On the Aesthetics of Mathematical Thought*

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1. Introduction

One of the major goals of mathematics teaching is to lead students to appreciate the power and beauty of mathematical thought The aesthetic aspects of mathematics are, to varying degrees, also relevant to professional mathematicians. Aesthetics may well be the driving subconscious force behind mathematical creativity. Poincaré, for example, thought that "the distinguishing feature of the mathematical mind was not logical but aesthetic." [Papert, 1980, p. 190]. Papert and Poincaré [1956] believe that aesthetics play the most central role in the process of mathematical thinking. To Poincaré, decisions on aesthetic grounds necessarily enter into the creative process of the professional mathematician. More precisely, there are stages of mathematical research activity, during which the problem at hand is dealt with in the unconscious, and resurfaces to the conscious domain as a sudden illumination. This illumination underlies the aesthetics of the solution processes. Among the many products of unconscious activity only those will become conscious which:

> ... affect most profoundly our emotional sensibility. .. the feeling of mathematical beauty, of the harmony of numbers and forms, of geometric elegance. This is a true emotional feeling that all real mathematicians know. [Poincaré, 1956, p. 2047]

Hofstadter [1979] describes a situation in which Hardy was presented with the necessity to generalize an equation of the form $u^2 + v^2 = x^2 + y^2$. Hofstadter asks himself why Hardy immediately chose $u^3 + v^3 = x^3 + y^3$ as a generalization, rather than choosing one of the equally natural generalization $r^2 + s^2 = u^2 + v^2 = x^2 + y^2$ or $u^2 + v^2 + w^2 = x^2 + y^2 + z^2$, or even the "grand" generalization $r^3 + s^3 + t^3 = u^3 + v^3 + w^3 = x^3 + y^3 + z^3$. Hofstadter concluded that:

> There is a sense, however, in which Hardy's generalization is "the most mathematician-like". Could this sense of mathematical esthetics ever be programmed? [p. 565]

Papert, on the other hand, investigated the role of aesthetics in the mathematical activity of non-mathematicians. He came to the conclusion that his

> ... subjects in the experiment clearly proceed by a combinatoric, such as Poincaré postulates in his second stage (the unconscious one), until a result is obtained which is satisfactory on grounds that have at least as much claim to be called aesthetic as logical. [Papert, 1980, p. 197]

It follows then, that aesthetics should be an integral component of a student's mathematical education. But from a teacher's point of view:

> If mathematical aesthetics gets any attention in the schools, it is as an epiphenomenon, an icing on the mathematical cake, rather than as the driving force which makes mathematical thinking function. Certainly, the widely practiced theories of the psychology of mathematical development (such as Piaget's) totally ignore the aesthetic, or even the intuitive, and concentrate on structural analysis of the logical facet of mathematical thought. [Papert, 1980, p. 192]

Papert's claim certainly opens more questions than it answers What is mathematical aesthetics? How can it drive mathematical thinking? How is the aesthetic related to the intuitive? And why is structural analysis as seen by Papert contradictory to aesthetics: Isn't metricizing a structure necessarily one of the main components in assessing aesthetic value?

In the first part of this paper an attempt will be made to partially answer some of these questions. Specifically, what does the term "aesthetics" refer to within the realm of mathematical reasoning? Following this, the relationship of aesthetics to problem solving will be discussed, as well as the results of a study which investigated how students assess aesthetic aspects of mathematical reasoning. The results of a second study, in which the proof of the irrationality of 2 was used to assess the working mathematician's perception of aesthetics will also be discussed, and then constrasted with those of the students. The paper will conclude with some recommendations on how to build in students an appreciation for the power and beauty of a mathematical argument.

2. Towards a definition

"Aesthetics" may be defined as "the branch of philosophy that provides a theory of the beautiful ..." [Webster, 1984]. Probing deeper, Birkhoff [1956] looked at the problem of quantifying aesthetics in a general way. In his classic paper on the mathematics of aesthetics he developed the formula $M = O/C$, where O is a measure of order, C a measure of

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complexity, and M a measure for the aesthetic value of the object or argument under consideration. It is of interest to note that to Birkhoff, structure as a component of order has a direct relationship to aesthetic value: More structure—more aesthetic value; but the complexity of the object or argument erodes this aesthetic value.

But on the other hand to Hofstadter:

> ...there exists no set of rules which delineates what it is that makes a piece beautiful, nor could there ever exist such a set of rules. [p. 555]

Hofstadter believes that it is impossible to define the aesthetics of a mathematical argument or structure in an inclusive and exclusive way. What does it mean, for example, that "this proof is elegant" or "that theorem is beautiful"? Such assessments are very personal; nevertheless, there is far-reaching agreement among scholars as to what arguments are beautiful and elegant. We will now explore the aesthetics of mathematics by circumscription. We start by considering the following problem which could come from many a teacher's repertoire of "neat" problems:

> Suppose you decided to write down all whole numbers from 1 to 99999. How many times would you have to write the digit 7?

Although this problem can be solved rather easily, it exhibits a number of features which are relevant to the discussion of the aesthetic aspects of mathematical thought. The problem is easily stated and easily understood. Its solution does not require anything beyond counting and organization of thought; not only is there no need for advanced mathematics, but there is even no need for elementary algebra. There is, however, more than one natural approach to the problem's solution. Moreover, it is the aesthetics which characterize and distinguish the arguments from one another for explaining the same mathematical situation. We will look at two approaches which differ in the way the solution organizes the given information.

> *First solution:*
>
> The digit 7 appears once between 1 and 10, once between 11 and 20, in fact once in every "regular" 10-plet of numbers; here "regular" means that there are no digits 7 in the tens or higher places. Between 61 and seventy, there are two digits 7; between 71 and 80, there are 10; collecting all of these, one concludes that the digit 7 appears 20 times between 1 and 100, and thus 20 times in every "regular" 100-plet. In the 100-plet from 601 and 700 there is an additional 7, i.e. the digit 7 occurs 21 times, and in the following 100-plet there are 99 additional ones, yielding altogether 300 digits 7 between 1 and 1000. Proceeding in a similar way, one finds that the digit 7 appears 4,000 times between 1 and 10,000, and it appears 50,000 times between 1 and 100,000. As it does not appear in the number 100,000, the correct answer is that the digit 7 appears 50,000 times when listing all the integers from 1 through 99,999.

The organizational structure of this argument can be somewhat improved for instance by replacing the 10-plets 1...10, 11...20, etc. by the 10-plets 0...9, 10...19, etc., and similarly for 100-plets, 1,000-plets, etc. But in spite of this and similar improvements, the argument is prone to retain its basic heaviness which is linked to the fact that at every stage the deliberations have to be repeated; this hampers generalizations of the technique used and effectively prevents one from understanding more than superficially why the answer is correct.

> *Second solution:*
>
> Include 0 among the numbers under consideration—this will not change the number of times the digit 7 appears. Suppose all numbers from 0 to 99,999 are written down with five digits each, e.g. 306 is written as 00,306. All possible five digit combinations are now written down, once each. Because in this set of all possible combinations every digit will take every position equally often, every digit must, overall, occur the same number of times. Since there are 100,000 numbers with five digits each, that is 500,000 digits, each of the 10 digits appears 50,000 times. In particular, this is true for the digit 7.

The second solution starts by organizing the numbers in such a way that a general pattern can be observed. It recognizes and uses the fact that the digit 7 is in no way different from the other digits, and thus appeals to the symmetric role played by all nonzero digits in the list of numbers. Adapting the role of the digit 0 to be analogous to that of the other digits, is the only part of the argument which needs a little acrobatics of the mind. Although it is not easy to state why, mathematicians generally agree that the second proof is aesthetically more appealing than the first one.

In the above problem the difference was in the approach to a solution with the level of prerequisite knowledge being essentially the same for either solution path. But for other problems, different approaches to a solution may also differ in the level of prerequisite knowledge. We will analyze this point later, but in the meantime we consciously disregard it. The path on which a mathematical argument leads from the givens to the result determines the aesthetic rating of the solution. Several properties of this path enter into play: Its level of prerequisite knowledge, its clarity, its simplicity, its length, its conciseness, its structure, its power, its cleverness, and whether it contains elements of surprise. The more you have to put into an argument, in terms of prerequisite knowledge, the more elegance the argument loses. A cumbersome line of thought is clearly unattractive; clarity is easier to achieve with a simple argument than with a complicated one. Brevity is considered an attractive quality in the eye of the beholder: Why should one work through sixty pages of argumentation, if the same result can be obtained on six? In this context brevity can not always be measured in number of words or pages; an important aspect of brevity is the number of logical steps and the step-size. It is not our aim to deal with a quantification of this aspect. [See Rimer, 1970, for such a quantification] In the first solution to the above problem, there are a large number of steps to be done, none of them standing out or being more essential than any other one in bringing us closer to the goal. In the second solution however, there is only one important step, namely the conclusion from the symmetry between the digits to the equality of their number. The other steps are limited in number with each contributing to the realization of the central step. This feature provides the solution path with a structure that lets it appear superior to the first solution from an aesthetic point of view. A path that is built around a small number of powerful arguments appeals to our sense of elegance more than a lengthy and formless, albeit carefully sequenced progression of steps. The power of a single argument is another factor to be taken into account [Dienes, 1964]; an agument is powerful if it leads to a far-reaching conclusion on the basis of few elementary assumptions. That is, if the conclusion follows from a clever manipulation of the assumptions rather than their content. The conclusion of such a powerful argument tends to contain an element of surprise for anyone who did not know the argument before. This surprise, in turn, is a further contributor to the aesthetic appreciation of the argument; mathematicians, similar to the spectators of a magician, like the unexpected, at least as long as they consider they have a fair chance at understanding the reasons behind the surprising conclusion. The factors contributing to the aesthetic appeal of a solution or proof are thus connected to each other; they almost follow naturally from each other: clarity → simplicity → brevity → conciseness → structure → power → cleverness → surprise. While being centered on different main ideas, this chain has some links in common with Hardy's generality - depth - unexpectedness - inevitability - and - economy-model which he developed in his essay, *A mathematician's apology*. [Hardy, 1956; see Gardiner, 1983/84, for a recent discussion and illustration of Hardy's ideas.]

In the preceding paragraph, structure is seen as giving a mathematical argument a measure of unity, giving the beholder the possibility of seeing how a line of thought is composed from smaller elements and thus perceiving it as a whole. It is thus not so much the fact that there is structure but rather the simplicity of the structure that contributes to the aesthetic value. This partially resolves the conflict between the positive contribution which structure makes to aesthetics according to Birkhoff and the negative one which it makes according to Papert.

In recent years several scholars have made reference to aesthetic aspects in mathematical reasoning. Essentially, what they are driving at is a metric of elegance which can be placed upon a completed solution. But Halmos' discussion of Appel and Haken's [1977] lengthy computer assisted solution to the four-color map problem encapsules the essential:

> ....I am much less likely now, after their work, to go looking for a counter-example to the four-color conjecture than I was before. To that extent, what has happened convinced me that the four color theorem is true. I have a religious belief that some day soon, maybe six months from now, maybe sixty years from now, somebody will write a proof of the four-color problem that will take up sixty pages in the Pacific Journal of Mathematics. Soon after that, perhaps six months or sixty years later, somebody will write a four page proof, based on the concepts that in the meantime we have developed and structured and understood. The result will belong to the grand glorious architectural structure of mathematics.... [Albers, 1982]

It may be concluded from this citation that to Halmos, as to many other scholars, brevity, and structural clarity are the signal characteristics of the elegance of thought. There are not clear-cut criteria for assessing these characteristics, nor for the other attributes mentioned earlier, but they appear to be measured against an inner standard which reflects a certain type of mathematical mind. [Hadamard, 1945]

Davis and Hersh [1981], in a similar vein, have tried to take some of the mystique out of mathematics by discussing several of the philosophical questions underlying the development of mathematical thought in their text, *The mathematical experience*. They take their reader right to the frontiers of certain research areas, all of the time explaining them in layman terms. They are, along with Halmos, Gardner, Polya, and others, among the great expositors of mathematics. Through their comprehensive knowledge of mathematics and gift for writing, even laymen can get a glimpse of what a mathematician does, how he does it, and why it is important. Without ever making it explicit, Davis and Hersh give the layman insights into the power and beauty of mathematics.

3. Relation to problem solving

The aesthetics of mathematical thought seems to involve the placement of a personal internalized metric upon a solution to a particular problem. One would expect that if students do develop and internalize such a metric, it initially happens through studying specific proofs and mathematical structures. This aesthetic development and internalized metric then naturally transfers to problem solving activities. How then, does one teach the aesthetic aspects involved in problem solving? The digit-counting problem analyzed above shows one place where this could happen, namely in analyzing different solution paths to the same problem. Although we know of not even a single case where this has been done explicitly in the framework of school activities, this comparison may conceivably be carried out according to the criteria cycle mentioned: to analyze the brevity → structure → power → etc. of different solution paths. At one time, an attempt in this spirit was undertaken by the problem editors of the *College Mathematics Journal*. In the beginning issues of this journal "neat" problem solutions were juxtaposed with "ordinary" solutions to the same problems. But this instructive procedure was soon abandoned, presumably for reasons of collegiality.

3.1 Aesthetics of process versus aesthetics of product

In this Section we consciously concentrate on problem solving as opposed to the aesthetics of mathematical structures, axiom systems, etc. For mathematics educators it is at least as important to determine the characteristics of problem solving aesthetics as to attack the larger problem of the aesthetics in mathematics itself. It is interesting to note that at the PME 9 conference there was quite a bit of conversation on precisely this point. It seemed to be the consensus of opinion there that the aesthetics of problem solving dealt with assessing the product (a finished solution) rather than the process (the thought patterns leading up to the solution). In other words, beauty appears in a mathematical argument after it has been polished for presentation, after it has become "ready made mathematics" [Freudenthal, 1973]. This consensus implied that beauty is not inherent in mathematics as it is being developed. In terms of a process-product view of mathematical problem solving, this view holds that while mathematical thinking is concerned with the process, aesthetic considerations should rather be concerned with the product. In reality, the situation is more complex. The majority view expressed at PME 9 is in direct contradiction to Poincaré's and more recently Hofstadter's interpretation of mathematical thinking. To them, both the process and the product have aesthetic value. And Papert has made it clear that Poincaré's and Hofstadter's deliberations are by no means restricted to the professional mathematician, but are equally valid for the average student of mathematics. The problem solving student who has to take a decision, when he sees two possibilities for continuing his computations, is likely to take aesthetic considerations into account. More than that, even if he does not consciously see various ways open before him, aesthetic values subconsciously guide him along his way.

To clarify the difference consider the following problem discussed by Burton [1984], and again at PME 9:

> At a warehouse I was instructed that I would obtain 20% discount on my purchase but would have to pay 15% sales tax. Which would be better for me to have calculated first, discount or tax?

To us, this problem does not elicit aesthetic mathematical thinking. Granted, one must know some mathematics and apply some problem solving strategies in order to correctly solve the problem, but the mathematics is really straightforward, and there is little room for mathematical thinking as viewed from an aesthetic point of view. The participants at PME 9 took issue with this, most claiming that a tremendous amount of mathematical thinking was involved in the construction of the solution. To them, mathematical thinking is synonymous with problem solving. Mathematical thinking and the aesthetics of mathematical thought are two completely different things. To Poincaré, Papert and Hofstadter, however, this dichotomy is quite unacceptable. It is unclear to us whether or not this distinction is more than a moot point.

3.2 Efficacy versus efficiency

Efficacy is usually defined as the "... power or capacity to produce a desired effect." [Webster, 1984] Efficiency, on the other hand, is defined as the quality of being efficient, i.e. producing an effect with a minimum of unnecessary effort. In more pedestrian terms, efficacious means "getting the job done" while efficient means "getting it done in a neat way." With regard to problem solving, the efficient solution is almost always the one with a higher aesthetic value. The clever problem solver produces a more efficient solution.

Most models of problem solving postulate as a last step: that the problem solver should look back on the solution path and try to solve the problem in a different, more efficient way [Polya, 1945]. Interestingly, to this day this is hardly ever done in practice. Even among professionals, the idea is to get the job done, to find a solution, upon which the problem is done with and the solver turns to some other occupation. Unless forced to present a particular solution, say for publication, mathematicians and mathematics students alike tend to not even finish solving a problem, but only advance it to the stage where they feel they "know how to do it." A perfect example for this is programming. Once a computer is programmed to produce the result we are looking for we are usually content, although more often than not we realize that there are more elegant, more efficient algorithms. But let's get closer to mathematics. Consider the following:

Problem: Construct the formula for finding the distance from a point P to a line l.

Solution 1.
Work in the plane defined by the line l and the point P. Let l have the equation $y = mx + b$ and let the point be P(a,c).

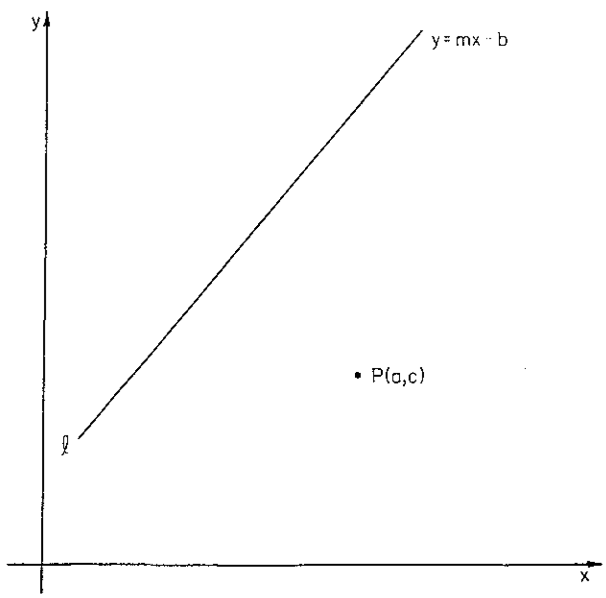


Figure 1
Distance from a point to a line.

1. Find the equation of the line through P which is perpendicular to l. Call this line q.

2. Solve simultaneously the equations for l and q. Call the intersection point Q.
3. Use the distance formula for finding the distance between P and Q. This will give the required result.

Solution 2 [Eisenman, 1969]:

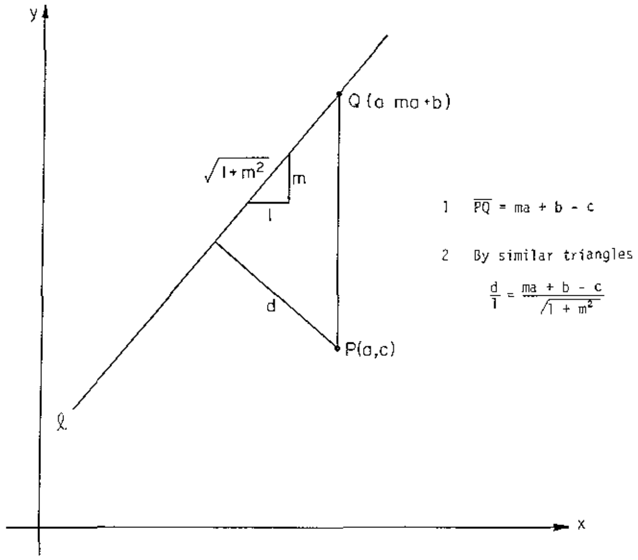


Figure 2
Distance from a point to a line—elegant solution.

Remarks: — There is a considerable amount of algebra involved in Solution 1 But in the end, the desired result is obtained.
— Vector methods also exist for solving this problem. E.g., it can be done with cross products or with Hessians. But the point is that these methods use high-powered mathematics, and are not more elegant than Solution 2.

Solution 2 is clever, slick, and elegant. But Solution 1 is more straightforward. Interestingly, Solution 1 or the more high powered vector methods are the ones first given by most "experts" when faced with the problem. They immediately jot down the main steps of the argument, omitting the algebra, and then consider the problem solved. They do not realize the amount of algebra involved in the problem solution, nor do they reflect upon the solution afterwards. Would they reflect more readily if they did carry out the algebra to its full extent?

That is, to the extent that the aesthetics enters mathematical problem solving through the last stage of looking back on the solution path—not even experts are doing it. This is certainly true when first encountering a problem. Even experts want first to get the answer. Often, they seem content with leaving it at that—and then it is impossible for aesthetics to enter the picture. In other words, having an immediate picture of a solution overrides aesthetic concerns, even for experts. The process of looking for an aesthetic solution must be trained—and constantly be kept in mind [Halmos, 1981].

4. A study of college students

In an effort to try to understand how aesthetics enters mathematical thought processes and how students develop an appreciation for elegance, we built a set of problems we thought would elicit elegant solutions in the sense explained above [Eisenberg and Dreyfus, 1985]. The following may serve as a prime example:

The buried treasure problem [Gamov, 1967]:

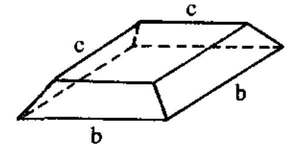
> A young man found among his great-grandfather's papers the location of a hidden treasure: "Sail to 16.7 northern latitude and 175.2 western longitude where you will find a deserted island. On the island is a large meadow with a lonely oak and a lonely pine tree. There is also an old gallows. Start from the gallows and walk to the oak counting your steps At the oak turn left by a right angle and take the same number of steps. Put a spike in the ground. Now, return to the gallows and walk from there to the pine counting your steps. At the pine turn right by a right angle and take the same number of steps. Put another spike in the ground. Dig half way between the spikes, the treasure is there." The young man found the island, the meadow, the oak and the pine, but to his sorrow, the gallows was gone. He started digging at random all over the island. But all his efforts were in vain, the island was too big. So he sailed home empty-handed. Had the young man known some mathematics, he could have found the treasure. How?

Some other problems we used are presented in Table 1.

Table 1
Problems with more than one solution

1. 463 tennis players are enrolled in a single elimination tournament. They are paired at random for each round. If in any round the number of players is odd, one player receives a bye How many matches must be played, in all rounds of the tournament together, to determine the winner?
2. Find the remainder when 10^{10} is divided by 98
3. Find all real numbers x such that $|3x+2|+|2x-5|<12$.
4. If x_1 and x_2 are two solutions of $ax^2+bx+c=0$, show that $x_1 \cdot x_2 = c/a$.
5. Find the area of the triangle if the grid points are separated by one unit of length.

6. The body in the figure has a square base of length b and a square top of length c. Its height is h. Find its volume.



7. The zeros of a function $f(x)$ are $x = -2$ and $x = 7$. For what values of t is $f(3t) = 0$?
8. Find the minimum of the function $f(x) = \sqrt{(2x^2 + 1)}$
9. Find the smallest integer k such that $1260k$ is a perfect cube (i.e. the third power of a natural number).
10. A boat, on its way upstream, passes a log after b miles, continues for c hours, turns around and reaches the log at the boat's starting point. Find the river's velocity.

Some of the common characteristics of these problems are that:

- they have more than one solution, one of which is considered by experts to be considerably slicker than the rest.
- neither the solution nor a way or technique to find it should be immediately apparent to the average student.
- the mathematics needed for their solution is not beyond the average high school student.

The problems were presented in interview and written format to two groups of university level mathematics students: third year teacher trainees in Israel, and first year graduate students in the USA. With very few exceptions, the students did not come up with the elegant solutions, nor were they expected to. But when the elegant solution was presented to them, their reaction did not parallel the experts' reaction: the students, although understanding that both paths constitute solutions to the problem, did not find the elegant any more attractive than the one they had come up with on their own—they failed to grasp its aesthetic superiority. A typical reply was: "Oh, that's how you do it. So what! My way works too." Probing didn't help either. Moreover, when faced with two possible solution paths, many of them were unable to distinguish the elegant from the pedestrian one. They did not base their decisions on aesthetic values, but rather practical ones—they wanted to immediately pick up the pencil and start working—without first reflecting upon different solution paths.

The objective of finding an aesthetically appealing solution to a mathematical problem was not detectable in these university level mathematics students, some of whom had received large doses of formal mathematics. Their aim was to solve the problem, not to solve it in an elegant way. Worse, the excitement of seeing the elegant solution was absent. They showed no inner sense of feeling for the cleverness of a solution.

Von Glasersfeld [1985] has made the point that we cannot expect children to show an appreciation for the beauty of mathematics. Specifically, it is a misconception of mathematics educators to expect that "... the beauty and elegance of mathematical solutions should be as obvious and entrancing to every lay human as is the beauty of a sunrise or the flight of a hawk." One would hope that such an appreciation of mathematical thought grows through experience and schooling, and that it should certainly be found in university mathematics students and mathematics teacher trainees. This hope is supported by the fact that following Polya's example literally hundreds of books have been written listing appropriate problems and their solutions. Thus the material to teach an aesthetic appreciation of mathematical thought is available, albeit haphazardly organized. The present study shows that in spite of this mathematics educators have failed miserably in achieving this aim. We are apparently succeeding in teaching techniques, and perhaps some of their underlying processes, but little beyond that seems to be going on in the mathematics classroom [see also Rosnick and Clement, 1980; Vinner, 1984]. In particular, any kind of reflective thinking beyond the actual solution process is lacking. Over forty years after Polya's "How to Solve It" [1945] appeared, even the best students, and many of their teachers, turn away from a problem as soon as they get a hold of its solution. Higher order thinking skills in textbooks have been on a decline for the past three decades [Nicely, 1985]. But lack of reflective thinking is not limited to mathematics education. Indeed, in a scathing article Feynman [1985] has exposed rote learning as a world wide problem. With the situation as it is, any aesthetic feeling for mathematical thought is achieved serendipitously, if it is achieved at all.

5. The experts' views

Mathematics is in a constant state of flux, but there are certain problems, solutions, and proofs which remain awesome in their simplicity, elegance, and beauty. Among them, Euclid's proofs of the infinitude of primes and the irrationality of $\sqrt{2}$ are known. A number of them were collected by Harris [1971], and several more were printed in subsequent issues of *The Mathematics Teacher*. Five of these proofs were chosen for a comparison of experts' assessments of their aesthetic value. These proofs are presented in Table 2.

Table 2
$\sqrt{2}$ is not rational

A. Assume that the $\sqrt{2}$ is a rational number. Then there exist two integers p and q, $q \neq 0$ such that $\sqrt{2} = p/q$ where p and q are relatively prime: $(p,q) = 1$.

Since $\sqrt{2} = p/q$

$$2q^2 = p^2 \quad (*)$$

2 divides the left hand side of (*), so it must also divide the right hand side of (*). That is:

2 divides p^2.

That is, p must be even.
Since p is even we know that $p = 2k$ for some integer k. Substitute this into (*):

$$2q^2 = (2k)^2$$
$$q^2 = 2k^2$$

Here we see that 2 divides q^2 and thus 2 divides q. In other words, the fraction p/q was not in its lowest terms; we thus have a contradiction. Hence our assumption that $\sqrt{2}$ is rational must be wrong.

B. It is known that if a polynomial of the form

$a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0$ (where a_i is an integer)

has a rational root p/q ($q \neq 0$, $(p,q) = 1$), then p divides a_n and q divides a_0.

Let $x = \sqrt{2}$, and consider the polynomial $x^2 - 2 = 0$. If this has a rational root of the form p/q, then p must be ± 1 or ± 2 and q must be ± 1. We form all possible values for the fraction p/q. In each case the rational root does not satisfy the equation $x^2 - 2 = 0$. Hence $x^2 - 2 = 0$ does not have a rational root, which implies that $x = \sqrt{2}$ is not rational.

C. Assume that $\sqrt{2}$ is a rational number of the form p/q where p and q are integers, $q \neq 0$. Since p and q are integers they can be expressed as the product of prime numbers raised to an appropriate power.

$$p = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \dots$$

$$q = 2^{\beta_1} \cdot 3^{\beta_2} \cdot 5^{\beta_3} \dots$$

$$p^2 = 2^{2\alpha_1} \cdot 3^{2\alpha_2} \cdot 5^{2\alpha_3} \dots$$

$$p^2 = 2^{2\beta_1} \cdot 3^{2\beta_2} \cdot 5^{2\beta_3} \dots$$

Since $2q^2 = p^2$ we get

$$2^{2\beta_1+1} \cdot 3^{2\beta_2} \cdot 5^{2\beta_3} \dots = 2^{2\alpha_1} \cdot 3^{2\alpha_2} \cdot 5^{2\alpha_3} \dots$$

This implies $2\beta_1 + 1 = 2\alpha_1$, which can never happen. So $\sqrt{2}$ is not rational.

D. Assume $\sqrt{2} = p/q$ where p and q are integers, $q \neq 0$. $2q^2 = p^2$ cannot have a non-zero solution in the integers, because the last non-zero digit of a square written in base three must be a one, but the last non-zero digit of twice a square, written in base 3, is 2. Hence we have a contradiction. The $\sqrt{2}$ is not rational [Gauntt and Gustave, 1956; this proof was published in this form. It is incomplete, but sufficient for the ensuing discussion.]

E. Assume $\sqrt{2} = p/q$ where p and q are integers, $q \neq 0$. Since $p > q$ there is an integer a greater than zero such that $p = q + a$ and $2q^2 = q^2 + 2qa + a^2$.
This implies $q > a$.
Consequently, there exists an integer $c > 0$ such that $q = a + c$. Hence $p = 2a + c$ and $2(a + c)^2 = (2a + c)^2$.
This last equality implies $c^2 = 2a^2$, so the entire process may be repeated indefinitely, giving a sequence of integers

$$p > q > c > a > \dots$$

But every non-empty subset of the positive integers must have a least element. So we have a contradiction, i.e. $\sqrt{2}$ is not rational. [Halfar, 1955]

All five proofs in Table 2 are indirect. Method A makes use of the fact that an integer is divisible by 2 either an even number of times or an odd number of times, but not both. The contradiction then appears in simple terms: assume that p and q have no common divisors; ... then 2 divides both p and q. Method B is brief; it is based on an easy but relatively unknown result from the theory of polynomials. Method C uses the Fundamental Theorem of Arithmetic, i.e. the uniqueness of the prime factorization of any natural number. In contrast to the theorem used in Method B, the Fundamental Theorem of Arithmetic is intuitively understood by students. Note that the proof is slightly more general in the sense that it does not require that the greatest common divisor of p and q is 1. Apart from that Method C is closely related to Method A. In fact, the divisibility result used in Method A is an immediate corollary of the uniqueness of the prime number decomposition. Method D is as brief as Method B, but very different in its approach. It uses properties of the representation of certain numbers in base 3. It thus makes use of some extraneous knowledge, albeit knowledge more readily accessible than in Method B. Method E, finally, uses the well-ordering principle, namely that every non-empty subset of the natural numbers must have a least element. This principle, although deep, is very intuitive, and therefore readily used by students.

These proofs were shown to mathematicians who were asked to rank them according to elegance. They were then asked to give reasons for their rankings and to decide which method(s) they would use to teach this theorem to high school students. On both counts, Methods A and C were chosen consistently for two reasons: Their simplicity and the fact that they can be completely understood with a minimal amount of mathemtical background. It is interesting that the lack of need for prerequisite knowledge was given as a reason to choose a particular proof not only for the high school students, but also concerning the experts' personal preferences. In fact, when they were asked to decide again, under the assumption that the students would be familiar with the mathematical content and machinery underlying methods B, D, and E, they still chose A or C. This might be partly due to the fact that they themselves were more familiar with these proofs than with the others.

It is of interest to note that with the exception of Methods A and C the technical machinery and underlying concepts used in the proofs were not available to the ancient Greeks. In other words, those proofs that are older, and more elementary, were judged to be more elegant.

6. Conclusions

The views expressed in this paper deal with an aspect of mathematical activity that goes beyond the usual problem solving activities. They related particularly to an extension of the final phase of problem solving which is usually referred to as "Looking Back" [Polya, 1945]. We are thus concerned with "problem solving plus", with an aesthetic assessment of the processes and outcomes of problem solving which should follow the usual phases.

The metric used by experts for such assessments combined brevity of argument with the amount of prerequisite knowledge required. Even when this prerequisite knowledge could be assumed to be well ingrained in the student, it did not influence the solution path being selected. For example, for proving the irrationality of $\sqrt{2}$, Method B is certainly terse, but it seemed to appeal least to the mathematicians. Thus the principal factors that should guide us in developing an appreciation for mathematical thought— are the conciseness, clarity and simpicity of the argument. There are, however, quite a few other relevant aspects:

structure, power, cleverness, and surprise, have all been shown to contribute to the aesthetic appeal of mathematical thought.

We believe that an aesthetic appreciation for mathematics can be nurtured. It is possible to learn how to appreciate art, music and poetry by developing an inner metric against which to measure them [Bernstein, 1959; Vitz and Glimcher, 1984]. Hofstadter [1970] has worked out far-reaching parallels between mathematics, art and music. Just as individuals learn to appreciate music, art and poetry, by understanding their underlying structures, so one can learn to appreciate mathematical thought. But as witnessed in this study, this has yet to be achieved in the mathematics classroom. It may be an overstatement, but something is terribly amiss in the mathematics curriculum.

In order to find out which methods could possibly contribute to fostering a feeling for mathematical aesthetics in our students, we should ask ourselves two questions: (i) how is a feeling for aesthetics fostered in other domains such as art or music, and (ii) how did we, ourselves, develop a feeling for the power and beauty of mathematical thought?

Here are some recommendations: Of prime importance is problem solving exposure. The "aha" of problem solving activities [Gardner, 1982; Pederson and Polya, 1984; Dunn, 1980 and 1983] should be exploited to develop "the mathematical unconscious" in the sense of Papert [1980]. Rarely is more than one solution path discussed in the mathematics classroom before pursuing it (nor after pursuing it, for that matter). From the above, it follows clearly that arguments must be compared and contrasted before one can assess them. Aesthetic aspects should be discussed and compared. Quite apart from that, consideration of two or more solution paths could bring very practical benefits: developing a familiarity with different solution methods and deeper conceptual understanding, e.g. in the case where an equation is solved both by graphical and by analytical methods. Moreover, discussing different solution paths affords an opportunity to take individual differences into account. Not only mathematics educators but also cognitive scientists have recently stressed the need for this. With reference to differences in cognitive style, de Ribeaupierre [1985] recently stated:

> Individual differences represent more than errors of measure or "noise"... Different individuals of a similar general level use different processes for solving the same problem. [pp. 1-3]

Some appropriate problems for use in this spirit are the ones in Table 2. There is, however, a critical need for more texts pointing out different approaches to the same problem.

Appreciating the aesthetics of mathematical thought can be trained. Lester [1984] has shown that problem solving activities do indeed contribute to fostering the aesthetics and the accompanying excitement, if the problems are sequenced and used correctly. But at present, developing an aesthetic appreciation for mathematics is not a major goal of school curricula [NCTM, 1980]. This is a tremendous mistake.

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It is useful to distinguish, however tentatively, between the two major Western approaches to the problem of reality. Most people think that reality is physical, public, external, and somehow "hard," and they think that what is not real is mental, private, internal, and somehow "soft." The terms hard and soft are also used to distinguish people who accept this distinction (hard thinkers) from those who feel that all phenomena (physical or mental, public or private, external or internal) are equally hard and soft (soft thinkers). Hard thinkers think that you should always define, sharply, at the start, what you think and that you should always continue to think it; soft thinkers feel that you can play it by ear and shift your definitions as your understanding grows. Hard thinkers think that you cannot believe two contradictory things at once; soft thinkers think that you can. These definitions are self-referential: soft thinkers do not think that they, or hard thinkers, exist as a separate category; hard thinkers think that they do. I would be called a soft thinker by people who think that there is such a thing as a soft thinker as opposed to a hard thinker — a distinction that I would challenge, as a soft thinker should....

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