Routine and Meaning in the Mathematics Classroom*

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1. Introduction: What comes first: rules of calculation or the meaning of concepts?

"First of all, and this is most important, the pupils have to learn the several steps of the calculation exactly. It is only on this basis that they will perhaps attain deeper insights or conceptual understanding . . ."

"Especially with weaker pupils it is necessary to clarify the rules very carefully, step-by-step, and to repeat them always in great detail: only a total mastery of rules will perhaps later on lead to a justification or to the meaning of these rules . . ."

"Only on the basis of a complete and clear definition of mathematical concepts and procedures does it make sense to approach the understanding of conceptual relationships in mathematics teaching . . ."

In such or similar ways teachers answer when asked about the conditions and possibilities for the development of mathematical knowledge in teaching. In their view, "knowledge" first of all seems to be some matter or stock in hand consisting of many single elements which have to be handed over to the pupils one-by-one and in an adequate manner. In this respect, it is necessary for communication in the classroom to have definite expressions for mathematical concepts and to make clear and detailed descriptions for mathematical operations in order to avoid misunderstandings and problems of comprehension from the beginning. Only after the individual elements of knowledge and elementary operations and procedures concerning this knowledge have been conveyed may one turn to discovering the structures and networks of relations between these elements. In short: in the opinion of many mathematics teachers, the process of developing mathematical knowledge rests upon well-defined operations and algorithmic procedures and upon clear-cut formal definitions of concepts. Meaning, mathematical insight, and understanding, in the learning process of pupils can only evolve on this basis [cf. Steinbring, 1987].

> The "algorithm" is a tool which enables the clearing of obstacles and the solving of didactic conflicts in the sense that it momentarily allows a clear apportionment of responsibilities. The teacher shows the algorithm, the pupil learns it and "applies" it correctly: if not, he must exert himself, but his uncertainty is nearly null. He is firmly told that there is a whole class of *different* situations to which the algorithm gives a solution (the conflict will resume when a choice of algorithm must be made for a given problem).
>
> The algorithm is practically the only "official" method of release; this means that it has been the subject of making the teaching methods relating to it explicit. And it is used as a unique or nearly unique model for all the sub-cultural approaches in teaching. [Brousseau, 1986, p. 30]

In the frame of the cooperative project of teacher in-service training PROFORMA (praxisorientierte Fortbildung für Mathematiklehrer, i.e. practice-oriented in-service training for mathematics teachers) this problematic relation between formal routines and conceptua meaning has been the object of common work and reflection. The participants understand the project as ar attempt at a cooperative in-service model which first o all does not intend to manage the conveyance or th "transport" of new and interesting subject matter an teaching procedures (in the way that research deliver new content according to the requirements of practice) which aims at an equal and joint reflection on the prob lems of teaching and learning mathematics [for a detaile description of the PROFORMA project see v. Harten Steinbring, 1986].

The concrete topic for the common work in our projec was "Problems of understanding and development asso ciated with the introduction of decimals in mathematic teaching". For the treatment of this topic and its prepara tion for teaching a small group of teachers developed, i cooperation with me, proposals for materials containin a *conceptual didactical orientation* with regard to th epistemological problems of teaching and learning dec mals, the description of many different, especially *expe imental types of learning activities* in this area, and series of *exemplary work sheets* for pupils (for instance some adaptations of Swan's teaching materials [Swar 1983]. This material consisted of various modules fc different purposes and served all participating math matics teachers as an orientation and a basis for prepa

*Paper presented to the Third International Conference on "Systematic Cooperation between Theory and Practice in Mathematics Education", London, Great Britain. April 10-15, 1988

ing their "own" teaching unit: "Introduction to decimal numbers". The performance of the teaching sequence was observed in many classes and the lessons were recorded on audio-tape. These records served to supply interesting teaching episodes—in the same way as in previous in-service seminars of the PROFORMA project—in order to have transcribed documents of teaching phenomena to analyze and discuss during the in-service training. These transcribed teaching episodes offered promising opportunities to develop careful reflections about classroom conditions, changes, and didactical perspectives on teaching, while maintaining a "positive distance" from the very concrete everyday events in the mathematics classroom.

The idea underlying the following considerations is not to give priority to a didactical analysis of decimal numbers or to provide an alternative way of introducing and developing the concept of decimal number in mathematics teaching additional to the approaches found in Brousseau [1980, 1981] or in Swan [1983]. The main objective will be to take the problems which arise in the actual teaching or the decimal number concept and which express personal, social, and epistemological, difficulties of learning as exemplary cases allowing us to investigate how the specific conditions and mechanisms of teaching/learning processes and their influences on the understanding and the development of mathematical knowledged can be analyzed and clarified in a framework of cooperation between didactical research and teaching practice.

2. The obligation to teach and the algorithmization of knowledge

In the view of many teachers, and according to the description in most textbooks, decimal number calculus seems to be basically a very simple mathematical domain. The four fundamental operations of arithmetic with decimals are presented in textbooks as a rather direct "generalizations" of the operations with natural numbers. The only thing to do—this is often stated in textbooks—is to define exactly the correct use of the decimal point in the representation of the new numbers; apart from that, the pupil may calculate in the same way as with natural numbers. This is relevant especially for written calculations such as addition, subtraction, multiplication and division.

In contrast to such a technical reduction of decimal numbers to natural numbers and their arithmetical operations, intended to deliver at the same time a quick and easy comprehension of this concept, the important conceptual difficulties and misconceptions pupils actually have concerning decimals are known from many teaching experiences. Mathematics educators judged these problems of understanding to be the result of a predominance of algorithmic procedures and an underestimation of the development of conceptual meaning in mathematics teaching.

> When treating decimal numbers . . . a deficiency in our mathematics teaching becomes visible. The formulation and drilling of operational procedures are placed in the foreground, the passage to formal mechanised drill is much too quick . . . Too little methodological creativity and reflection, patience, and time, are spent in stimulating and securing the conception of concrete quantities represented by decimal numbers and the understanding of decimal numbers as a number representation in a place value system . . . It must surely be . . . intended that the pupils develop a greater conceptual understanding and a better idea of decimal numbers." [Günther, 1987, p. 25]

It is surprising that a conception as technically simple as decimal number implies such big problems of meaning and comprehension for many pupils. Apparently it is not sufficient to supply pupils with elementary techniques and rules such as "Write one decimal point exactly beneath the other and then operate in the same way as you already do with natural numbers!"

The decimal numbers and their introduction in mathematics teaching exemplify certain problems in the relationship between meaning and routines in teaching/learning processes. Their cause shows precisely that routines and operational procedures are frequently given priority within the process of knowledge development when teaching mathematics. First of all the pupil has to know the arithmetical operations off pat, and then perhaps may think of going on to slowly develop meanings and insights—this is the opinion of many mathematics teachers.

This understanding of and this approach to the learning and the mediation of mathematical knowledge in the classroom is not just a deliberate simplification of mathematical teaching/learning processes by the teacher; this reduction of learning processes to operational procedures, rules and techniques is also a result of specific social and epistemological constraints and dependencies as they become effective in communicative and interactive teaching processes. The social obligation on the mathematics teacher to *teach* mathematics—Brousseau [1984] even introduces the concept of a *didactical contract*—that is, the expectation that the teacher will "give" mathematical knowledge in an appropriate way to his/her pupils, together with the conception of school mathematics as "matter", leads to the wrong idea that mathematical knowledge can be conveyed directly to the pupil if only enough methodological simplifications and didactical elementarizations have been made.

> The well-meant intention of many teachers to describe simply and to explain directly any unknown mathematical knowledge to the students, even the obligation they assume in interactive teaching processes to make all meanings explicit, leads to the effect that by the total reduction of the new knowledge which is to be learned to knowledge already known nothing really new can be learned. The tendency to the complete explication and reification of mathematical knowledge makes it impossible to really acquire new knowledge in teaching. This process may be interpreted as a self-reinforcing *vicious circle of teaching practice*: an increasing evacuation of the theoretical meaning of concepts through the

> growth of methodological simplifications . How can one escape from this circularity of the increasing methodological simplification and at the same time decreasing meaning of concepts? Evidently not by further methodological means or rules [Steinbring, 1986, p. 5]

This "disappearance" of theoretical meaning caused by an excessive algorithmization of mathematical knowledge corresponds on the side of communicative interactions to the conception of "routine and being forced to act" ("Routine und Zugzwang") [Voigt, 1984] as well as to the conception of a "funnel pattern of interaction" ("Trichtermuster der Interaktion"), that is, "a narrowing of action by the anticipation of an answer" [Bauersfeld, 1978]. ". . . The image of a funnel represents by its mechanical nature the idea of a fixed course given *a priori*, which seems to be prescribed from the beginning in this and no other way, and which takes place according to an inescapable principle of cause and effect. But there are many reasons why this is a step-by-step, still consolidating process of interaction, in which one stake determines the next one but where at each stage of development another than the described stake is possible even though the degrees of freedom are steadily decreasing The effect of the funnel emerges from the interplay between the several actions and reactions of the participants, by moves and counter-moves, by anticipations and anticipated anticipations, etc.—that is, as a result of the process but not as its cause." [Bauersfeld, 1978, p 162/3] The vicious circle of teaching practice is a self-reinforcing "automatic control system" that the teacher cannot immediately escape because of his obligation to teach the pupils something.

The teaching of the "Introduction to decimal numbers" that was observed showed up many situations with "circular patterns" of interaction. The negotiation of meaning and the explicitation of knowledge performed step-by-step between teacher and pupils often produces an increasing algorithmization and thus a reduction to "naked" rules and routines The compulsion teachers often feel to tell their pupils finally "how the rule works"—the pupils naturally knowing that the teacher knows how—implies in this "automatic control system" of teaching practice that the mathematical knowledge which should be learned autonomously by the pupil is more and more directly provided by the teacher—and simultaneously transformed into another "type" of knowledge

> The teacher must succeed in getting the pupil to solve the problems which he has given him in order to ascertain and to enable others to ascertain that he has fulfilled his own task . .
> The teacher has the social obligation to *teach* all that is necessary concerning knowledge. The pupil—especially when he is at an impasse—requires it of him . .
> So, the more the teacher gives in to these requests and reveals what he expects, and the more he tells the pupil precisely *what* the latter must do, the more he risks losing his chance of securing and ascertaining objectively the learning that he must in realit[y] aim at. [Brousseau, 1986, p. 35]

Such phenomena leading to an algorithmization [o]f knowledge were observed in the mathematics lessons [o]f the participating teachers The following episode [is] exemplary evidence for a communicative pattern be[-]tween pupils and teacher in which the emphasis shif[ts] more and more to the formal rule about the knowledge [in] question

"A rule for rounding decimal numbers"

The following long division has been calculated on the blac[k]board:

540 2 : 13 = 41.5538

(The calculated result expresses the average weight of [...] pupils)

T: What do we have to calculate when we want to rou[nd] the third digit after the point?
P: Yes, four
T: Four, that's clear Well, Sonja could you repeat it on[ce] again?
P: Greater than five, rounded up and four, ahem, less th[an] five, rounded down.
T: Okay, and what will now happen with this three here?
P: Rounded down!
P: Rounded up!
T: H'm, but what did you just say?
P: Here, rounded down, up,
T: Well, what is it then here, the fourth?
P: Eight!
T: Yes, and now?
P: Rounded up!
T: Yes. and to what does it correspond, to which weight d[oes] it now correspond, Adrian?
P: Forty-two point . .
T: No, no, no, where is the two. I thought it was a one? Isn'[t it] a one?
P: Forty-one point sixty
T: No . how many places after the decimal point, I ha[ve] said before, do we want to have? One kilogramme, tha[t is] how many grammes?
P: Three places.
T: Yes, would you please say it then?
P: Forty-one point six. nought, nought . .
Teacher writes at the same time on the blackboard: 41 .
T: Where do we round then, at which position, Timmy?
P: Forty-one point five, five, four
Teacher writes the whole result.
T: Well, at this position!
P: Oh, I see
T: That is the decisive figure here now.
Teacher points to the third place after the decimal poi[nt]
T: Three places after the point. This corresponded to [the] average weight of the boys, okay One kilo has a thous[and] grammes, therefore we have three positions after the po[int] and the third position after the point must be rounded [...] for this I need the fourth, this is bigger, equal to or big[ger] than five, therefore it is rounded up. Is this clear?
P: I still don't get it
T: Well, what do you not understand? I want to know no[w]
P: . with rounding down or up . . .
T: Okay, . three places after the decimal point, [...] understand?

P: Yes

T: One kilogamme, there we have a thousand grammes, h'm, then I can write *(teacher writes on the blackboard: 1 kg)* one kilogramme What could I write if I want to write a decimal point in this, or a decimal number?

P: One point nought

T: One point nought, nought, nought *(writes it at the same time on the blackboard).* It's the same, yes, marvellous. Three places after the point we want to have, then four have to be calculated to be able to determine the third. Whether it will be rounded up or down Do you know when to round? Sonja just said it

T: Vivien, say it please again!

P: When the figure is bigger than five it will be rounded down, and when it's smaller . . .

P: Oh, . no!

T: Don't mix it all up now!

P: I mean, when the figure is bigger than five it will be rounded up, and when it's smaller than five, rounded down

T: Is it correct this way?

P: Yes.

T: And what will we do with the five? . bigger than five, rounded up, smaller than five, rounded down, and what will we do with the five? Doesn't it exist at all?

P: Rounded down

T: It will remain the same? Marietta?

P: Well, bigger than five, when it's bigger than five it's rounded up, h'm I think, this was .

T: Peter, stop it and pay attention Peter!

P: Bigger than five it's rounded down

T: Could *you* formulate it someway?

P: H'm, smaller than five is rounded down, five and more is rounded up.

T: Well done, good We can leave it this way

The reduction to routines and to operational procedures plays an important role not only on the teacher's side The pupils are also aware of the specificity of operational rules and calculations in mathematics teaching

"How to add decimal numbers?"

The pupils have to add the following three decimal numbers:

$$\begin{array}{r} 2.37 \\ +\ 13.731 \\ +\ 0.2 \\ \hline \end{array}$$

Several pupils have "filled up" these numbers with noughts:

$$\begin{array}{r} 2.370 \\ +\ 13.731 \\ +\ 0.200 \\ \hline \end{array}$$

T: Alexandra!

P: Well, I have put in the noughts with a pencil and afterwards I erased them.

T: H'm. Tobias!

P: I think this doesn't work, because they are all beneath the, the decimal point is equal, then they are like, they are really equally big, only these do have the noughts back there, perhaps.

T: Lars!

P: Doesn't matter at all for the calculation, whether there are noughts or not, it's nothing to do with adding up.

T: Yes . .

P: But when one looks at it, the numbers are bigger now

T: What do you mean by bigger?

P: Well. for instance. two, three, seven. h'm, two, three, seven, nought, this would become three hundred and seventy now

T: H'm, well, Alexandra!

P: I wanted to say the same.

T: Marc, please! *(Pupils are putting their hands up)*

T: Peter!

P: When we add up. it makes no difference at all, because nought plus nought remains nought.

The described self-reinforcing circle of teaching practice produces an alteration in the character of mathematics in the classroom: the result of social, communicative and epistemological constraints and mechanisms in the process of teaching In view of this negative, vicious circle, how is it possible to produce the conditions and possibilities which may lead to a positive self-reinforcing "system" in which the learning of mathematical knowledge can be conceived and developed as an interrelation between conceptual meaning and operational procedures?

3. The relation between conceptual and procedural knowledge in school mathematics

How can one motivate mathematics teachers to reflect on the process of the "algorithmization of knowledge" in teaching from a detached perspective? It's not enough to develop an understanding in the teachers that the reduction of knowledge to its operational procedures is insufficient. The teachers should also get an idea of the very complex structure of this problematical relationship between routine and meaning in the teaching and learning of school mathematics.

To develop teachers' awareness of these problems we have attempted in our project to involve them in getting an understanding of the new knowledge to be learned in the following way. With regard to the main problems of comprehension for pupils treating and using decimal numbers as we observed them in the classroom (for instance, the method of long division, the role of the figure 0 at different places in the representational system of decimals and its meaning, the possibility that some numbers can be represented in different ways in the decimal number system, e.g. 1 and 0.9999 . . , etc.), the teachers were asked during an in-service seminar to undertake a series of tasks similar to those of their pupils, with one significant difference. The teachers had to solve the problems given to the pupils not in the decimal number system but in other positional number systems (for instance, the binary, ternary, quinary or septenary number systems) This different arithmetical frame for the same problems first of all induced irritation and detachment in the teachers, but then became the starting point for mathematical work and problem solving which not only yielded calculated results, but also new insights, assessments and understanding with regard to the possible problems pupils encounter when trying to solve the same tasks in the decimal number system.

It became evident—during the ensuing discussion with the teachers—that the often implicitly made assumption, "Try to apply the formal rule mechanically", does not lead directly to the goal in this case. It was, for instance, not sufficient for long division simply to proceed according to the method explained in detail by the teachers to their pupils: taking down the figures, noting some figures, shifting them to other positions in the system according to special rules, etc. It was not sufficient, the teachers themselves remarked, to keep strictly within the mechanical structure of the formal rule pattern. In order to be able to operate the rule an explanatory counterpart, a meaning for the rule was necessary, which had to be represented in some other context.

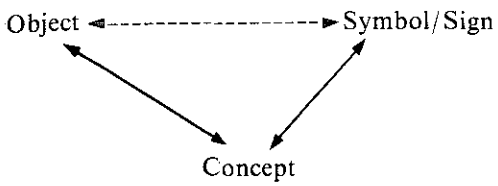
The teachers drew the meaning for the new sign systems embodied in the rules from the tables of the place values for the different number systems. These number systems also provided a relational structure and the possibility of comparison with the routinely mastered decimal number system. Only on the basis of a comparison between the operational rules and the possible meaningful interpretations of these abstract figures in the frame of a place value system were the teachers able to begin the solving process and, finally, to obtain the results. It became clear that not the "naked" rule alone but only *a structural context of meaning*—even such a simple one as in this case—together with the rule made possible the idea and the method of solution. Later it became increasingly easy for the teachers to return, by means of analogies between different number systems, to a quick and routine treatment of the task even though entering into these seemingly simple operational procedures required a relatively large amount of time for reflection and the construction of meaningful networks.

Besides these rather technical operational aspects, some mathematical problems also require comparisons and relations between mathematical domains seemingly totally separate and far removed from one another to be established. One example is the connection between an infinite geometrical series and the corresponding representation of a decimal number in a specific number system. In the ternary system, for instance, a ternary number may "represent" an infinite series whose "result" can be calculated as a fraction of the ternary number. Working on such a problem the teachers were able to imagine the difficulties pupils might encounter when having to construct relations between different domains, even in elementary school mathematics. Before, the teachers had always considered these domains to belong directly together and to be related to each other in a natural way.

This way of working with specific in-service education material which tries to "alienate" certain mathematical tasks (as well as learning situations, the understanding of concepts, etc.) by placing them in different contexts (for instance, varying the methods of work as compared to those of the pupils, changing the mathematical frames, differentiating learning opportunities by means of task-systems, etc.), offers many possibilities for starting a conscious reflection by teachers about the problems of mathematics teaching which have otherwise dissolved into teaching routines and into explicit rules for treating operational problems. It also offers opportunities to explain to teachers in a tentative way the complex and multi-layered nature of the processes concerned with the development and understanding of theoretical knowledge. In such alienated situations, teachers themselves can experience that learning does not simply begin with routines and then reach meanings, but that meaning must be incorporated into the process of learning from the beginning in order to arrive—on this basis—at quick and effective routines for the later solution of problems. Routines are not the starting point but the final point of learning. They should be "flexible" and allow a reference back to meaningful structural networks when the context of problems changes in order to permit varying and adapting the routines according to changed or novel requirements.

These considerations concerning the interrelation between routines and meanings in mathematics teaching were supported by epistemological orientations about the conceptual nature of decimal numbers in the proposals for the teaching material. The decimal number concept was not "defined" as a "natural number with a point in its representation", but decimals and their conceptual meaning were placed in a manifold structural network [cf. Brousseau, 1981]. Essentially, the epistemological characterization referred to two sides of this concept. On the one hand, the decimal number concept with its specific type of representation was linked with concrete *problems of measurement*: decimals could be seen as elementary mathematical models of measurement procedures in which the measuring unit varies according to specific requirements for accuracy. The decimal number system is used for this by choosing the units to be powers of 10 or 1/10. On the other hand, decimals play an important role in the *development of the number concept* in mathematics teaching: in this context, the decimal number is not simply a "further development" of the natural number but also relates to rational numbers and fractions and can be interpreted as a "precursor" of the real numbers. Decima numbers are important in school because they contain for the first time discrete and continuous aspects of number which show up both in the extension of the numbe concept and in problems of measurement. [For furthe considerations of the didactical and epistemologica characterization of decimal numbers see Brousseau, 1981 Günther, 1987; Swan, 1983; Steinbring et al, 1987]

The rough outline of an epistemological characteriza tion of decimal numbers is in relative contrast to th formal definition of "natural number with a point in it representation". In the frame of our description, decima numbers are conceived of as representational systems, o as systems of figures used to characterize aspects of th number concept, and to offer a means to manipulate an operate with the number by using its representation. A with any mathematical concept one must also distinguis between the *representational form given by the symbo* (the figures and the decimal point here) of a number an its meaning as it is constituted by reference to certai domains of "application". The epistemological characte ization of the decimal number concept has to keep apa the symbolic representation (in the form of specific sign and the intended "applications" of these symbols (certain domains of application, objects, problems, or even other representational systems) and from the respective conceptual understanding of the number concept. This epistemological framework can be outlined as a relational triangle: object—symbol—concept.



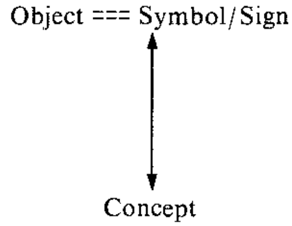
The "symbol" or "sign" means the specific way of writing decimal numbers with figures and a point, the "concept" in this case is the number concept (or certain conceptual aspects of it), and the "object" could contain, for instance, certain concrete contexts of measurement.

The distinction between the representations of knowledge (in the form of signs) and the knowledge itself or the meaning of the knowledge is an important characteristic of theoretical mathematical knowledge [cf. Kaput, 1987; Otte, 1984a]. The development of the decimal number concept, one could say, takes place as an attempt to relate the actual symbolic representation of decimals to an intended field of application and thereby to extend the operative manipulations of the symbols. Therefore it is really important to distinguish between the aspect of the application and the aspect of the representation or the modelling of knowledge; only by relating these two sides to each other without identifying them directly can the development of knowledge advance. Domains of measurement and of application possess qualities of their own and can in some respects be described by means of the number concept and of arithmetical operations; on the other side, the formal structures of symbols and rules are autonomous with regard to the domains of application but open up a relational perspective on these domains.

How can this characterization of conceptual mathematical knowledge organized in the frame of the epistemological triangle help to discuss, analyse, and perhaps better understand the problematic relation between routine and meaning in mathematics teaching?

The self-reinforcing circle of teaching practice, i.e. the increasing methodological simplication of knowledge and its simultaneous algorithmization, can be explained in terms of the epistemological triangle as follows: the well-meant methodological simplifications of the teacher, together with the motivations of the pupils coming in, often cause in this "automatic control system" the "object" of knowledge to be identified with the "symbol" or "sign" of knowledge. "The practice of mathematics, particularly in school, is ... induced by automatization, by the algorithm expressed in a formula as a procedure for calculating, to identify sign and signified, or, if the threefold distinction between concept, sign, and object is made (which is in principle necessary), to identify the sign and the object while neglecting the conceptual aspects which are independent of it" [Otte, 1984a, p. 19].

Accordingly, the structure of the epistemological triangle is restricted to a linear consecutive course:



Mathematics teachers often prepare problems and application examples for teaching in a way which is intended to ensure that the mathematical structure directly flows out of them. On the basis of explicit procedures, rules, and operational techniques, they hope to elaborate the mathematical concept and its meaning. This consecutive and step-by-step modelling of the learning process corresponds to the idea held by many mathematics teachers that mathematical knowledge is a "material" stock consisting of single elements which must first be learned in a cumulative way in order to make later discoveries of the underlying structural relations and meanings possible.

Acceptance of a necessary separation between sign/model and object/problem helps to prevent a reduction of mathematical meaning to its algorithmic and formal aspects. This acceptance requires that teachers become aware, within the teaching process, of the constraints and the positive opportunities offered to the learning processes by direct teaching aids. It also requires that teachers provide learning situations, problem domains, and activities for pupils which assist them to reconstruct in a relatively independent way understanding and meaning of mathematical concepts by establishing relations between aspects of the signs or models and the domains of problems or applications.

The relational character of mathematical knowledge is constituted in a two-fold manner: on the one hand, knowledge is constituted as an *objective* relation between "sign" and "object" which, on the other hand, must be constructed by the learning *subject*; the objective structural facts represented in this relation are not unchangeable or given *a priori* in a fixed pattern, they must be subjectively chosen, constructed, changed, adapted, and developed further. Mathematical knowledge can be characterized as a *simultaneously epistemological and social relational form* [cf. Seeger, 1987].

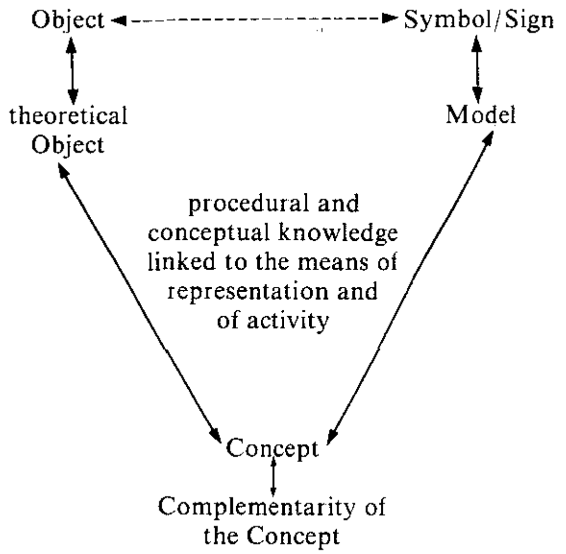
The personal reconstruction of the relations embedded in mathematical knowledge requires *means of representation and of activity*. For the developing of the relations between the representational system of decimals and possible intended applications, the means of a scale of measurement (in the context of measurement problems) and of a place-value system (in the frame of mathematical modelling) are helpful in making it possible to act upon conceptual aspects of the number concept. On the one hand, scales are important practical instruments for the activity of measuring; on the other hand, scales can be seen as "ancestors" of the symbolic representation of decimal numbers. A scale has practical meaning, it describes lengths, it permits the comparison of different lengths in practical measurements; it also makes it possible to understand elementary conceptual aspects of decimal numbers and to organize them into a specific representational system in order to act upon them The "table of a place-value system" is a means of representation *and* of activity. It belongs to the level of symbolic representation and stands closer to an interpretation of decimal numbers which refers them to other conceptual aspects of number such as fractions or rational numbers. As compared with a scale the table of a place-value system contains all the relevant figures organized in a systematic way (at least in principle) according to the order of powers of 10 The table of a place-value system represents decimals as specific fractions: decimals are written down as symbolic figures, and every figure is related to its specific category of fraction. Tables of place-value systems do not directly refer to measurement situation but show what can be done with the help of scales and measuring tapes.

Consideration of these two means of representation and of activity, i.e. scale and table of a place-value system shows in an exemplary way the role of these means for development of knowledge and meaning. These means are important "nuclei of crystallization" for the pupil's development of understanding when learning the decimal number concept; they permit fixing in an adequately tentative way certain local conceptual aspects, linking them to potential operational procedures in view of some intended fields of application. With these means, the pupil may in part immediately manipulate and operate, and with their help he can express simple ideas and meanings of concepts (of decimal numbers) The means of representation and of activity enable the learner always to develop in a meaningful manner, according to the progress of his learning, and to develop further ideas and meanings

The relationship between conceptual and procedural knowledge contains fundamental problems for the learning of mathematics. [cf. Hiebert, 1986] Does conceptual knowledge take priority over procedural knowledge in the frame of learning and understanding, or is procedural knowledge a "precursor" of conceptual knowledge? Silver [1986] expresses the conviction that it would be very difficult to distinguish clearly between procedural and conceptual knowledge. "The intertwining of procedural and conceptual knowledge is evident even when that knowledge is being applied to apparently simple tasks . . . Our major conclusion is that pure forms of either type of knowledge are seldom, if ever, exhibited and that it is the relationship between the knowledge types that gives one's knowledge the power of application in a wide variety of settings." [Silver, 1986, p 183]

The means of representation and of activity are possibilities for the learning subject to "reconcile" procedural and conceptual knowledge and to develop them both in relation to each other The means permit the subject to connect procedural and conceptual knowledge because they contain exemplary local aspects of the concept in question These means are "theoretical" *tools* which at the same time permit the subject to experience *knowledge* when applying and using them operationally. [cf. Douady, 1984, and Otte, 1984b]

This idea of a permanent interplay between procedur and conceptual knowledge which is to be developed a controlled by the means of representation and of activit produces a *systemic* characterization of the epistem logical nature of school mathematical knowledge



In this epistemological triangle, the processes of learnin and understanding are interpreted in such a way th changes and extensions in the interrelation betwee symbol/sign and object/problem will emerge in th course of time. Signs and operational procedures becom more extensive, more structured and varied, and, in rea tion, the domain of objects and problems is simult neously enlarged and modified. In addition it must b remarked that "objects" cannot any longer be conceive of as *empirical* objects; they have been transformed int *theoretical* objects in the course of their developmen through teaching. This implies, in particular, that it is no simply the empirical quality which is interesting for mathematical investigation, but the hidden potential rela tional structures in and between objects that are funda mental and have to be established.

The development of mathematical knowledge as th extension, differentiation, and growth, of the varieties o representational structures and domains of objects mean that—comfortably or not—meanings together with proce dures always play a fundamental role. When beginning t learn new knowledge, when trying to enter new struc tures, it is necessary consciously to include possible mean ings and to reflect on them When the process of under standing advances, when the first structural contexts ar revealed, both on the side of signs and models and on th side of objects, it becomes possible to use more and mor routines and economical solution procedures.

Routines are not a precondition for meaningful con ceptual understanding but, on the contrary, can only b the result of such learning processes. Routines are no totally isolated from conceptual meanings. The linkage t meaning in routine is reduced to a "minimum"; some times it is even sufficient if routines refer only to relative abstract structures of objects (or other representational systems) [cf. Kaput, 1987] What characterizes "good" routines is the fact that when confronted with an "alienated" problem structure they can be referred back to broader contexts of meaning in order to vary and adapt the routines according to the new requirements; these routines may then again be "restricted" to narrowly limited and abstract areas of meaning. (This has been in principle the "effect of learning" the teachers of our project have experienced when trying to solve the alienated arithmetical problems.)

The analysis of the problematic relationship between routine and meaning in mathematics teaching in the frame of considerations concerning the introduction of the decimal number concept in the classroom, together with the observed difficulties in actual mathematics teaching, has produced an *"inversion"*: routine is not the basis for meaning but, conversely, a certain amount of meaning is involved in every learning process at any point in time; the routine work can only be the result of the process of learning

For mathematics teaching and its actual concrete problems, these reflections cannot provide direct help or a pat solution; rather, they raise new questions and problems The conclusion which must be drawn is that it is in principle impossible to convey theoretical knowledge directly to pupils, or to base theoretical knowledge and its meaning solely on routines and formal rules The actual process of learning mathematics is governed by specific epistemological and social conditions of the institutionalized teaching processes; developing and learning knowledge does not take place in a linear way, it is much more complex than teachers often suppose. In discussions with mathematics teachers it is very important to emphasize that even in mathematics teaching with its logically structured knowledge, the subject matter cannot be learned in a direct and linear way The teacher must become aware of his basic limitations with regard to the possibility of directly influencing the pupil's learning, limitations which are dependent on the theoretical nature of the knowledge in question and on the specific constraints of interactive social processes. Taking into consideration these fundamental limitations, the teacher should provide opportunities for learning which require the use of means of representation and of activity in such a way that pupils may learn the knowledge by actively reconstructing mathematical relations. The teacher must learn to restrain himself; he is not the central person who can dominate and determine the learning process directly and in every detail. The social obligation for teachers to teach must not be interpreted too narrowly, otherwise the nature of school mathematics will gradually change into a body of rules and procedures.

4. Some consequences for the teacher's professional knowledge

The analysis of the relationship between routine and meaning in mathematics teaching has the consquence that mathematical knowledge cannot be conveyed in the form of elementary elements and simple operational rules as a prepacked product for the pupil. Mathematical knowledge and its meaning is constituted in a twofold *relational form*: an *objective* relation between symbolic representations and intended objects which must be established by the learning *subject* Theoretical knowledge cannot be presented, it must be reconstructed. This reconstruction requires meanings at all levels of development together with operational procedures. Formal rules and routines alone are insufficient.

The difficult relation between routine and meaning discussed here with respect to the problems of teaching and learning mathematics can also be "applied" to the relation between didactical research and the mathematics teacher's practical work. Here, too, specific theoretical knowledge is exchanged between research and practice. This knowledge contains, for instance, interpretations of the nature, amount, structure and frame of (school) mathematics, and also knowledge concerning mathematical teaching/learning processes and their difficulties. The knowledge treated in this cooperation between theory and practice is much more complex and multi-layered than the school mathematics which in a certain sense it comprises.

As in the case of our reflections about the relation between routine and meaning, it seems to be consistent that the exchange of knowledge between research and teaching practice cannot be conceived of as a linear process either, a process in which concrete recipes, methodological rules or procedures for teaching are offered to the teacher to help him solve practical problems in the classroom Didactical research cannot invent, construct, analyze and evaluate optimal teaching recipes and rules which automatically solve all commonly occurring teaching problems Didactical research and the practice of the teacher are relatively autonomous domains with appropriate frame conditions and different goal perspectives. The exchange of knowledge between these two domains can basically only be organized as an autonomous reconstruction in the respective framework of professional activity Didactical theory is not an area of knowledge hierarchically superior to the practical wisdom of teachers, from which they immediately derive useful rules for teaching by concretization, elementarization and explication. It would be a mistake to require didactics to supply directly workable procedures for teaching as this would be in conflict with the complex theoretical nature of mathematical knowledge. Didactical research cannot provide direct support for every practical teaching problem; as a theoretical discipline it may "only" contribute towards enabling the practice of teaching to help itself in a certain way. This is not a restriction but a basic result of the fundamental problem in the relation between didactical theory and mathematical teaching practice.

What kinds of cooperation and exchange between didactical theory and teaching practice require such a "help for self-help"? The complex professional knowledge of mathematics teachers which is necessary fo teaching processes has a theoretical nature—in a way similar to mathematical knowledge itself, even induced by the latter—and therefore allowance should be made for a distinction between the domain of "objects and problems" and the forms of representing this knowledge. With respect to the theoretical knowledge concerned with teaching/learning situations, the domain of "objects and problems" contains, for instance, the learning processes of mathematics and the problems of teaching, phenomena which have to be "represented" in different manners and models according to the specific circumstances in order for it to be possible to investigate a selected concrete problem.

This fundamental distinction between the domain of real problems and the forms of their representation also implies, in respect of the teacher's professional knowledge, an avoidance of the attempt to explain and judge teaching events only as isolated empirical cases. Basically it permits teaching situations to vary in the sense that a concrete case of teaching can be "generalized" in some respects within the frame of variation offered by the representational system. This possibility to vary and generalize otherwise individual empirical teaching events or isolated personal difficulties of learning and understanding is central to the communication and exchange of knowledge between researcher and mathematics teacher, and among teachers themselves as well.

The common work in our cooperative in-service project PROFORMA concentrated on the development and exploration of such forms of "representation and modelling" of teaching/learning problems as they emerge in the mathematics classroom. On the one hand, such representations refer to problems observable in *actual teaching practice* and on the other hand the models relate to *intended teaching events*. The "representation" of *actual* problems includes, for instance, transcribed teaching episodes, video-tapes of teaching episodes, transcribed interviews with mathematics teachers concerning real questions of learning and teaching, recorded observations of pupils working on mathematical tasks, theoretical representations (codings and modellings of different types for different purposes) of actual teaching and learning situations [cf. Bromme/Steinbring, 1987; Voigt, 1984].

The "representation and modelling" of *intended* teaching situations includes, among other things, teaching materials—as, for instance "task-systems" [cf. v. Harten/Steinbring, 1985], "teaching units" [cf. Wittmann, 1984], and the cooperatively constructed material in our own project. These "models" are not intended to simply express normative wishes and beliefs about mathematics teaching but to take into account experience with specific problems in the actual practice of teaching. This means, for instance, that "models" for intended teaching situations must respect the fundamental difference between theoretical representation and the practical use of such models. Teaching material for an intended teaching situation cannot anticipate every practical difficulty in detail. Teaching material must be flexible and changeable according to different practical requirements; it should be constructed in a modular way on different conceptual levels which will be integrated only in actual teaching/learning situations and in diverse concrete forms.

A variable perspective on individual concrete learning problems in mathematics teaching is an important component of the professional theoretical knowledge of mathematics teachers. The distinction between the domain of an object or problem and its representation or model must also be "generalized" to the knowledge teachers need, which is the result of communication between didactical theory and teaching practice. The elaboration and analysis of potential forms of representing or modelling real teaching problems contributes both to the professional activity of teachers and to that of researchers (with their respective proper goal perspectives). Didactical research cannot produce detailed and ready made "models" for the practical solution of concrete teaching problems. The cooperation between theory and practice must not be understood as a channelling of perfect teaching rules and infallible knowledge; there can only be an exchange between theory and practice about the relational forms in which knowledge is constituted. How these relations will be constructed, justified, reconstructed, changed, adapted or generalized, etc., depends on the respective specific frames, conditions and goals of teaching practice and of didactical research.

The fundamental limitations in the direct influence of didactical research on the actual practice of the mathematics teacher is, among other things, a consequence of the theoretical nature of mathematical knowledge. Schoo mathematical knowledge is subject to the particula appropriate conditions of its teaching/learning processes Even the teaching and learning of mathematical subjec matter as seemingly simple as "decimals" shows, upo closer scrutiny, how limited the teacher's possibilities o directly influencing the pupil's learning are. The produc tive power of didactical research for teaching practic does not lie in elaborate practical rules for teaching, bu rather in the development of variable (i.e. "theoretical" perspectives towards the different domains of "objects" o "problems" occurring in real situations where pupils lear mathematics or in classroom situations where teacher teach mathematics

The theoretical nature of mathematical knowledge and, consequently, of teachers' professional knowledge, i the very reason why there cannot be a clear definition c "theory" on the one side and of "practice" on the othe which asserts that theory is prior to practice and "dete mines" it. Both domains contain theoretical and practic aspects. "We suggest ... that intelligent reflection on th actual and potential relationships between researche and practitioners may be better achieved by locating th nature of both theory and practice residing in each c these communities rather than by dichotomizing them [Brown/Cooney, 1986, p. 1] The relation between theor and practice as two "independent" domains, each co taining its proper theoretical and practical aspects, has t strive for cooperative forms of communication an exchange. Joint efforts with groups of teachers must n be organised according to the "watering-can" model c traditional teacher-inservice training: from above (fro theory) water (new knowledge) is delivered to the teac ers. It is important to develop the forms and materials c communications and common work between theory and practice; this will also be useful for a professional cooperation between teachers themselves which respects the relative autonomy of the specific practical frames of different partners.

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