fact which is becoming increasingly clear; that the mathematical competence which is developed through exposure to algebraic symbols is not as closely related to instruction in the schools as mathematics educators might like to imagine.

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Four Operations? or Ten?

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This contribution aims to relate two recurring themes in the literature. One concerns the way so many children start school quite experienced in mathematical thinking (including problem solving), and with quite a valuable outfit of mathematical ideas. This theme laments the way school fails to build on these foundations, and substitutes instead a "subject" which many children find meaningless, so that they can no longer use their thinking powers and instead take refuge in rote-learning.

The other concerns the extraordinary difficulties children seem to encounter in coming to terms with the ideas of ratio and proportion. The second of these themes is the subject of high-powered research, but the first is an area ripe for development. Current research [e.g. Carpenter et al, 1982] seems to be on quite a different track.

The operations of everyday life with everyday things

Beth Blackall gave a very stimulating talk at the 1975 annual conference of the Mathematical Association of Victoria. She described how in her junior primary classroom she dealt with nine operations: three forms of subtraction, two of division, one each of addition and multiplication, and also two others, namely doubling and halving. This struck a chord with me as I had recently contributed an article about seven of them to *Mathematics in School* 4, 5.

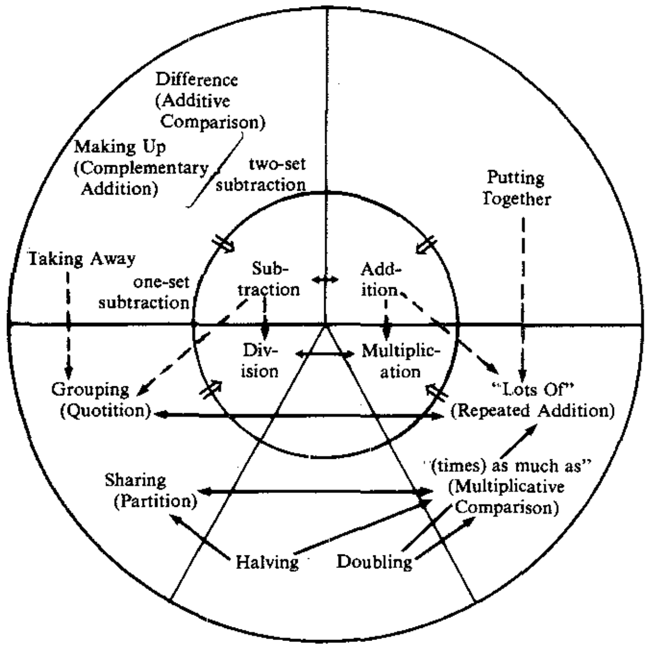
Clearly doubling and halving are valuable, even essential, additions to my seven. They make use of the very special number *two* and provide a way forward for mastering numbers of all kinds, both large and small. As an example now well known to most adults the "rule of 72" gives you a quick guide to how long it takes for prices to *double* (or for the value of money to *halve*) due to inflation. At the beginners' level, facility with doubling and halving, together with multiplication by ten, reduces the number of "table facts" that need attention to a mere ten: and if the square numbers are known as friends this number is further reduced to five, namely those with products 21, 27, 42, 54 and 63.

The surprising thing is the way so many people, including teachers, are quite unaware that subtraction comes in several forms in real life. Over the years I have introduced teachers and future teachers to some of these forms in various ways. The most successful way seems to be to treat the various manifestations as *different* parts of the furniture of the world and them, only then, after plenty of discussion and practical work, to point out their connection with subtraction (which everyone knows is exactly the same as "take away"!)

Similarly with the two forms of division. For some reason adults find it hard to distinguish between them, or to recognise them in real life, even before their connection with division is pointed out. And as in the case of subtraction it does seem necessary to point it out. Accommodation to this new way of thinking about the operations does not come easily. When I, or others, have used a straightforward analytical approach it seems to lack the impact needed to change well-established understandings about the nature of these operations.

Another everyday operation with everyday things

Since those days it has become apparent that the number of everyday operations is ten, not nine. The extra one (see figure) is multiplicative comparison. On the figure this has been shown with the form of words *(times) as much as.* The "times" is in brackets because it is only used when the number concerned is greater than one. Thus "petrol costs seven *times as much as* it did twelve years ago" but "twelve years ago petrol cost one seventh *as much as* it does now". Even then "times" is omitted if the number is two (twice *as much as* it was) or if the suffix "-fold" is used.



Outer ring represents the world of things
Inner circle represents the world of numbers
↔ connects operations which "undo" one another
⇒ indicates "becomes by abstraction"
→ indicates "becomes by generalisation"
--→ shows result of repetition

Perhaps it is not surprising that so many people, including teachers, are unaware of this form of multiplication. Before the mathematical idea of a *field* was available to illuminate the connection between + and − on the one hand, × and ÷ on the other, language worked against discriminating between additive and multiplicative forms of comparison. Words like more, less, increase, decrease, larger, smaller, go well with additive comparison. But then they are apt to appear again in multiplicative comparison, with confusing effect. Thus "The cost of living has increased four-fold"; "the cost of calculators is less than it was..... by a factor of thirty"; and so on.

We also have the confusing struggles of the journalists and sub-editors. "Japanese cars have five times fewer faults in manufacture compared to....."; "The cost of bread has gone up seven times since..." (Does this mean on seven occasions?); "Accidents are up by 300%" (*by a factor of three* intended, usually); or even "Accidents have been cut by 400%" (!)

Those with a good understanding of this multiplicative comparison often use the word "factor" as in the example above, but this does not seem suitable at the five-year old level. Nor yet are such long words as "multiplicative comparison" suited in the least to classroom use at any primary level. Their place is in discussions between teachers.

Yet the idea itself, of multiplicative comparison, is readily available to five-year olds, because it is part of the everyday world of everyday things. For example, if three children share a pile of (interesting) things between them each person gets a third *as much as* is available, and if they all put their shares together again then altogether there'll be three *times as much as* one child's share

Four steps take you four *times as far as* one step. The essence of measurement is multiplicative comparison with a basic unit — *times as far as, times as heavy as, times as long as, times the area of, times the volume of.* Since the idea is all around us in everyday life a classroom which relates to everyday life cannot be short of opportunities to help the growth of the idea. Appropriate language for the classroom uses forms of words like "(times) as much (far, heavy, long) as" etc.

A major step

Evidently a major and long overdue step in mathematics education is to reintroduce the idea of multiplicative comparison into the thinking of those who teach mathematics to children. (*How* it can be done is another matter!) But the benefits will be extensive.

In the early stages the ten operations of the everyday world (see figure) will enable children to use their thinking abilities to cope and develop. A little later on, and not much later either in these days of the electronic calculator, this idea will provide a firm foundation of meaning for the multiplication of decimals. Williams and Shuard [1982 p. 262] use it for just this purpose. But it needs to be there all the time, growing alongside the *other* idea of multiplication-as-repeated-addition.

How can a person who *identifies* multiplication with repeated addition suddenly change ideas? This identification is explicitly stated in some textbooks, and it is as crippling to development as the equally common identification of subtraction with take-away. Indeed it is more crippling, since many people have a good deal of difficulty forming mental images for the multiplication of decimals and the multiplication of fractions.

In passing, it is interesting to notice from the figure how multiplicative comparison is related to both doubling and halving (it is a generalisation of both) and how it "undoes" sharing, and is "undone" by it (in those cases where sharing is possible.)

Finally how does all this relate to the second theme mentioned in the introduction? Such a change in the thinking of primary teachers will lay the foundations for an easy intuitive grasp of the ideas which now seem to be found so hard in "Ratio and Proportion" [Hart 1981 Chap 7]. The brilliant CSMS research programme investigated children's understanding *in the light of the teaching they had received.* How much of the findings reflect children's thinking, and how much the thinking of their teachers?

Once this change has been wrought in teachers' thinking, there will be a drastic change in the "child methods" used for ratio problems. But how to make the change?