Popularizing Geometrical Concepts: the Case of the Kaleidoscope*

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> "La date de l'année 1823 était pourtant indiquée par les deux objets à la mode alors dans la classe bourgeoise qui étaient sur une table, savoir un kaléidoscope et une lampe de fer-blanc moiré."
>
> Victor Hugo, *Les Misérables* (II, III, I)

Presenting a given mathematical topic to a wide public is usually not an easy task. But such attempts can be facilitated when it is possible to capture the attention of the target audience through striking concrete phenomena produced by some instruments. The aim of this paper is to examine how the kaleidoscope can be considered as a particularly successful example of such an instrument.

Among devices based on a substantial mathematical phenomenon having enjoyed a wide dissemination through popular enthusiasm, the kaleidoscope stands as a rather unique case. Ever since its invention more than 170 years ago, it has been an inexhaustible source of fascination for people of all ages filled with wonder at the infinite richness of the pictures created by the interplay of mirrors. The French writer André Gide has aptly described the intense delight he took in the kaleidoscope as a six-year-old child:

> "Une autre jeu dont je raffolais, c'est cet instrument de merveilles qu'on appelle kaléidoscope: une sorte de lorgnette qui, dans l'extrémité opposée à celle de l'oeil, propose au regard une toujours changeante rosace, formée de mobiles verres de couleur emprisonnés entre deux vitres translucides. L'intérieur de la lorgnette est tapissé de miroirs où se multiplie symétriquement la fantasmagorie des verres, que déplace entre les deux vitres le moindre mouvement de l'appareil. Le changement d'aspect des rosaces me plongeait dans un ravissement indicible. (...) Bref, je passais des heures et des jours à ce jeu."
>
> André Gide, *Si le grain ne meurt* (I, I)

The curiosity and interest aroused by such an instrument can serve as a motivation for the need of a clear understanding of the underlying principles. It is thus instructive to look more closely at the kaleidoscope from the point of view of the popularization of mathematics. This shows how some elementary concepts of geometry can be brought into play to create an attractive context suitable for significant mathematical observations. Although the kaleidoscope was simply conceived from basic properties of mirrors, new technologies, and in particular computer graphics, now allow for a more thorough exploration of the kaleidoscopic phenomenon.

Historical notes on the kaleidoscope

From the misfortune of Narcissus, enamoured of his own image seen in the water of a fountain, to the adventures of Alice exploring the Looking-Glass House, the mirror has always been a source of mystery and fascination. (A study of various myths surrounding mirrors and interpretation of their effects can be found in the essay [Bal].)

While reflection in a single plane surface generates an identical image, different and more complex patterns are produced by the use of multiple mirrors. The effects of combining mirrors so as to produce a multiplication of images have been described by early writers on optics, such as Giambattista Della Porta (1535-1615) or Athanasius Kircher (1601-1680): the former presented in his *Magia naturalis* (1558) experiments done with two rectangular mirrors joined by one of their sides so they could be opened or shut like a book, while the latter stressed in his *Ars magna lucis et umbrae* (1645) the connection between the angle of the mirrors and the number of images formed. Applications of such combinations of mirrors have been described by the botanist Richard Bradley (1688-1732) in his book *New Improvements in Planting and Gardening, Both Philosophical and Practical* (1717), where sets of mirrors are used for preparing symmetrical designs for formal gardens.

In 1817, the Scottish physicist Sir David Brewster (1781-1868) patented a "philosophical toy" which he called the kaleidoscope (a name Brewster derived from the Greek words «kalos», *beautiful*, «eidos», *aspect*, and «skopein», *to see*). This optical instrument, which Brewster "invented for creating and exhibiting beautiful forms" [Br2, p. 443], was a direct result of his studies of the theory of polarization of light by multiple reflections. In the Supplement to the 6th edition of *The Encyclopaedia Britannica* (1824), the kaleidoscope is described as

> "an optical instrument, invented by Sir David Brewster, which, by a particular arrangement of mirrors, or reflecting surfaces, presents to the eye, placed in a certain position, symmetrical combinations of images, remarkable for their beauty and the infinite variations of which they are susceptible" [EB, p. 163].

(This article on the kaleidoscope, written by P.M. Roget, of *Thesaurus* fame, runs for more than nine pages, including pictures, and appears also in the 7th edition (1842); the kaleidoscope occupies less than two pages in the 9th edition (1880) of

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Britannica and a mere half a column in the 15th edition (1985)) Figures 1 and 2, taken from [EB], show respectively the arrangement of mirrors in a basic kaleidoscope and a typical kaleidoscopic rosace (or "rose-pattern") obtained by the coalescence of the images of a basic motif reflected in the two mirrors AC and BC.

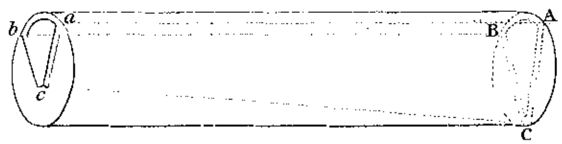

Figure 1

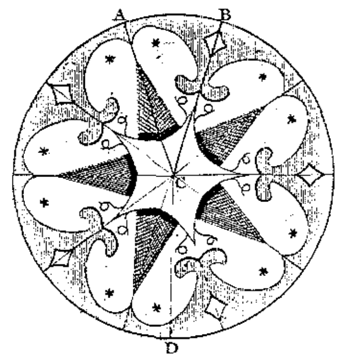

Figure 2

It has been reported in many places that the kaleidoscope was phenomenally popular as soon as it started being manufactured. It is said that no fewer than two hundred thousand instruments were sold in London and Paris in the space of three months [Br1, p. 7]. The article in *Britannica* comments that

> "the sensation it excited in London throughout all ranks of people was astonishing. Kaleidoscopes were manufactured in immense numbers, and were sold as rapidly as they could be made. The instrument was in every body's hands, and people were every where seen, even at the corners of streets, looking through the kaleidoscope. It afforded delight to the poor as well as the rich; to the old as well as the young. Large cargoes of them were sent abroad, particularly to the East Indies. They very soon became known throughout Europe, and have been met with by travellers even in the most obscure and retired villages in Switzerland." [EB, p. 163]

And Brewster himself states in a letter to his wife from London in May 1818:

> "You can form no conception of the effect which the instrument excited in London; all that you have heard falls infinitely short of the reality. No book and no instrument in the memory of man ever produced such a singular effect." (Quoted in [Wad, p. 196])

It should be mentioned, for historical record, that Brewster was rather bitter about what happened in relation to the kaleidoscope. Not only was there a controversy about the originality of his invention (a full chapter of [Br1] is devoted to the defence of his priority — a similar controversy took place over another of his inventions, the stereoscope, as is reported in [Wad]), but also his patent seems to have been mishandled, resulting in an important loss of revenue at a time when his financial position was not secure (see [Wad]). Moreover Brewster complains that out of the immense number of instruments produced, there are very few "constructed upon scientific principles, and capable of giving any thing like a correct idea of the power of the kaleidoscope" [Br1, p. 7].

The immense popularity that the kaleidoscope enjoyed at that time is not without reminder of a mathematical "toy" of a more recent vintage, the Rubik's cube. It should be noted however that both instruments, once the objects of a craze, came to a more proper popular status. After the fad of the early 1980s, the Rubik's cube lost some of its fascination, and many cubes (and analog offsprings) are now probably sleeping in the back of drawers. Something similar seems to have happened to the kaleidoscope. A chronicler of the last century even reported the following, probably with a certain exaggeration:

> "Vers 1820, le kaléidoscope fit fureur à Paris; on en trouvait alors dans tous les salons, sur les tables où les *Albums* et les romans du jour les ont remplacés depuis; et il y a longtemps qu'il est tombé dans l'oubli. C'est tout au plus si quelque marchand forain se hasarde à en offrir à ces chalands comme prix à gagner à l'un de ces jeux *où à tout coup l'on gagne*." [En]

Despite the fact that the kaleidoscope is no more a source of wild excitement as it was at the time it was invented, it is nevertheless an extremely attractive instrument which is still commercially offered, after more than 170 years, in many versions at various prices. And it infallibly fascinates its users, as can be easily judged by letting people actually handle it. The kaleidoscope, as well indeed as Rubik's cube, offers a rich occasion to involve a wide audience in some nice mathematics. Even if both the kaleidoscope and the cube can be looked at from a group-theoretic point of view (see for example [Co2] and [Ban] respectively), they are essentially geometrical devices, the former being concerned mainly with reflectional symmetry while the inventor of the latter, Ernö Rubik, was aiming at the development of spatial intuition through his articulated cube.

Understanding the kaleidoscope

> "J'étais autant intrigué qu'ébloui, et bientôt voulus forcer l'appareil à me livrer son secret."
>
> André Gide, *Si le grain ne meurt* (I, I)

What advantage is to be gained from an instrument, like the kaleidoscope, built from given mathematical principles? On the one hand, it can make easily accessible phenomena otherwise difficult even to conceive; Charles Wheatstone (1802-1875), a contemporary and at times opponent of Brewster, spoke in the following terms of one of his inventions, the kaleidophone (having no similarity but the name with Brewster's invention and serving to illustrate acoustical phenomena): it "renders obvious to the common observer what has hitherto been confined to the calculations of the mathematician" (quoted in [Wad, p. 205]). But conversely it can also attract the observer and induce him or her to investigate the causes of the phenomena with sustained interest; it can provoke involvement. In his treatise on the stereoscope [Br3], Brewster opens a chapter on the "Applications of the stereoscope to purposes of amusement" with the following comments, surely valid with respect to the kaleidoscope:

> "Every experiment in science, and every instrument depending on scientific principles, when employed for the purpose of amusement, must necessarily be instructive 'Philosophy in sport' never fails to become 'Science in earnest' The toy which amuses the child will instruct the sage, and many an eminent discoverer and inventor can trace the pursuits which immortalize them to some experiment or instrument which amused them at school. The soap bubble, the kite, the balloon, the water wheel, the sun-dial, the burning-glass, the magnet, &c., have all been valuable incentives to the study of the sciences." [Br3, p 204]

In the case of the kaleidoscope, the astounding effect produced by the interplay of mirrors leads to a need for a better understanding of the way mirror-images are being produced. One is thus introduced to the mathematical model of geometrical reflection, the mathematical equivalent of the mirror effect This need for a careful study was described as follows in *Britannica*:

> "This circular arrangement of the images, however legitimately it may have been deduced from the simplest law of optics, appears to be so extraordinary an illusion of the sense, as to call for somewhat further examination before we can feel perfectly assured that it is a necessary consequence of that law Perhaps the most satisfactory method of prosecuting their examination is to investigate separately the mode in which each of the images results from the successive reflections by the two mirrors" [EB, p 164]

But mirrors being omnipresent in our daily lives, we all feel quite comfortable about the way reflection acts. Or do we really? Even with a single mirror, some confusing situations can occur. In his charming book *The Ambidextrous Universe* [Ga], M Gardner asks for example the following naïve question: "Why does a mirror reverse only the left and right sides of things, and not up and down?" [Ga, p. 6] And if this seems to be a rather simple-minded or even silly question, Gardner then suggests the following experiment with two mirrors: we are not surprised by the fact that rotation of a single mirror by a quarter turn clockwise will not turn the image of our face upside down; but if two mirrors are joined at a right angle by one of their sides (thus becoming a 90°-kaleidoscope), they produce an image which is not reversed, and a quarter turn rotation will then turn this image upside down (as illustrated in Figure 3, showing two pictures† taken from [Ga]).

Giving a clear explanation of such phenomena is not self-evident and involves some good mathematical thinking As Gardner says, adults "take mirror reflections for granted without attempting to get clear in their mind exactly what a mirror does" [Ga, p 7]. Understanding how two mirrors will act upon the images produced by each of them calls for a sound knowledge of the mathematical notion of reflection. Basic manipulations of mirrors are thus excellent starting points in the exploration of certain fundamental concepts of geometry, as is so well illustrated by the "Mirror Cards" and other works of Marion Walter about mirror geometry (see for instance [Wal]). Such exploration then leads to the identification of pertinent mathematical principles and the building of a good mathematical model. (For the interested reader, a discussion

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Figure 3

of the left-right but not up-down reversal is presented in [Ga, p. 23].)

Once the interplay of images produced by each of the mirrors is understood, it is a direct matter to derive general rules about the generation of images in a kaleidoscope. Although any angle between the two mirrors can result in interesting reflected patterns, consideration of various possibilities leads to the identification of the principles that were so dear to Brewster for the production of "perfectly beautiful and symmetrical forms": he held that, in a "true" kaleidoscope,

> "the reflectors should be placed at an angle, which was an *even* or an *odd* aliquot part of a circle, when the object was regular, and similarly situated with respect to both the mirrors; or the *even* aliquot part of a circle when the object was irregular, and had any position whatever" [Br1, pp. 4-5]

And these principles, clearly emerging from actual manipulations, can be fully justified on a theoretical level: properties of geometrical reflection show the inconsistency in orientation occuring with an odd aliquot part.

Besides the typical kaleidoscope, in which the objects being looked at are fragments of coloured glass placed in a case that can be revolved at one end of the mirrors, Brewster also constructed "telescopic" kaleidoscopes [EB, p. 170], having a convex lens in place of the case so that any object can occupy the field of vision, as well as "polyangular" kaleidoscopes [EB, p 171], in which the angle between the mirrors can be altered at pleasure. Other variations include "polycentral kaleidoscopes" [EB, p. 167], built from a greater number of mirrors and thus producing groups of images around several centres spreading in all directions. Configurations suitable for the production of symmetrical combination of images are either the square (or rectangular) polycentral kaleidoscope or the three cases of triangular polycentral kaleidoscopes built respectively on triangles of 60°—60°—60°, of 90°—45°—45° and of 90°—60°—30° (see [Co2]) Figure 4, taken again from [EB], illustrates typical patterns generated by such arrangements of mirrors.

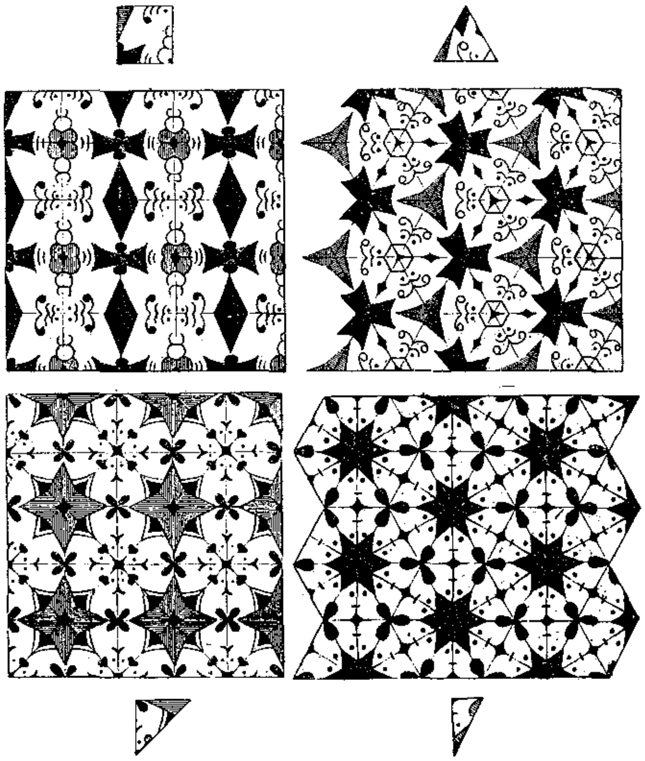

Figure 4

Early expositions on the kaleidoscope, like [Br1], [EB] or [EE], took great care to give an abundance of technical details about the construction of instruments producing the best possible effects Such considerations concern mostly the practical use of the kaleidoscope and can be of great importance in applications, for example to the ornamental arts, a case strongly supported by Brewster But with regard to an appreciation of the theoretical principles underlying the kaleidoscope, these concerns can become a distraction from the mathematical content. This might explain why Brewster's contribution seems to have found little echo in the mathematical literature, possibly being considered as of strictly physical interest A notable exception in this respect is Coxeter's remarkable study [Co1], where proper credit is given to Brewster for his work on the kaleidoscope and "mirror geometry"

Observing the kaleidoscope phenomenon

Attempts at popularizing mathematical concepts using a given instrument greatly depend on the attractiveness and the availability of the instrument The kaleidoscope, in this respect, can serve in many different contexts.

First of all, as is testified to by the early works of Della Porta, Kircher or Bradley, the kaleidoscope is easily made from standard mirrors (although possibly not quite correctly built according to Brewster's stipulations!). In fact, we frequently encounter kaleidoscopic phenomena in our daily lives, be it with hinged mirrors in bathrooms or with the limiting case of parallel mirrors at the tailor or the hairdresser. Kaleidoscopes are thus there around us; it belongs to the mathematician to provide a context where a fruitful and informative exploration can be made. Devices suitable for studying the properties of reflection in multiple mirrors can easily and inexpensively be built. Appropriate manipulations can then be made, even with young children in the classroom context, and the basic rules gradually identified (see for instance [Ho] for such an exploration of the "kaleidoscope geometry").

It should be mentioned that at the other extreme, the kaleidoscope can become a very sophisticated and expensive device, a collector's delight found only in specialized stores. Illustrations of such instruments and their visual effects are given in the book *Through the Kaleidoscope* by C. Baker [Bak]. (This book also presents a variety of information on the kaleidoscope both from an arts and crafts and from a popular point of view. Information can be obtained from the author about the so-called Brewster Society, a "kaleido-club for designers, collectors and lovers of kaleidoscopes" (sic!).)

Although all that is needed for basic exploration is a simple kaleidoscope made of two small mirrors, it is interesting to note that various modifications using modern technology are possible, thus revamping the kaleidoscope into an up-dated instrument for a high-tech society. For example, the paper [AS1] introduces some "present-day" kaleidoscopes in which polarizing filters and light-emitting diodes are used to produce spectacular effects. But such modernizations are to a certain extent superfluous: encountering the effects of plain mirrors is still the crux of the matter and no fancy sophistications, however interesting they might be, can prompt one to develop a sound understanding of geometrical reflection better than basic experiments with mirrors The paper [AS2] illustrates this point very clearly, using as a motivation the bewilderment provoked by mirror mazes. The feelings aroused by actual involvement in these explorations can also be nicely conveyed through animated films: the geometrical properties of mirror interplay suit very well to the potentialities of such a medium, as is illustrated by the films *Dihedral Kaleidoscopes* or *Symmetries of the Cube* [UM], produced under the guidance of H M S Coxeter. Animated films like these can be an excellent way of having a large number of people experience a succinct and systematized exploration of a given mathematical phenomenon with a minimum of material support

Among such "modern" approaches to the kaleidoscopic phenomenon, there is one which stands out as particularly interesting in that it provides simultaneously new stimulation and deeper understanding Graphic capabilities of computers, and even of standard microcomputers, allow for effective simulation on the screen of mirror reflections and of iterations of these reflections. A complex kaleidoscopic rosace can then be generated either all at once, as seen in real mirrors, or in a step by step manner, leaving control of the generation of each of the images to the user The next section presents an implementation of such an approach (see [Gr1], [Gr2]) based on the LOGO language.

The kaleidoscope and the computer

a. Didactical background

As explained above, the different types of kaleidoscopes are powerful means to start investigations of mathematical problems, geometrical ones as well as analytical ones. The intention is always to arrive at some small or larger formal theory on iterated axial or rotational symmetries, on regular polygons (from kaleidoscopes with an angle dividing 360 °), on regular stars (angle not dividing 360 °), on parquetting or tiling a plan (by applying special "polycentral" kaleidoscopes), etc.

In mathematics education, before reaching this final formal level of description and problem solving, different media are used for investigation. Learning is first supported in enactive phases of action, then the learner goes on to iconic ones and finally he or she ends up with symbolic ones. Very often the iconic means are rather poor or limited and so the learning step to the formal level is rather big. In this section we want to point out that computer graphics is an excellent tool that can be used to fill this gap.

The mathematical problems connected with kaleidoscopes can be worked on at the following five levels:

1. looking through the real kaleidoscope;
2. reducing the kaleidoscope to a model with two or more real mirrors placed on a sheet of paper carrying some figure;
3. abstracting the mirrors and their effects to reflections in straight lines (axial reflections), constructed with ruler and compasses;
4. transferring these constructions to a computer graphics display;
5. using formal methods to describe the phenomena, like calculi of different complexities up to analytical geometry or linear algebra.

There should always be a very careful examination of the advantages for learning before the computer is used in some field of mathematical education. There is no use in transferring manual or mental activities (like constructions with ruler and compasses) to the computer unless this brings about more efficiency in learning. Another good reason may exist if the computer allows activities which the students cannot achieve with their hands or brains. Then the computer is like an additional tool, increasing the traditional abilities of the students. We think that our kaleidoscope software is a good example for such a kind of tool. It offers additional help in exploring mathematical problems. It allows

- a great variety of investigations with little effort;
- easy experimentation
- a huge spectrum of complex geometrical constructions to turn up when studying many iterations of reflections or complicated figures to reflect;
- doing manually impossible constructions like the pointwise (pseudo-) simultaneous construction of two or more figures, e.g. one original object and its two images under two axial reflections;
- studying "fictional kaleidoscopes" which have no real material equivalent, namely mathematical models of kaleidoscopes using central reflections instead of axial reflections;
- introducing and exercising simple methods of CAD, a technique which has replaced manual technical drawing to some extent.

Whereas in the first years of computer application in mathematics education the numerical power of computers was dominant, we are now in a position to use its graphical power wherever and whenever helpful from a mathematical or methodological point of view.

The remainder of this paper is devoted to the presentation of some software developed for simulating kaleidoscopic phenomena on the computer and examples of patterns that can thus be produced

b. Two-mirror kaleidoscopes

The development of our software started in a course at Freie Universität Berlin for teacher students and in-service teachers on "computers in mathematics education", using LOGO as a programming language. We selected LOGO as an adequate tool for the programming of graphics without caring too much about a general philosophy of LOGO. This language allowed us to construct a very transparent package of educational software, open to comfortable adaptation by each teacher using it in his or her class. Our software has been developed for an IBM compatible PC with graphics card and display, not necessarily with colour (which is helpful, however)

Besides the correct control of the different kaleidoscope constructions, we took great efforts to get safe interaction between the user and the computer. The dialogue is controlled by menus which allow very flexible investigations. When looking at the pictures the user can easily switch between full graphics screen and mixed screen. There is a choice of colours for the different objects in the kaleidoscope, which can also be easily changed via the main menu. Moreover, this menu allows one to switch between manual control of the growth of the images in the kaleidoscope and automatic generation

The main menu for simulation of two-mirror kaleidoscopes, which is shown in Figure 5, offers a choice of four different types of kaleidoscope. *Mode 1* leads into a dialogue about forming a kaleidoscope with an arbitrary angle. The user gives the positions of the axes and then the position of the object selected from another menu (square, triangle, point, cross, line, etc.; see Figure 6) to be reflected in the kaleidoscope

```
SIMULATION EINES KAIEIDOSKOPS
- - - - - - - - - - - - - - -

1 .     ACHSEN FREI WAEHLBAR,
 .  .   SPLITTER FREI WAEHLBAR

2  .  . ACHSEN 45 60 72 90 120 GRAD,
    .   SPLITTERAUSWAHL EINGESCHRAENKI

3   ..  HODGSON - BIIDER

4  .  . DEMO VON 3 NACHEINANDER
    ..  AUSGEFUEHRTEN SPIEGELUNGEN




T     . TASTENSTEUERUNG NEU FESTLEGEN
F ...   FARBENTSCHEIDUNG NEU TREFFEN
#   .   ENDE DER ARBEIT

DEINE WAHI :
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Figure 5

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WAS FÜR EIN SPIITTER?

1 - PUNKT
2 - KREUZ
3 - GLEICHSEIT  DREIECK
4 - RHOMBUS
5 - QUADRAT
6 - HAKEN
7 - BRÜCKE
8 - STRECKE ( ACHSEN 60 GRAD )
9 - STRECKE ( ACHSEN 45 GRAD )
10 - STRECKE ( ACHSEN 90 GRAD )
11 - SCHRÄGSTRECKE
12 - KREUZUNG
13 - WINKEL ( ACHSEN 90 GRAD )
14 - WINKEI ( ACHSEN 60 GRAD )
15 - ECKE
16 - GLEICHSCH  DREIECK ( 72 )
17 - GLEICHSCH  DREIECK ( 120 )
18 - STREIFEN
19 - BAND
20 - WOLKEN

DEINE WAHI :
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Figure 6

After this setting the computer displays the two axes on the screen and shows the original object. It then constructs and displays one reflection after the other until the pattern is complete. This can be done with a pause after each image, allowing localized study, or in an automatic mode. It is also possible to fill the kaleidoscope with several objects of different colours. When controlling the construction manually, the user can always interrupt and start a new construction

Mode 2 allows one to select a kaleidoscope with angle 45°, 60°, 72°, 90° or 120° and then proceeds as above. Figure 7 shows some steps in the development of a 60° pattern in *Mode 2* and Figure 8 shows the same process for a 70° pattern in *Mode 1*. It becomes quite clear that repeated reflections run through a circle several times and, of course, the question arises: How many reflections before the pattern begins to repeat itself? *Mode 3* produces images corresponding to the situations used in the exploratory approach suggested in [Ho] to answer this question. Figure 9 gives two examples from this mode. Finally, the last mode, *Mode 4,* generates an example of the image obtained under 60° reflection with three different objects and three different colours.

We have shown in Figure 10 some more examples of patterns with decorative and geometrical properties that can be generated with our software; such images have proved to provide an excellent context for provoking mathematical discussions.

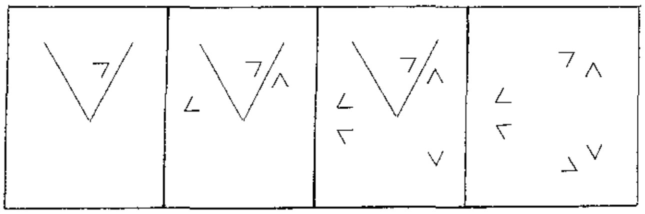

Figure 7

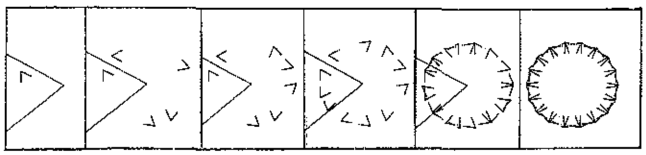

Figure 8

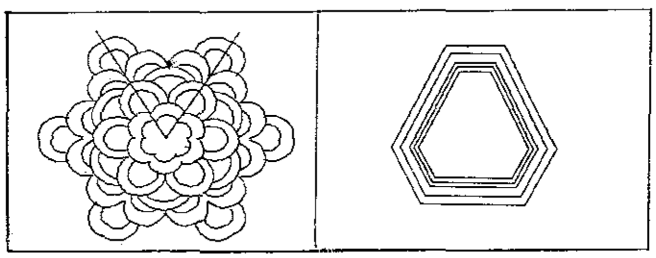

Figure 9

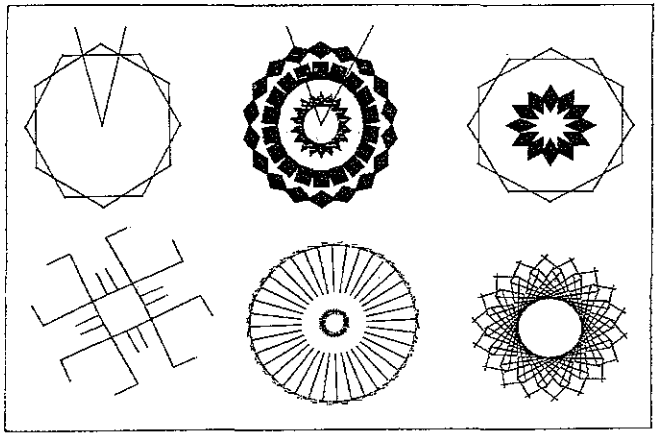

Figure 10

c. Polycentral kaleidoscopes

It is also easy to program "polycentral" kaleidoscopes (see the discussion in Section 2 above). The most simple example uses three mirrors, here three axes or three sides of a triangle. Instead of circular symmetry as with two mirrors and, say, angles of 60° or 45°, we discover another mathematical phenomenon: in the case of three equal sides we end up with a tiling of the plane. Indeed, the continuous reflections of mirror triangles on the original triangle determine an algorithm for this tiling. (It is not a very efficient one, unfortunately, since the same triangle in the plane can be covered again and again.) Figure 11 shows the growth of the tiling after 1, 2, 3, 4, 5, 6, 7 reflections. New questions are obvious: which constellations of mirrors, which triangles, will bring about real tilings? Not every triangle is a good starting-point, as Figure 12 shows.

What happens with four mirrors? You may succeed in tiling with special quadrangles, as shown in Figure 13, but not with all kinds, of course. (The interested reader will find the theoretical background in [Co2].)

d. Fictional kaleidoscopes

So far we have considered the transfer from real kaleidoscopes to mathematical models, combining axial reflections and varying the types of kaleidoscope. Why not vary the mathematical model and forget about reality? For instance *central reflections* (i.e. half-turns) will give us a model of some "fictional" kaleidoscopes having no physical counterparts. One case is to look at a triangle again, determined by three centers of reflection, and see what happens after repeated reflections. We can get a pattern extending in the plane (see Figure 14), leaving some blank spaces. In most cases the pattern will "fail", i.e. the images start overlapping. Similar developments can be studied starting with four centers, as shown in Figures 15 and 16.

A new situation occurs if we start reflecting a triangle not in its corners but in the midpoints of its sides, and go on reflecting the images in these midpoints. A new tiling of the plane develops and — what is really surprising for the beginner — this works with any starting triangle (Figure 17). (And you need only one type of this triangle, not additionally its mirror-symmetric mate as in the case of the real kaleidoscope. As a matter of fact, the mirror-symmetric type spoils the tiling in this

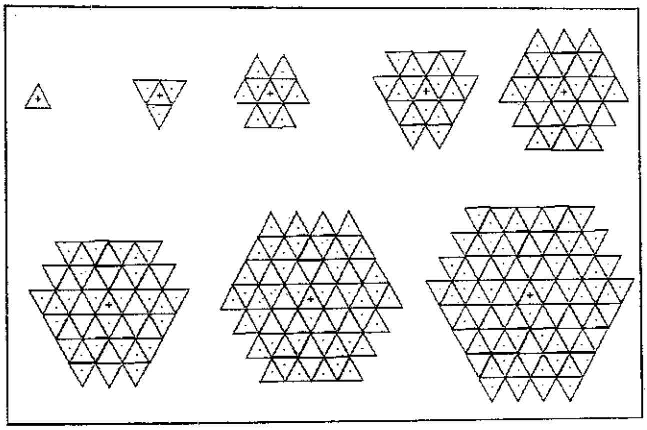

Figure 11

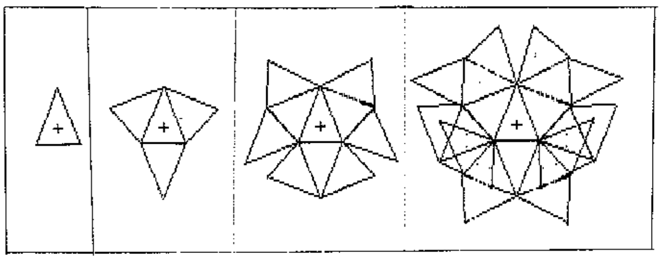

Figure 12

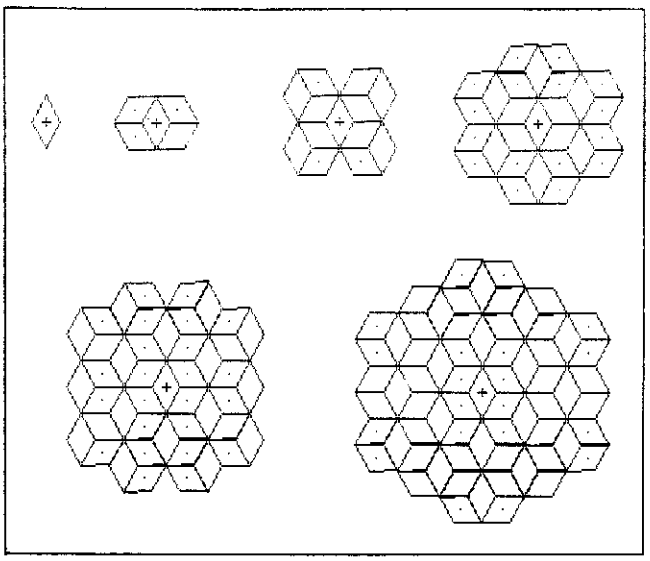

Figure 13

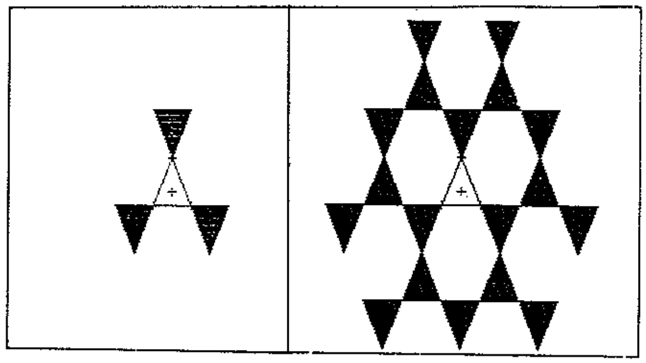

Figure 14

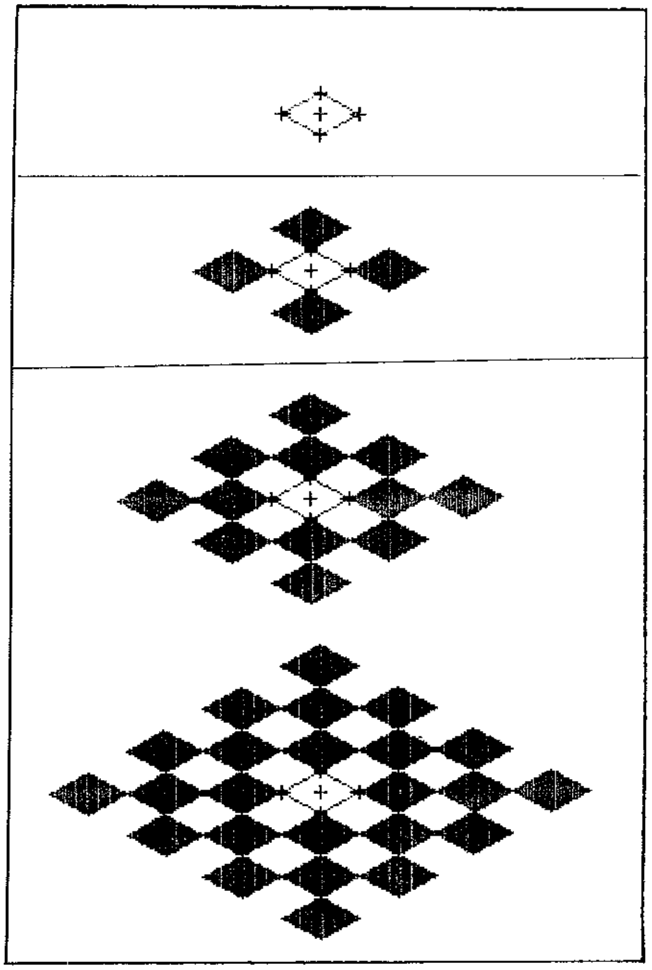

Figure 15

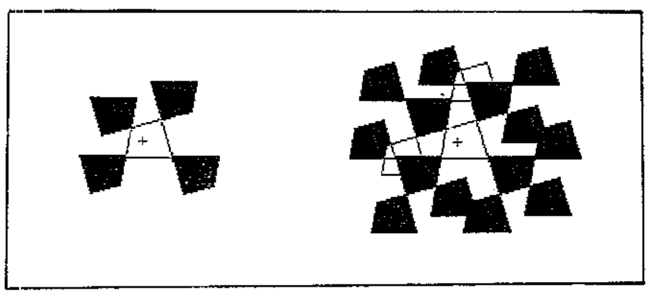

Figure 16

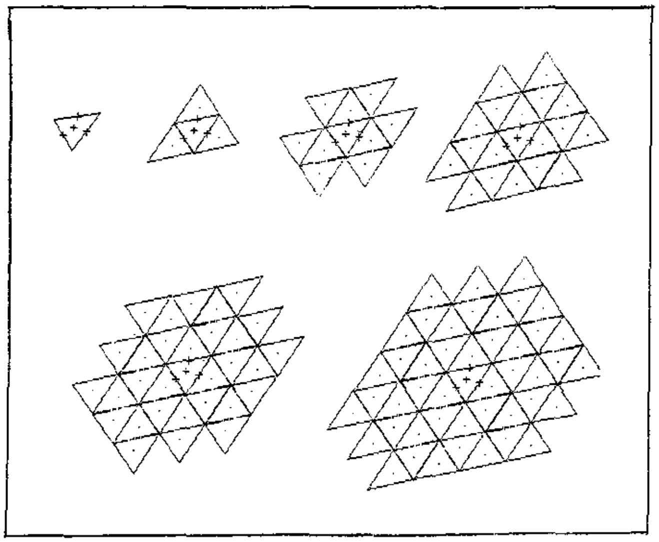

Figure 17

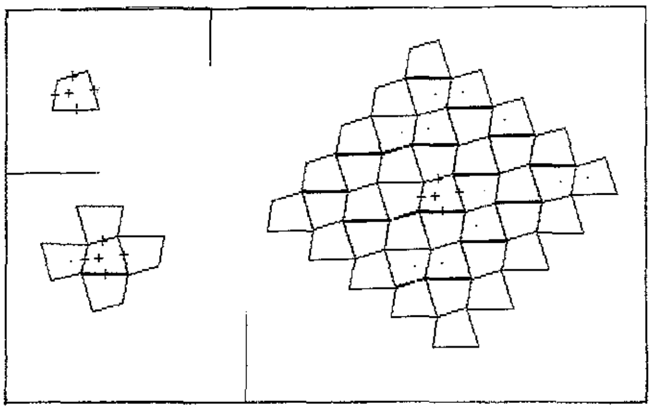

Figure 18

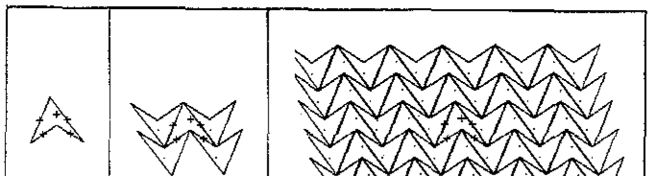

Figure 19

case.) Next you can turn to quadrangles and again you discover immediately that a perfect tiling can be completed with any quadrangle (Figure 18), even a non-convex one (Figure 19) So the fictional kaleidoscope brings you back to a real problem and the search for its solution.†

† After a presentation of these fictional kaleidoscopes in Gießen during a lecture, Günter Pickert remarked: "I have always said that more attention should be given to central reflection in geometry education".

Conclusion

Part of the previous discussion is related to material developed or used in the context of teacher training activities. Such a link between the mathematical preparation of teachers and the popularization of mathematics is to a certain extent natural, since the two fields have many points in common In both cases the main objective is to develop an awareness for the omnipresence and usefulness of mathematics as well as an appreciation of its intrinsic beauty. Mathematics teachers, especially at the elementary level, should themselves be extremely sensitive to the importance of a "mathematical vision" of the world which, although not replacing other essential visions like the historical, political, religious or artistic ones, has nonetheless a crucial rôle to play in our society. In order to make the general public comfortable with various mathematical paradigms, early and various contacts with *bona fide* mathematical ideas are necessary In this respect, encountering and truly mastering a phenomenon like the kaleidoscopic multiplication of images can be an extremely important experience. Firm knowledge about such a situation can illustrate the way mathematics works and prepare citizens to appreciate the significance of the numerous mathematical models so commonly used nowadays The satisfaction gained through a full understanding of the mathematical principles behind a phenomenon could help develop a positive attitude towards mathematics and could even eventually attract some new advocates to the field. Such concerns are essential ingredients of the task of a teacher of mathematics

The kaleidoscope represents a truly interesting case in the popularization of mathematical concepts While the simplicity of its construction contrasts with the richness of patterns it can generate, it is an instrument remarkably suitable for a first contact with a non-trivial mathematical phenomenon. Considering the fascination infallibly aroused by the interaction of mirrors, it is then no surprise that the kaleidoscope has been present for such a long time

Acknowledgment

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