

Cognitive Psychology and Mathematical Thinking

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Cognitive Psychology has been defined as "the study of knowledge and how people use it" [Glass, Holyoak and Santa, 1980 p.2]. As such it is a very wide and ill-defined field, embracing such diverse topics as neurophysiology, perception, language and reasoning. Of necessity, then, the coverage in this review is selective, the aim being to illustrate, rather than exhaustively catalogue, aspects of Cognitive Psychology relevant to the understanding of how people think mathematically.

Historical background

Research on cognitive processes was for a long time subject to the effects of behaviorism, particularly in America. Papert [1973] referred, not entirely tongue-in-cheek, to m*d, "an Anglo-Saxon four-letter word subject to widespread taboos in contemporary society". There were exceptions, of course, notably Bartlett, Piaget and the Gestalt psychologists. Historical milestones are arbitrary, but on the road back from behaviorism, a lot happened in the period 1956-1960. 1956 saw the publication of Bruner, Goodnow and Austin's "A Study of Thinking", which has spawned a lot of research on concept formation, and made the notions of "hypothesis-testing" and "strategy" common currency. The model of the brain as an information-processor was gaining ground, as exemplified by Miller's famous paper "The Magic Number 7 ± 2 " also published in 1956. This period saw the emergence of Chomsky's influence, including his review [1959] of Skinner's "Verbal Behaviour", which exposed many of the limitations of behaviorism with clinical precision. It also saw the beginnings of Newell and Simon's work on the General Problem Solver, a very important early contribution of Artificial Intelligence. In 1960 an early "Cognitive Party Manifesto" was published — Miller, Galanter and Pribram's "Plans and the Structure of Behavior".

Since then cognitive theories have virtually wiped the board. The extent of their domination may be seen by considering a recent review of "Conditioning" by Mackintosh [1978] in which he says:

We can now see that conditioning is not reducible to the strengthening of stimulus response associations by the automatic action of a process called reinforcement. It is more profitably viewed as a matter of detecting and learning about relations between events. It is the process whereby organisms typically discover what signals or causes such events of consequence to themselves as food or water, danger or safety

This has been accompanied by the resurgence of topics such as imagery and consciousness, and by revolutionary changes in other areas, notably memory. Associated methodological changes have included, in particular, a return to the method of asking the subject to "think aloud" and analysis of the verbal protocols thus obtained

Having achieved predominance, Cognitive Psychology is now undergoing something of a crisis of confidence. One of the first signs of this was a paper by Newell [1973]. Reviewing the "state of the art" he saw Cognitive Psychology as "phenomenon driven". That is to say, experimenters discover interesting phenomena (he lists 59), each of which generates its own body of research, but such research efforts lack co-ordination, with the result that there is little cumulative growth of psychological knowledge. A related anxiety is about the lack of "ecological validity" [Neisser, 1976] in much of the research. That is to say, the tasks given to subjects in laboratories, in investigating psychological processes, are so artificial that they tell us little about how those same processes operate under normal conditions.

A similar line has been taken by Donaldson [1978] in discussing the relevance of research in cognitive development to education. She criticizes the emphasis in studies of children's thinking (particularly by Piaget) on "disembodied modes of thought":

By the time they come to school, all normal children can show skill as thinkers and language-users to a degree which must compel our respect, so long as they are dealing with "real-life" meaningful situations in which they have purposes and intentions. These human intentions are the matrix in which the child's thinking is embedded [p 21]

On the other hand:

Education, as it has been developed in our kind of culture, requires him to be able . . . to call the powers of his mind into service at will and use them to tackle problems which do not arise out of the old familiar matrix but which are . . . presented in abrupt isolation [pp.121-2]

So, while recognizing (1) the dangers of extrapolating unjustifiably from abstract tasks to everyday thinking, and (2) the need to give children more help in disembedding their thinking, it must be remembered that abstract thinking is important and worthy of study in its own right. This applies *a fortiori* to the study of mathematical thinking; indeed, mathematics could almost be defined as "the disembedding of thought".

Psychological studies of mathematical thinking

The interest of psychologists in mathematics has been patchy and they have tended to use mathematical topics as grist to their theoretical mills rather than study them for their own sake. Thus we find Thorndike giving an associationist account of learning arithmetic, Gestalt psychologists finding mathematical illustrations for concepts like "restructuring" and "set", Piaget studying the development of mathematical concepts. Traffic in the opposite direction shouldn't be

forgotten — many mathematicians such as Poincaré, Hadamard, Polya and Dienes have written on psychological aspects of mathematics.

While there is plenty of current research on mathematical topics [Greer, 1980b], there is little by way of general theory. Signs of increasing interest among cognitive psychologists in mathematical thinking can be detected (though it still looks suspiciously as if they are motivated more by convenience than by interest in the subject itself). Among the reasons which may be advanced for this are:

- (1) Mathematical processes, by their nature, are very amenable to representation by information-processing models, since they break down into sequences of operations, transformations, logical steps etc. There is an insidious danger here of assuming that the formal expressions of these sequences necessarily mirror cognitive processes.
- (2) The role of imagery in thinking is a current focus of interest for cognitive psychologists; its place in mathematical thinking has long been recognized through anecdotal accounts.
- (3) The general notion of different representations of a given problem, and translations between them, is a shared interest, illustrated, for example, by studies of how mathematical representations of problems are abstracted from their verbal statements.

In the next sections, certain topics selected to illustrate various aspects of current developments in Cognitive Psychology are considered in relation to their implications for mathematical thinking.

Artificial intelligence

There are various definitions of Artificial Intelligence; perhaps the most straightforward is Minsky's [cited in Boden, 1977] that it is "the science of making machines do things that would require intelligence if done by men". Within AI there is a basic distinction between those who seek ways of making machines do something intelligent, and those who specifically try to model human thinking processes. It is the latter with which we will be concerned here, but it is worth pointing out that a case can be made for developing a general theory of intelligence which is independent of what does the thinking — man, child, animal or machine. As Papert [1973] put it, we came to understand bird flight by studying flight, not birds, and perhaps the same will prove to be true for human intelligence.

Computers are ideally suited, of course, for carrying out algorithmic procedures. A computer programmed to do so would not be considered as showing intelligence. Rather, we must turn for examples to areas of mathematics which require heuristic thinking. Newell and Simon [1973] wrote a program called the General Problem Solver (GPS) which, as its name implies, is written in such general terms that it can tackle a wide range of problems. The program is conceptualized in terms of:

- (1) *Objects* which characterize (i) the current situation (ii) the goal situation and (iii) intermediate situations

- (2) *Differences* between pairs of objects
- (3) *Operators* which operate on the objects

Any axiomatic system can clearly be represented in this way — the axioms are the initial situations; the theorems to be proved the goal situations; the operators the transformation rules of the axiom system. The problem in general is to apply transformation rules to get from the axioms (or already proved theorems) to a new theorem. GPS uses a general heuristic called "means-end analysis". This works backwards from the goal situation, making comparisons between it and the current situation, identifying the differences, then applying appropriate operators to remove those differences. Another heuristic it uses is that of constructing a general plan of the proof before working out the details.

One axiom system explored in depth with GPS is that for the calculus of propositions. Moreover, human subjects were tested with the same set of problems and their performances compared in various ways with that of the program. Basing their analysis on the comparison of verbal protocols from the subjects with detailed output from GPS solving the same problems, Newell and Simon claimed substantial correspondence between human and artificial solution methods, though, as discussed later, the problem of defining and measuring such a correspondence is a difficult one.

Opinions differ as to the importance of AI for cognitive psychology and there are complex arguments about the formal limitations of computer intelligence. Johnson-Laird and Wason [1977] suggest that general theories of thinking are not amenable to computer simulation, but that programs are ideal for modelling local aspects. Certainly AI models have many appealing features. They force the modeller to make his assumptions explicit; they provide a sufficiency proof (i.e. if the program works, then it embodies at least a *possible* explanation of the process); they frequently point up hidden complexities.

The most significant drawback of such models is the difficulty of evaluating them — what does it mean to say that a program solves a problem in the same way, to some extent, as a human would, and how can that extent be quantified? Certainly it has been demonstrated that computers can be programmed to use heuristic methods, and it is plausible to suppose that a program armed with a battery of such heuristics may tackle geometrical problems in a way that is similar, in some sense, to that of a human who has carefully studied Polya. Indeed, programs are often made to perform like people by explicitly identifying the heuristics people use and incorporating them into the program. For example, an early geometry theorem program written by Gelernter [described in Apter, 1970] works backwards from the theorem, a popular method for humans, and also makes use of diagrams, for example to identify line segments which appear to be equal in length. Nevertheless, such programs tend to produce surprises from time to time. Another geometry program [written by Minsky — see Boden, p 332] produced the proof of Euclid's *pons asinorum* theorem shown in Figure 1. This proof was discovered by Pappus, but virtually forgotten, and very few human subjects (not including the writer of the program) could have been expected to produce it. Doubts therefore remain

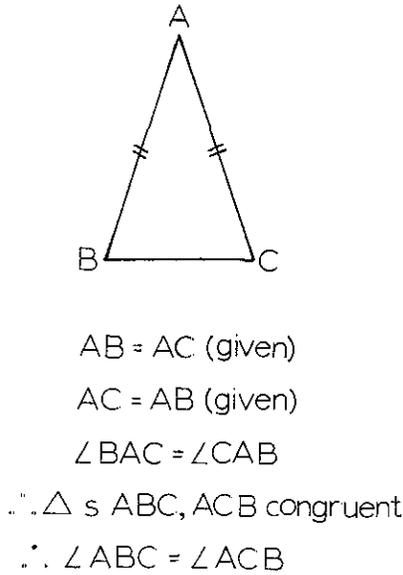


Figure 1
The computer as geometer: how to cross the pons asinorum without a construction

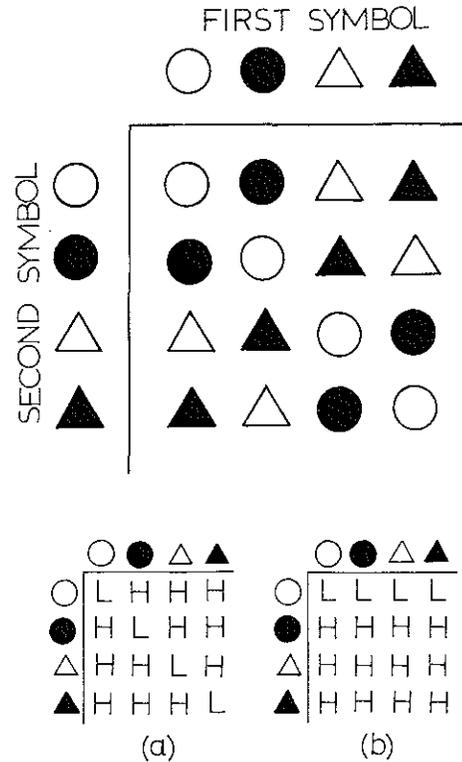


Figure 2
Klein 4-group task: differential patterns of retrieval times

as to the extent to which such programs truly parallel the thinking of human mathematicians, and also as to whether computers can ever model the more intuitive and creative aspects of mathematics

Some more recent AI work on mathematical topics will be considered later in the section on problem-solving

Making inferences about internal processes

As Chomsky [1959] pointed out, the issue between cognitive theorists and behaviourists is not over what constitutes valid experimental evidence:

... anyone who sets himself the problem of analyzing the causation of behaviour will (in the absence of independent neurophysiological evidence) concern himself with the only data available, namely the record of inputs to the organism and the organism's present response, and will try to describe the function specifying the response in terms of the history of inputs. This is nothing more than the definition of his problem

The difference lies, rather, in the belief of cognitive psychologists in mental processes which critically intervene between stimuli and responses. Theories about these mental processes have to be developed on the basis of indirect inferences from observed behaviour (which may include what the subject says about his own thinking). Many methods and paradigms have evolved for this purpose; two very contrasting ones which will be considered here are the use of verbal protocols, and chromometric analysis.

Both methods were used in a study involving the learning of a Klein 4-group structure [Greer, 1980a]. Subjects who had successfully learned the structure (in the sense that given the first symbol and the second symbol they could give the correct answer as determined by the multiplication table

in Figure 2) were tested to see how quickly they could retrieve the answer for each of the 16 combinations. At the end of the experiment, they were then asked to give a verbal description of the structure. It was found that there were two usual ways of doing this:

- (1) *Relational description.* If the two symbols are the same, the answer is the yellow circle; if the two symbols are the same colour, but different shapes, the answer is the yellow triangle, etc.
- (2) *Operational description.* If the second symbol is the yellow circle, the answer is the same as the first symbol; if the second symbol is the yellow triangle, the answer is the same colour as the first symbol, but different in shape etc.

If these verbal descriptions are an accurate reflection of the actual thought processes involved in representing the structure, then this should be reflected in patterns of retrieval latencies. Specifically, for a relational description the pattern should be as illustrated in Figure 2(a) with lower times for the four cases where the two symbols are the same, whereas for an operational description, the pattern should be as shown in Figure 2(b) with lower times for the four cases where the second symbol acts as an identity operator. It was found that the verbal descriptions did consistently relate to the latency patterns as predicted. While this is not a surprising finding, it does provide evidence that the verbal

descriptions elicited from the subjects do reflect the internal representations used to generate responses in the retrieval task. Such evidence is necessary in view of controversy as to when, and to what extent, verbal reports can be accepted as valid descriptions of mental processes [e.g. Evans, 1980]

Another area in which chronometric analysis has been used is in the study of arithmetical calculation, specifically addition and subtraction of small positive integers. As an example, let us look at Svenson and Hedenborg's [1979] model for subtraction. In their experiment, 10 children aged 9-11 solved all subtraction problems $M - N$ with $0 \leq M, N \leq 13$ and $M \geq N$. On the basis of solution times the model shown in Figure 3 was constructed. Memory is taken to be either *reproductive*, as reflected in short solution times for the special cases $M = N$ (mean solution time 1.14 secs.), $N = 0$ (1.25 secs.) and $M = 2N$ (1.38 secs) or *reconstructive*, as reflected in longer solution times in other cases. The reconstructive processes are postulated to correspond to counting up from N when $M < 2N$ and counting down from M when $M > 2N$ or when $M > 10$ and $N < 10$ (When $M = 10$, it is assumed that the result is sometimes retrieved directly from memory, sometimes worked out.) The evidence for the counting components of the model comes from a correlation between the number of units to be counted and the solution time — the time was an increasing function of $M - N$ for cases with $M < 2N$ and of N for cases with $M > 2N$

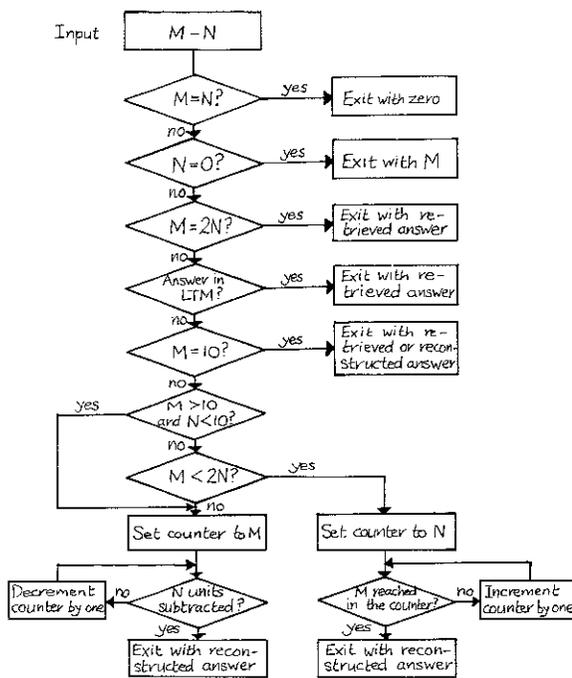


Figure 3
A process model for integer subtraction (after Svenson and Hedenborg, 1979)

What are we to make of this? The model shown in Figure 3 fits the data for the 10 subjects quite closely (with 7 parameters fitted, it accounts for 84% of the variance in the mean latencies for the 105 instances). So the approach can shed some light on very simple arithmetic processes. Trying the same model with younger or older children, for example, would be one way of investigating developmental changes. On the other hand, there are severe limitations in models of this sort. It is difficult to accommodate individual differences and idiosyncratic solution methods. Even in quite elementary arithmetic, such variability is well documented. For example, Jones [1975] identified seventeen different methods by which the subtraction $83 - 26$ was correctly evaluated [see also Ginsburg, 1977]. Two other criticisms apply generally to models of this type. Firstly, the flow chart approach implies sequential processing, which is often (as in this case) implausible. Secondly, there is no representation within the model of how the subject monitors the process itself.

Methods such as chronometric analysis are impeccably objective, but of limited applicability. Conversely, asking subjects to describe their own thought processes, or using an approach along the lines of Piaget's 'clinical' method, is likely to yield much more insight, though the findings must be treated with circumspection because of subjectivity in interpreting them, and because of the lack of control and standardization. There is also the unavoidable problem that the act of verbalizing cognitive processes itself affects those processes, and there is the whole question of validity, i.e. whether subjects can report accurately on their own thinking [see, especially, Evans, 1980]. Nevertheless, many cognitive psychologists are turning to such methods — for example, Dominowski [1974] put forward a very strong case for using verbal reports in studying concept learning. There is clearly a very important role for this type of investigation in studying mathematical thinking, as shown by Ginsburg's [1977] book, to name but one example.

Developments in memory research

The study of memory is another area of cognitive psychology revolutionized in the last decade or so. Three particular aspects of this revolution which I will consider here are:

- (1) The shift from the "Ebbinghaus tradition" to the "Bartlett approach" [Baddeley, 1976, Ch 1]
- (2) The realization that memory cannot be studied in isolation from learning and thinking
- (3) A new interest in developmental studies of memory

Ebbinghaus sought to study memory in a pure form, by stripping the material to be remembered of its associations and meaning. This approach was compatible with the behaviourist approach and for a long time the study of memory was largely concerned with sterile research with nonsense-syllables and the like. However, Bartlett [1932] swam against the general current and stressed the artificiality of the Ebbinghaus approach in excluding (or, rather, attempting to exclude) meaning. He stressed the *reconstructive* nature of memory — what is remembered is not an exact copy but a stripped-down version of the original. On recall, it is reconstructed from this version — and preexisting

knowledge may play an important part in the reconstruction.

The non-verbatim nature of memory for sentences was neatly demonstrated by Bransford, Barclay and Franks [1972]. Subjects were asked to remember sentences of the following two types:

- (1) Three turtles rested beside a floating log, and a fish swam beneath them
- (2) Three turtles rested on a floating log, and a fish swam beneath them

They were later presented with sentences like:

- (3) Three turtles rested beside a floating log, and a fish swam beneath it
- (4) Three turtles rested on a floating log, and a fish swam beneath it

Subjects who had heard (2) originally tended to falsely recognize (4); subjects who had heard (1) originally showed no corresponding tendency to falsely recognize (3). The implication is obvious — the sentences are not remembered verbatim. In the former case, what is remembered (perhaps something like an image) is compatible with (4) because of the inference:

(Fish beneath turtles) and (Turtles on log) therefore (Fish beneath log)

whereas no such inference holds in the latter case.

In Mathematics, it is certainly the case that reconstruction plays a vital role in memory. It is often easier to reconstruct a formula (e.g. for a straight line with given gradient passing through a given point) than to remember it. Moreover, mathematical formulae have patterns which can be used like error-correcting codes. Sawyer [1955, p. 21] gives an example. The condition for two roots of the cubic equation $ax^3 + bx^2 + 3cx + d = 0$ to be equal is:

$$a^2d^2 - 6abcd + 4b^3d + 4ac^3 - 3b^2c^2 = 0$$

By checking that the equation holds for $a = b = c = d = 1$ (i.e. that the coefficients add up to zero) and that each term has the same "weight" (defined by setting $a = 0, b = 1, c = 2, d = 3$ and then adding powers), and by various other checks, mistakes and gaps in memory can be emended.

In Mathematics, also, the important things to remember are general patterns, rules, procedures, heuristics etc., rather than specific cases. Krutetskii [1976] found that inept and capable pupils differed in this respect:

Most capable pupils remember the type and the general character of the operations of a problem they have solved, but they do not remember a problem's specific data or numbers. Incapable pupils, on the other hand, usually recall only specific numerical data or specific facts about a problem.

... the essence of a mathematical memory consists in the generalized recollection of typical schemes of reasoning and of operations [p. 299]

The second feature of recent memory research listed above has been summarized thus by Baddeley [1976]:

... whereas the characteristic 1960s view of memory was that of a store that had the sole function of holding information which might or might not be used in other information-processing tasks, the tendency in the 1970s is to regard memory as an integral part of other information-processing tasks, such as perception, pattern recognition, comprehension, and reasoning. This is reflected both in a growing interest in working memory — that is, in the role of storage processes in other information-processing tasks — and in a growing awareness of the importance of coding. Coding is essentially the other side of the same coin, and it represents the mnemonic consequences of processing information in different ways. [p. 187]

Working memory is discussed by Hitch [1980], including its role in elementary arithmetic. In doing a calculation such as:

$$(3 + 2) \times (2 + 6 + 1)$$

memory is involved in several ways:

- (1) If the problem is presented and solved verbally, it has to be remembered
- (2) Addition and multiplication facts are retrieved from long term memory
- (3) Intermediate results are stored in short term memory
- (4) The whole process has to be monitored

Workers like Hitch are only beginning to study the complexities of such aspects of memory processes and their interactions.

The third striking feature of recent memory research to be considered is the sudden explosion of interest in developmental changes. A very consistent theme of such work is that the development of memory is characterized by changes in strategies. Thus, according to Harris [1978]:

... one could claim that there really is no change during development in the basic capacity of memory. Instead, as children get older, they put their memory systems to work in a more strategic fashion. [p. 133]

What is of particular significance from the educational viewpoint is that there is evidence that suitable training can produce strategic memorization earlier than it would appear spontaneously. This has been shown, for example, for the most basic strategy, rehearsal [see Kail, 1979, Ch. 2].

Another resolution of the apparent contradiction between postulating a constant basic capacity of memory, and the obvious developmental changes in performance on memory tasks, is put forward by Case [1978], who proposes that:

- (1) There is one central memory which can serve as a space for storing information or for operating on it
- (2) The underlying capacity of the working memory does not change with age
- (3) The measured increase in capacity within each stage is due to a decrease in the capacity required to execute the operations which are characteristic of that stage. [p. 58]

A lot of the current literature on memory presents, to the non-specialist, a bewildering selection of models of memory — several new models are unveiled each year. However, I have avoided these technical details here and concentrated on the new cognitive emphasis in memory research, which, it seems to me, makes it much more relevant to memory in mathematical contexts

Visually mediated processes

This title covers a number of related topics — spatial ability, imagery and the role of diagrams in mathematical thinking. Spatial ability in relation to mathematics education has been excellently reviewed by Bishop [1980], so I will not attempt to cover the same ground here.

The history of imagery in psychology is an interesting one [Kaufmann, 1979]. It was one of the topics exiled during the long reign of Behaviourism. For example, Kessel [cited in Kaufmann, p. 9] has pointed out that in the period 1940-1960 *Psychological Abstracts* contained only five references to imagery. Since the start of the 1970s however, imagery has enjoyed a remarkable comeback. In contrast to the statistic quoted above, *Psychological Abstracts* for the first five months of 1980 has 64 references to imagery. Of these, however, only two have any bearing at all on mathematics — one dealing with children's drawings of geometric forms, and the other discussing Einstein's use of imagery.

It is perhaps surprising that those studying imagery have as yet taken little interest in mathematics, considering the part that imagery plays in mathematical thinking, as shown, for example, by anecdotal accounts and case studies of eminent mathematicians [Hadamard, 1945; Shepard, 1978]. To broaden the context, mathematical thinking relies even more heavily on the use of diagrams. Again, very few studies have been done on this by psychologists — we have to turn to the educational literature to find research on the topic [e.g. Fischbein, 1977]. One of Fischbein's examples concerned the problem:

On a farm there are hens and rabbits. Altogether there are 13 animals possessing 36 legs. How many hens and how many rabbits are there in the farm?

This can be solved pictorially by drawing 13 bodies with 2 legs on each, then adding extra pairs of legs until the total reaches 36. The use of a diagram in this way interacts in complex ways with other aspects of the problem. For example, this pictorial strategy would be less obvious for an analogous problem concerning coins of different values — coins don't have legs. Again, if the numbers involved were large, it would be impractical to draw the total number of bodies and legs, yet a picture could still help to guide the solution process. Clearly there are fascinating psychological aspects here which as yet have received little attention from cognitive psychologists.

Aspects of problem solving

Problem-solving is perhaps the most obvious topic of mutual interest to cognitive psychologists and mathematicians, and there is a lot of familiar literature on the topic. Rather than cover this ground again, I will discuss a theme which

seems to underlie a great variety of work in recent cognitive psychology, namely the relation of the problem in its abstract form to the real-world context in which it is framed. This is clearly highly relevant to mathematical thinking.

Let us begin with a well-known example. Wason [1968] introduced what has become known as the "4-card problem". Subjects are shown four cards, on which they can see, respectively: *A*, *B*, 3 and 2. They are informed that each card has a letter on one side and a digit on the other, then asked the following question:

Which card or cards is it necessary to turn over in order to check the truth or falsity of this statement: "If a card has an *A* on one side, then it has a 3 on the other side"?

Perhaps surprisingly, this logical problem proves very difficult — typically less than 10% of university students get the answer right. The task has produced a large literature (which is still flourishing) but, for our purposes, we will look at one particular ramification. Johnson-Laird, Legrenzi and Legrenzi [1972] used the original 4-card problem, and an isomorphic but realistic task, in which subjects were asked to imagine themselves as post-office workers sorting letters according to the rule that a 40 lire stamp was all right for an unsealed letter, but sealed letters required 50 lire stamps. Four letters were depicted showing respectively:

- (1) The front of an envelope with a 40 lire stamp on
- (2) The front of an envelope with a 50 lire stamp on
- (3) The back of an envelope, unsealed
- (4) The back of an envelope, sealed

Subjects then had to say which of the letters would need to be examined. The results were clear-cut. Almost all the subjects succeeded on the realistic task; most of the same subjects failed on the abstract task. Moreover, there was virtually no transfer between tasks, or awareness of the underlying similarity (as judged from introspective reports).

It is, at the very least, an oversimplification to say that these results merely indicate that concrete problems are easier than abstract problems [Manktelow and Evans, 1979]. Recent evidence suggests that subjects failing the abstract task often show what has been called "matching bias" [Evans and Wason, 1976]. In other words, they give the answer "A and 3", matching the terms named in the rule stated. This could be considered as an example of a general phenomenon — responding to "surface features" of a problem rather than its logical structure. This is a common theme in Krutetskii's book [1976] and another striking example is given by Janvier [1978]. Pupils were asked to decide which shape of racing circuit from a number of alternatives corresponded to a speed/time graph (see Figure 4). Weak pupils tended to choose alternative (b).

Manktelow and Evans suggest that the success obtained with the "envelopes" task is a matter of memory. Subjects are familiar with the idea of differential postal charges, and know without having to think about it that a letter with a cheaper stamp must be unsealed to be legitimate and that an unsealed letter with a dearer stamp may be unnecessarily extravagant, but is legitimate. The corresponding inferences for the abstract task — that a digit other than 3 must have a letter other than *A* on the other side, and that a letter other

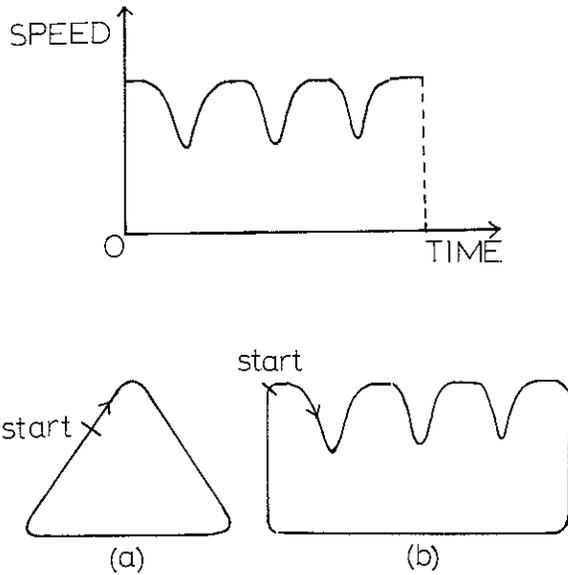


Figure 4
 (After Janvier, 1978) The task was to decide which of a number of possible circuits, which included (a) and (b), corresponds to the speed/time graph for a complete circuit shown

than 4 can have a 3 on the other side without violating the rule — have to be logically derived from the problem as stated. Viewed in this light the results are not surprising.

Another aspect of interest is the change from one representation of a problem to another. A particular case of this which has been studied intensively is the interpretation of verbally stated problems in mathematical terms. Programs have been written which can accept as input problems stated in natural English (subject to some restrictions) and convert them to mathematical form before solving. This exercise shows up many of the hidden complexities of the process, and may throw some light on the difficulties children have in the same situation.

As another example of AI work paralleling direct research into pupils' mathematical performance, we can cite the question of how the relevant information is abstracted, and irrelevant information ignored, when a problem in verbal form is encountered. This has been studied by Robinson and Hayes [1978], who compared the performance of a computer program and human subjects in picking out relevant information from algebra word problems. The program didn't do very well, but this approach represents a good way of pinpointing the difficulties of the task. Krutetskii [1976] tackled the same question with a series of "Problems with Surplus Information" [pp. 109-11, 227-229, 235, 295]. He defined four levels of perception of these problems ranging from:

- (1) Does not single out unneeded data and does not separate out the redundant, even with the experimenter's help, because does not catch on to relations

in problem, perceiving only disconnected data

to:

- (4) Immediately indicates unnecessary, superfluous data, grasping problem's structure, its relations, "on the spot" [p. 235]

Moreover he later remarks [p. 295] that superfluous data are usually forgotten immediately by capable pupils.

The relationship between concrete embodiments of a given problem or structure and the corresponding formal, abstract representations is at the core of mathematics. There is a wealth of evidence from studies of problem-solving that people in general do not readily abstract the mathematical essence of a problem stated in realistic terms, and that invalid inferences (and, indeed, often valid ones) are made on the basis of surface features irrelevant to the logical structure of the problem. Relationships between isomorphic problems differing only in surface features are not recognized. Conversely, the power of mathematics comes from the recognition of underlying relationships, and the isolation of form from content. Dienes [e.g. 1970] has proposed systematic training for children in recognizing isomorphisms through investigation of rules, abstracting the core structure, representing it graphically, and finally producing formal symbolization.

Overview

Those involved in teaching mathematics are entitled to ask how they can benefit from psychological theory and research. This particularly applies to pure research — contrast the psychologist's interest in "natural" cognitive development with the educationalist's interest in what can be achieved through intervention in that development (a major limitation on the relevance of Piaget's work for education). The relevance of much of the work reviewed here is not immediately obvious, perhaps, for teaching — it must be assumed, however, that knowledge of how people think mathematically is bound to have some educational pay-off in the long run.

What is particularly needed, in my view, is more research which occupies the middle ground between straightforward applied research (is method A or method B better for teaching subject C to pupils of type D?) and pure psychological research. Two good examples of what I have in mind are Krutetskii's [1976] massive research program, and the work of the Shell Centres for Mathematics Education in Britain. These combine obvious relevance to normal mathematics teaching with considerable psychological sophistication.

While there is a lot of research on mathematical thinking going on in psychology, it is noticeable that very few attempts have been made to provide a coherent psychological theory of mathematical thinking in general. It also seems to me that there is a lack of communication between those cognitive psychologists who study mathematical thinking, and researchers involved in mathematics education. (As a case in point, take the work on how the relevant information is abstracted from algebra word problems, discussed above. In the paper by Robinson and Hayes [1978], who looked at the problem from an AI viewpoint, there are six references,

all of which are to previous AI studies. No reference is made to relevant work by Krutetskii and others). I remember Alan Bell, of the Shell Centre for Mathematics Education at Nottingham, commenting on a draft of my review of "Mathematical Thinking" [Greer, 1980b]: "you seem to read a different set of journals from me". Perhaps this situation is changing. In the UK, the recently-formed British Society for the Psychology of Learning Mathematics provides the sort of link that is needed. Given communication channels of this sort, I think cognitive psychology can be tapped to the benefit of mathematics education. Conversely, psychologists could benefit a great deal from a closer acquaintance with the research of those involved directly with mathematics.

References

Recommended reading in Cognitive Psychology

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