what Lakatos calls a *progressive change in a problemshift,* with the consequent increasing of heuristic power of the new research programme This increase is corroborated by the appearance of new types of problems, of new auxiliary theories and the appearance of new facts and phenomena.

For the first time, the notion of a didactic phenomenon which cannot be reduced to associated psychological, sociological or linguistic phenomena emerges. In this way, one of the basic implied postulates of the cognitive programme - in which it was assumed that all phenomena related to the teaching and learning of mathematics were reducible, in the final instance, to specific psycholinguistic phenomena - is contradicted This is the reason why, while inside the frame of the cognitive programme didactic-mathematical facts are used to explain certain psycholinguistic phenomena; in the epistemological programme, on the other hand, the psycholinguistic behaviour of the subjects is used to explain didactic-mathematical phenomena.

The emergence of didactic-mathematical phenomena is a characteristic of the epistemological programme and is followed by the emergence of didactic-mathematical problems, since in every scientific discipline each type of problem makes reference to (some aspect of) a phenomenon The hard core of this new research programme is initially constructed by a general epistemological model of mathematical activity at school and, at the same time, by specific epistemological models of the various 'branches' of school mathematical activity. The educational model (and in particular, what is understood by teaching and learning mathematics) will therefore be formulated by making use of the primitive terms of the epistemological models which have been mentioned

Historically, the epistemological programme was inaugurated by the theory of didactical situations (Brousseau, 1972, 1986, 1997). [10] As the programme developed, it became evident that it was not possible to interpret school mathematical activity adequately without taking into account phenomena related to the school reconstruction of mathematics, which has its origin in the institution that produces the mathematical knowledge. Therefore, the phenomenon of the *didactic transposition* (Chevallard, 1985) appeared and, as a natural consequence, the anthropological theory of didactics. In this theory, institutionalised mathematical activity is taken as the primary object of research and, therefore, a general model of institutionalised mathematics (which includes, as a particular case, school mathematics) and a model of institutionalised mathematical activities (which includes, in particular, the teaching and learning of mathematics at school) have to be made explicit.

In the most recent developments of the epistemological programme, this model articulates the notion of *praxeology* (mathematical and didactic) and establishes the hard core of the current version of the anthropological theory. [11]

3. Two incommensurable research programmes?

The case in which two theories cannot be compared in any way is talked about in terms of *incommensurable theories* Kuhn (1962/1970) utilises this notion (which has also been incorporated by Lakatos with reference to his work on scientific research programmes) to refer to the gap between two *normal* scientific traditions which are prior and subsequent to a *scientific revolution*. The corresponding communities of investigation "practice their professions in different worlds" In particular, two incommensurable theories *disagree* regarding the following:

- the types of problems that need to be discussed;
- the nature of its discipline and the basic norms for its functioning;
- the relationships that are established among the primitive terms that are utilised, the concepts that are constructed and the experiments that are designed;
- the need to demonstrate a law that for a given community can be unprovable and for another can be taken as intuitively evident.

It can be said that between two incommensurable theories, or even better, between the corresponding scientific communities, a radical misunderstanding exists, one which cannot be resolved through proof. On the contrary, two theories (or two programmes of investigation) are said to be *rivals* if, despite having the same hard core, they nevertheless propose different *positive heuristics,* another technical term of Lakatos' (1978, p. 49) account of scientific research programmes.

In this sense, for example, the theory of didactic situations and the anthropological theory of didactics are potential rivals, given that both share an epistemological model of mathematical activity in their hard core (which at least permits them to be compared) and that they both propose different positive heuristics. The integration of these two theories requires, in the first place, a detailed comparison of the corresponding *hard cores* with the objective of analysing the possibility that they could be developed with a unified common hard core In the second place, it would be necessary to revise the coherence of the corresponding positive heuristics in line with this new unified hard core.

The topic I now turn to with more precision is the following: to what extent can APOS theory [12] and the anthropological theory be considered rivals? Or, on the other hand, are these theories incommensurable? What relationships can be established among the types of problems that they address and between their respective hard cores?

The relationships between the types of problems that construct both theories

One could claim that different theoretical perspectives in the didactics of mathematics encompass different problems. One could even talk about a phenomenon of dispersion and isolation of the didactic problems. But one should never forget that, just as with other scientific-experimental disciplines, the didactics of mathematics also constructs its own problems and that these, far from being eternal and immutable, evolve in conjunction with the evolution of the corresponding didactic theory (Gascón, 1993). This does not impede the search for points of contact between the didactic problems encompassed by proceptualist perspectives and by the anthropological theory.

Consider an ensemble of 'didactic facts' (recognised as such by both theoretical perspectives) Once the problems - relative to such a *problématique* in each of the perspectives - have been stated, they can be contrasted or at least compared Take, for example, the initial *problématique* of the proceptualists: namely, the great difficulties that teachers encounter when teaching (and the students when learning) the basic concepts of calculus, within the frame of secondary school and the first year of college. We have already seen the way in which different proceptualist perspectives conceptualise this *problématique,* how they use the notions they construct to hypothesise the existence of different 'phenomena' (like, for example, the alleged 'transition between two levels of mathematical thinking' - from EMT to AMT) and which are the 'didactic problems' that they state in order to give an account of such phenomena. Therefore, for example, they offer the problem of how concept image and concept definition relate to each other or that of how students can be helped to construct the necessary schemes for the comprehension of a particular concept

In the frame of the anthropological theory, the conceptualisation of the facts that form the foundation of this *problématique* would be very different Above all, it would be 'depersonalised', in order to state it in institutionalised terms Consequently, it would be postulated that many phenomena, which are relative to the study of mathematics (Chevallard, Bosch and Gascón, 1997) and that appear in the first college cycle, can be explained in terms of the contradictions and sudden changes among the mathematical-didactic praxeologies of high school and college. Or, equivalently, there are those that can be explained among the clauses of the institutionalised didactic contracts that dictate the type of mathematical activity that each institution has the possibility of engaging in With the help of the notions of point, local and regional praxeology [13], the following general provisional conjecture could be stated, in order to explain many phenomena that characterise the transition from studying mathematics in high school to studying mathematics in college

> *The mathematical praxeologies that are studied in high school are point ones and very rigid. One of the main outcomes of this rigidity comprises increasing the difficulty or even impeding the study of local mathematical praxeologies in high school. Ignoring the absence of local praxeologies in high school, the study of regional mathematical praxeologies is proposed in college. I postulate that the absence of an institution, one in which the study of local mathematical praxeologies takes place, is the cause of many of the difficulties for both directing the study process and making it work in the first year of college*

This general conjecture concentrates a group of specific conjectures whose empirical contrast to, as well as its relationship with, other didactic phenomena could be stated as didactic problems. [14]

(a) Relative to the rigidity of the mathematical praxeologies that are studied in high school like, for example, the dependency of the nomenclature associated with a specific technique, the non-existence of two techniques for the same problem in some mathematical homework and the non-reversion of techniques in order to realise homework inversely, among others.

(b) Relative to the step from a 'showable' mathematical organisation (in high school) to a 'provable' one (in college) that is manifested in the change of roles in the definitions (from descriptive to constructive) and in the step from the preponderance of the problems to be resolved to the problems to be demonstrated (Gascón, 1997)

(c) Relative to certain general characteristics of the mathematical organisations that are studied in high school, like for example: the intrafigural character of the study of geometry (in high school, neither the relations between figures, nor the properties of geometric space are studied) and the pre-algebraic character of school mathematics (Gascón, 1999a)

Given the chasms among the phenomena that can be detected and described in each of the perspectives, as well as the different nature of the problems that are stated, it does not seem that the results that one could obtain can be comparable with each other Therefore, at this point at least, I have come to what appears to be an initially pessimistic conclusion regarding their potential commensurability.

Towards a confluence of the respective hard cores?

In contrast to the pessimistic conclusion that was stated before, a progressive approximation between the APOS theory and the anthropological theory is possible, in a prolongation of the rational reconstruction that I assumed before (Gascón, 1998) This approximation is reciprocal and in both instances flows towards specific, local epistemological models that, as I now show, occupy a place which takes on greater and greater importance in the hard cores of the respective programmes

It is from inside proceptualist perspectives that a critical attitude regarding the purely 'cognitive' nature of the didactic explanations begins to manifest itself:

> I am asserting, however, that not every event in a mathematics learning can be explained in cognitive terms, and that it is a fallacy to assume that the cognitive approach is adequate for almost every situation in mathematics learning. (Vinner, 1997, p 97)

Vinner specially underlines the limitations of explanations based on the 'pre-concepts' of the students:

> But it seems that whenever we want to explain students' statements in mathematical contexts, our immediate tendency is to describe their ideas about fractions, functions, continuity or isosceles triangles, ignoring other potential causes. (p. 98)

At this point, he states the notion of the didactic contract in relation to the question of how it is possible to distinguish genuine situations of learning (or problem solving) from false ones:

> [Students] are forced to learn certain topics or to solve certain problems. But nobody has control over their thoughts. Because of the didactic contract (Brousseau, 1988), they are not supposed to demonstrate their lack of interest in the activities imposed on them by the educational system. They will try to please the educational system with certain behaviours acceptable to the system. The most promising behaviour that students can think of is to give the correct answer to a question posed to them by the teacher. Therefore, instead of learning or problem-solving processes, they attempt other processes which can produce the correct answer. (p 98)

The indissoluble unity of the 'mathematical' and 'psychological' aspects of the basic components of APOS theory that constitute its hard core continues to defend itself explicitly.

> The processes to be discussed in this chapter are mathematical and psychological ones, and in many cases they are both; in fact, the mathematical and the psychological aspects of a process can rarely be separated. For example, when you build a graph of a function, you are executing a mathematical process, following certain rules which can be stated in mathematical language; at the same time, however, you are very likely generating a visual mental image of that graph; in other words, you are visualising the function in a way that can later help you reason about the function. The mental and the mathematical images are closely linked here. Neither can arise without the other, and they are in fact generated by the very same process; they are, respectively the mathematical and the psychological aspects of this process. (Dreyfus, 1991, p. 26)

But in scientific practice, there is a progressive movement towards the identification of such components with its 'mathematical' aspect. Therefore, the genetic decomposition of a concept in Dubinsky's model (Asiala *et al*, 1996) ends up manifesting itself in a group of mathematical assignments. Take, for example, the case of the concept of variable. Its decomposition contains three types of mathematical assignments:

- those which refer to the conceptualisation of the *unknown variable,* i.e.:

  > identify the unknown variable in a specific situation, and adequately symbolise it through an equation. (Trigueros, 1999, p 285)

- those relative to the conceptualisation of *variable as a general number,* i.e.:

  > simplify or develop algebraic expressions (p. 286)

- those which refer to the conceptualisation of the *variable in a functional relation,* i.e.:

  > determine the intervals of variation of one of the variables when the intervals of the other are known. (p. 286)

The anthropological theory, which emerged as a natural development of the epistemological programme, has had an evolution determined by a progressive concretisation and specification of the epistemological model that constitutes its hard core. [15] Initially, the theory of the didactic transposition proposed a vaguer modelling of mathematical knowledge, in terms of mathematical, paramathematical and protomathematical objects. The subsequent introduction of the ecological *problématique* permitted building into the study the restrictions and hierarchical interrelations that are created among the objects of mathematical knowledge and its teaching, in what constituted a first refinement of the epistemological model.

The next step was given by the theory of 'the relation to knowledge' *(le rapport au savoir)* that situates didactics in the field of cognitive anthropology, enlarging its domain (mathematical knowledge is a result of institutionalised human action and, as such, is utilised, taught and, in general, is transposed within institutions), while refining at the same time the axioms of the theorisation. But in this arena, a model of institutionalised mathematical practices that would enable their description and the study of their realisation was not deployed. The most recent developments in the anthropological theory, with the introduction of the notion of praxeological organisation or praxeology, fills this gap, thereby taking another step towards the concretisation of the epistemological model.

Many important differences still exist among the local epistemological models (explicitly in the anthropological theory and more implicitly in APOS theory) that are proposed. I will mention that in the anthropological theory, in order to reach a description of global mathematical activity, one has to depart simply from homework and mathematical techniques. From a point praxeology generated by a unique type of homework, we move on – by successive aggregations – to local praxeologies (which are centered in a technology) and to regional praxeologies (which are centered in a theory - see Chevallard, 1999) In APOS theory, on the contrary, one keeps on starting off from the same concepts in order to reach the phenomenon of mathematics homework that is supposedly required by the construction or comprehension of such concepts.

I suggest, in this synthesis, that the possible commensurability between both research programmes will depend, on the one hand, on the capacity of APOS theorists to integrate their 'cognitive', which becoming closer and closer to 'local epistemological models', into a global model of mathematical activity (the most recent works of Tall and Dubinsky seem to be oriented in that direction) On the other, it will depend on the capacity of the epistemological programme and, in particular, anthropological theorists to take into consideration the 'molecular' level of mathematical activity within their epistemological models. [16]

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Notes
[1] I would like to underline that, although I will inevitably use specific works to 'represent' the different perspectives, these are *theoretical constructions* and therefore they do not correspond to real, 'visible' historical facts It should be very clear that I am analysing and interpreting tendencies but in no way authors.

[2] In the book with the same name, *Advanced Mathematical Thinking,* edited by David Tall (1991a), a good compilation of works which are representative of the first developments of this approach can be found. The most recent developments of this approach is Action-Process-Object-Schema (APOS) theory (see, e g Asiala *et al* 1996) In Artigue (1998), these perspectives are integrated within *"les problématiques des rapports processus-objects"* (p 231)

[3] The latest developments of the anthropological theory of the didactic are presented in Chevallard (1992, 1997, 1999), Chevallard, Bosch and Gascón (1997), Gascón (1998) and Bosch and Chevallard (1999).

[4] The first point of inflection is constituted by the *International Group of the Psychology of Mathematics Education* (PME) (Bauersfeld, 1976), which identified the need to take into consideration a kind of "specifically mathematical knowledge" The investigators of this group took, as a new primary object of investigation, the processes of mathematical learning of a student and started to construct instruments to describe such processes

[5] Within this new perspective, I should note the work of Freudenthal (1983) that marks, by itself, a new line of development of the cognitive programme Given its importance, its complexity and its originality, it would require a special study which I cannot undertake here Therefore, I will not include him in my rational reconstruction Suffice it to say that his 'phenomenological analysis' of a concept or of a mathematical structure consists of the description of the 'phenomenon' (objects of our mathematical experience) for which the concept or the mathematical structure being observed is the "means of organisation" and the identification of the interrelations between the 'phenomenon' and the corresponding 'means of organisation'

Given that mathematical objects are incorporated into the world of our experience (in particular, through ostensive objects such as sounds, graphs, etc , that are manipulated in mathematical activity), they become new 'phenomena' that, at the same time, require new means of organisation, and so on. A stepped production of 'phenomenon-means' pairs of organisation are consequently observed (Puig, 1997), which reminds me of the recurrent character of the process of modelling and manifests the interdependence between mathematical activity and mathematical concepts and structures that it produces and that, at the same time, forms the base from which such activity is developed. Given the importance of the notion of 'mental object' in the construction of concepts, the work of Freudenthal can initially be situated in the frame of the cognitive programme, although it would also be possible to take from his work an epistemological model that is based on the phenomenological analysis and that clearly comes close to the epistemological programme.

[6] Although in algebra there are concepts that have to be taught such as *variable. operation, polynomial* and *equation,* it is also true that:

> *in algebra, certain teaching objects are not concepts* The writing $ax + b$ is a well-formed expression and, as such, it has a meaning. The question of knowing whether the student effectively attributes a meaning to this writing, and what that is, is crucial in the didactics of algebra. But $ax + b$ is not a concept in the same sense that multiplication or plane are (Drouhard, 1992, p 6; *my translation and emphasis*)

[7] Among such perspectives, the theory of conceptual fields stands out. Because of reasons analogous to the case of the phenomenological analysis, I will not include it in my rational reconstruction This theory is explicitly situated inside the cognitive programme:

> La théorie des champs conceptuels est une théorie cognitiviste, qui vise à fournir un cadre cohérent et quelques principes de base pour l'étude du développement et de l'apprentissage des compétences complexes, notamment de celles qui relèvent des sciences et des techniques Du fait qu'elle offre un cadre pour l'aprentissage, elle intéresse la didactique; mais elle n'est pas à elle seule une théorie didactique (Vergnaud, 1991, p. 135)

But that perspective includes in its hard core an explicit and operative modelling of the *concept* as a triad of groups: **C = (S, I, s)**, where S is the set of situations - which are identified initially with 'homework' - which give meaning to the concept (the *référence*); **I** is the set of invariants on which the operativity of the schemes (the *signifié*) rests; **s** is the set of ways of symbolically representing the concept, its properties, the situations and the procedures which are useful for dealing with such situations (the *signifiant)*

Also, and even though the theory of conceptual fields is not specific to mathematics, when it attempts to explain the progressive conceptualisation processes of additive or multiplicative structures, of spatial-numerical relations or of algebra, it takes epistemological models of the mathematical concepts involved as a starting point. Vergnaud goes on:

> Certains chercheurs privilégient, pour cette analyse, des modèles de la complexité relevant soit de la linguistique, soit des théories du traitement de l'information La théorie des champs conceptuels privilégie au contraire des modèles qui donnent un rôle essentiel aux concepts mathématiques eux-mêmes. (p 146)

In summary, the theory of conceptual fields must be considered as a perspective between the cognitive and the epistemological programme

[8] I label *didactic-mathematical* phenomena those which depend on the specific characteristics of the institutionalised mathematical organisation and of the current *institutional didactic contract* (Brousseau, 1997). They are irreducible to psychological, sociological or linguistic phenomena. The epistemological programme postulates that all phenomena relative to the learning and teaching of mathematics depend on certain didactic-mathematical phenomena

[9] A didactic analysis of these phenomena can be found in Chevallard, Bosch and Gascón (1997), Bolea, Bosch and Gascón (1998a, 1998b) and Gascón (1998, 1999a).

[10] Brousseau (1997) is an abridging, in English, of all of Brousseau's works published between 1970 and 1990 Given the privileged position that the theory of didactical situations occupies in my rational reconstruction of the evolution of didactics of mathematics, I believe that the development of such a theory and its diffusion can help clarify the relationships between both research programmes in the didactics of mathematics

[11] The first formulations and exemplifications of such models can be found in Chevallard (1997, 1999), Chevallard, Bosch and Gascón (1997), Gascón (1998) and Bosch and Chevallard (1999)

[12] The set of theories that I have been explaining in the *proceptualist perspectives.*

[13] The notions of 'point', 'local', and 'regional' praxeology are described and exemplified in Chevallard (1999)

[14] These conjectures are being effectively contrasted in a work in progress (Gascón and Fonseca, 2000), whose first results will be presented shortly

[15] The details of this evolution have been described in Bosch and Chevallard (1999, pp. 82-87)

[16] Both conditions are complex and are related to the role assigned, in

each case, to the semiotic instruments of mathematical activity From the perspective of the anthropological theory, any rival theory should also prioritise (take as a primary object of investigation) the study of institutionalised mathematical activity and the didactic-mathematical phenomena which emerge from such activity.

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