DIDACTICAL PHENOMENOLOGY: THE ENGINE THAT DRIVES REALISTIC MATHEMATICS EDUCATION

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The theory of Realistic Mathematics Education (RME) emerged from what is now known as the Freudenthal Institute in the Netherlands. It is meant to support the design of instruction consistent with Hans Freudenthal's notion that mathematics is fundamentally a human activity (Gravemeijer & Terwel, 2000). Freudenthal felt that students "should learn mathematics by mathematizing: both subject matter from reality and their own mathematical activity. Via a process of progressive mathematization, the students should be given the opportunity to reinvent mathematics." (Gravemeijer, 1999, p. 158). RME theory itself has grown over the years as a result of numerous design research projects and now includes several instructional design heuristics including guided reinvention, emergent models, and didactical phenomenology. Guided reinvention and emergent models have been substantively discussed in several publications (*e.g.,* Gravemeijer & Doorman, 1999; Gravemeijer, 1999). The intent of this article is to begin a similarly substantive discussion of the didactical phenomenology heuristic by highlighting its power to support instructional design and by exploring the relationships between didactical phenomenology and other design heuristics in order to further articulate the internal structure of RME theory.

Didactical phenomenology

As a design heuristic of RME, didactical phenomenology finds its roots in Freudenthal's (1983) description:

> Our mathematical concepts, structures, ideas have been invented as tools to organise the phenomena of the physical, social and mental world. *Phenomenology* of a mathematical concept, structure, or idea means describing it in its relation to the phenomena for which it was created, and to which it has been extended in the learning process of mankind, and, as far as this description is concerned with the learning process of the young generation, it is *didactical phenomenology,* a way to show the teacher the places where the learner might step into the learning process of mankind. (p. ix).

Gravemeijer and Terwel (2000) further articulate the role of didactical phenomenology in instructional design and it is their description that will form the foundation of my exploration:

> Situations should be selected in such a way that they can be organized by the mathematical objects which the students are supposed to construct. The objective is to figure out how the 'thought-matter' (*noumenon*) describes and analyses the 'phenomenon'. [...] For example, to construct length as a mathematical object students should be confronted with situations where phenomena have to be organized by length. (pp. 787–788).

The 'thought-matter' referenced here is the mathematics that the designer wishes the students to learn. The word 'organize' here means to mathematize (to make mathematical). Simply put, didactical phenomenology tells the designer that an instructional sequence meant to support the learning of a piece of mathematics should be situated in a context that can be productively organized *by students* using that piece of mathematics.

A typical first step in developing a didactical phenomenology for a concept is to investigate its historical phenomenology (which describes how the mathematical idea arose in history) and/or what Freudenthal (1983) refers to as a 'pure' phenomenology in which one describes a mathematical idea in relation to the phenomena that it serves to organize, "indicating which phenomena it is created to organise, and to which it can be extended, how it acts upon these phenomena as a means of organising, and with what power over these phenomena it endows us" (p. 28). Put simply, a pure phenomenology is focused on how the concept is used in mathematics. These kinds of phenomenological analyses can provide inspiration for developing a didactical phenomenology, in which one is concerned with describing how a mathematical idea could emerge in a learning and teaching process as a means to organize phenomena. The distinction between didactical phenomenology and these other phenomenologies is illustrated by Bakker's (2004) project focused on statistics concepts. He explored the historical phenomenology of the median and considered modern uses of the median, coming to the conclusion that a skewed data set could provide a context in which the median would be seen as useful. However, in his instructional design experiment, students did not see this usefulness and instead often relied on the mid-range. Thus, he concluded that more work needed to be done to develop a didactical phenomenology for median. Didactical phenomenology depends upon, but also must go beyond, pure and historical phenomenology.

My goal in this article is to explore how didactical phenomenology can provide the instructional designer with valuable insight into how students might be supported in developing mathematical ideas through a process of progressive mathematizing. I begin by briefly describing guided reinvention and emergent models, as my exploration will emphasize the ways that didactical phenomenology can be used productively in concert with these other design principles.

Guided reinvention and emergent models

Gravemeijer (1999) notes that *guided reinvention* is the first design principle of *RME* and states that, "the goal of the RME research program is to determine how mathematics education can be presented to students to facilitate their reinvention of mathematics" (p. 158). In this sense, guided reinvention functions as a mission statement for any RME design study. As a heuristic, guided reinvention does not give an instructional designer much specific guidance other than emphasizing that, "the idea is not to motivate students with everyday-life contexts but to look for contexts that are experientially real for the students" (p. 158). Thus, the guided reinvention heuristic provides a way to evaluate the appropriateness of a task context by attending to whether the students are able to engage with tasks because the situation is real and meaningful to them. However, the reinvention principle does not help the designer decide *which* contexts or tasks to consider.

In order to develop an instructional sequence to support the guided reinvention of a mathematical idea, the designer first engages in a thought experiment to imagine how the students might reinvent the idea. The result of this thought experiment is a preliminary version of what Gravemeijer (1999) refers to as a *local instructional theory* (LIT). Such instructional theories are local in the sense that they are focused on a specific mathematical idea. A LIT consists of a generalized sequence of instructional activities and a rationale for those activities. The sequence of tasks is meant to be general in the sense that it can be adapted to work in different instructional situations. The need for the instructional sequence to be adaptable is why the rationale is the most important aspect of an LIT. This rationale is a theoretical explanation for *how* the tasks support the students' reinvention process, and it is this explanation that guides the designer (or a teacher) in making the adaptations to the task sequence that are necessary to accommodate different instructional contexts. The primary goal of an instructional design experiment (Kelly, Lesh & Baek, 2014) in the RME tradition is the creation and refinement of a local instructional theory.

The *emergent models* heuristic can be a powerful aide in the creation of an LIT because it provides a tool for conceptualizing how students' informal mathematical activity can emerge from a starting point context and then develop into the more formal mathematics that is the goal of instruction. Specifically, the designer attempts to imagine informal mathematical activity in a starting point context that would anticipate the concept to be reinvented. For example, the designer might imagine students doing informal work with the symmetries of an equilateral triangle that an expert would recognize as involving core ideas from group theory. Thus, an early goal of the research would be to design tasks that could evoke this informal student activity. If successful, this would represent the initial emergence of the group concept as a *model-of* the students' activity. Then, the designer would imagine how this informal activity might evolve to be more general and more mathematically powerful as the students work through a sequence of tasks, until eventually the concept becomes something that they can *use* to engage in more general mathematical activity. This transition from informal activity situated in a starting point context to more formal and more general activity is referred to as the *model-of / model-for* transition. Note that there are two interrelated aspects of this transition: The students' activity advances in terms of the level of generality and the concept transitions in terms of its role in the students' work. When the concept initially emerges in the starting point context, it only functions as a model *for the observer* who can characterize this activity using their formal knowledge of the concept (*e.g.,* the observer can use the group concept to model students' activity as they work informally with the symmetries of the triangle). Later, the concept takes on more of an object-like nature and becomes a model *for the students* that they can use in subsequent mathematical activity (*e.g.,* the students can use the group concept to guide their partitioning of a group to construct a quotient group). In this way, the emergent models heuristic brings structure to an LIT by providing a framework for the trajectory of the guided reinvention process. For a more detailed discussion of emergent models, see Gravemeijer (1999).

Didactical phenomenology: the engine of RME

The guided reinvention heuristic provides the mission statement for the instructional designer while the emergent models heuristic provides structure for the LIT in the form of a trajectory from informal mathematical activity toward more formal activity. It is didactical phenomenology that provides insight into how one might promote the emergence of potentially productive informal activity and how one might promote the transformation of this activity into more formal activity. In this way, didactical phenomenology helps the designer provide momentum to the progressive mathematizing that constitutes the guided reinvention of the desired concept. Here I will illustrate this role of didactical phenomenology by drawing on a design study conducted by my research team, focused on introductory group theory. Note that a much more complete discussion of the creation of this group theory course can be found in Larsen, Johnson and Weber (2013). Here I will include only enough detail to support my goal of illustrating the role of didactical phenomenology. The first subsection will focus on the initial emergence of a concept as a *model-of* students' informal mathematical activity and the second subsection will focus on the transition of this informal model into a model-for more formal mathematical activity.

Emergence of the group concept as a *model-of*

A primary goal of our first sequence of design experiments was to learn how to support students in reinventing the group concept. For a more comprehensive discussion of the design process and the resulting local instructional theory see Larsen (2013). The process began with didactical phenomenology. We tasked ourselves with identifying a context that begged to be organized using the group concept. Fortunately, the founder of RME, Hans Freudenthal (1973) provided a starting point in the form of a pure (mathematical) and historical phenomenology. Freudenthal considered numerous applications from chemistry, celestial mechanics, geometry, physics, and algebra. After discussing these, he considered what they all had in common, arriving at the conclusion that, "groups are important because they arise from structures as systems of automorphisms of those structures." (p. 109). Furthermore, groups can be leveraged as tools to analyze and classify such structures. Consider the equilateral triangle. How can we account for its aesthetic appeal? One way is to observe that its shape is unchanged by any reflection across one of its angle bisectors, any rotation about its center that is a multiple of 120°, or any combination of these. In this way, the group of symmetries of an equilateral triangle provides a useful way to mathematize the aesthetics of this shape.

Drawing on Freudenthal's analysis, we identified the symmetry of an equilateral triangle as a promising starting point context for the reinvention of the group concept. However, we also needed to pose a task in this context that could promote the initial emergence of the group concept—we needed a task for which group theory could be useful in order to evoke informal strategies that anticipated the formal group concept. We settled on the task of identifying all of the symmetries of an equilateral triangle, because addressing it requires determining whether a given combination of two symmetries is a new symmetry or is in fact equivalent to one that has already been identified (which in turn depends upon the development of a meaning for equivalence of symmetries). Thus, it is a task that necessitates a focus on the operation of combining symmetries (which is the operation under which the collection of symmetries forms a group). Therefore, in our first design experiment, we began by engaging two students in the task of identifying and symbolizing the symmetries (six in total) of an equilateral triangle. This gave way to their analyzing each combination of two symmetries (36 pairs in total) to determine which symmetry was equivalent to it.

We expected the students to analyze combinations by manipulating a paper triangle we provided, and we were confident that they would eventually organize their results in the form of an operation table. Then our hope was that this table could become material for further mathematizing, resulting eventually in the development of the definition of group. However, our preliminary LIT contained a significant gap, because our didactical phenomenology was incomplete. While it was easy to imagine that the identity and inverse properties could emerge in some form given that they are readily apparent when examining the operation table of a group, this is not true for the associative property. So, we did not know how (if at all) the task could promote its development. To use the language of Freudenthal, we were unsure how to create a context or task that begged to be mathematized using the associative property. It is here that I come to an observation that anticipates a central point I wish to make in this article: While our initial designer-created context did not seem to beg to be organized using the associative property, the resulting student-created context did.

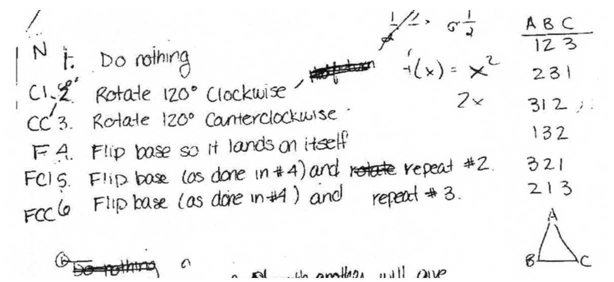


Figure 1. Students' description and symbolization of the symmetries of an equilateral triangle.

The students developed a set of symbols (Figure 1) for the symmetries of a triangle that included composite symbols (*e.g., FCL* represented a flip across the vertical axis of symmetry followed by a clockwise rotation of 120°.)

When they used these composite symbols and tried to analyze a combination of two symmetries, they wrote down expressions like $F\,(FCL)$ that begged to be calculated using associativity, inverses, and the identity property: $F\,(FCL) = (F\,F)\,CL = N\,CL = CL$. With these composite symbols, the set of symmetries of an equilateral triangle provided a context that was naturally and productively organized by a system of rule-based calculations. We capitalized on this opportunity by asking the students to develop (and then reduce to a minimal set) a list of computational rules that was sufficient for calculating all 36 combinations. This task proved to be successful in promoting the initial emergence of the group concept as a model of the students' activity. By this I mean that the students engaged in activity that I, as an observer, could explain (model) using the group concept. As they performed and recorded their calculations, the students were engaged in activity that I recognized as anticipating the group concept without them being aware (at that point) that such a thing existed.

Notice that the context that ultimately proved to be productive for designing tasks that supported the initial emergence of the group concept was one that was *created by the students' own mathematical activity.* It was the nature of the students' choice of symbols, along with their desire to look for shortcuts, that produced a context (a set of symbolically represented combinations begging to be computed symbolically), that in turn evoked informal mathematical activity truly anticipating the formal group concept. Expanding on Treffers' (1987) work on progressive mathematizing, Rasmussen, Zandieh, King, & Teppo (2005, p. 55) anticipated this possibility when they said that, "In more complex cases, progressive mathematizing refers to the fact that students' newly formed mathematical realities, resulting from previous mathematizing, can be the context for additional horizontal and/or vertical mathematizing." This example from the group theory design study suggests that these newly formed mathematical realities constitute material that can (and should) be productively subjected to (didactical) phenomenological analyses as an instructional design study unfolds.

Supporting the transition of the group concept to a *model-for*

In the previous section, I illustrated the role of didactical phenomenology in designing contexts and tasks that supported the initial emergence of the group concept as a model of the students' informal activity. I also introduced the notion that didactical phenomenology can be productively applied to student-created phenomena. In fact, the very nature of progressive mathematizing (key to the emergent models heuristic) makes it crucial for the instructional designer to continuously analyze students' mathematical activity to identify aspects that can be productively mathematized using the concepts students are reinventing. On the one hand, the designer has the goal of supporting the further development of the mathematics, so new design problems emerge. On the other hand, the students' mathematical activity results in the creation of new mathematical realities to analyze in order to solve these design problems. In this section, I explore this idea in more detail as we turn our attention to the transition of the group concept from a *model-of* the students' informal activity to a *model-for* more formal activity.

Let us begin by returning to the student-created context introduced above and conducting a deeper phenomenological analysis. Recall that the students invented a set of symbols that included composite symbols and developed a calculus for computing combinations. Consider the calculation: $F\,(FCL) = (F\,F)\,CL = N\,CL = CL$. Notice that this calculation requires the realization that $FF = N$, but does not require the awareness of the fact that each symmetry has an inverse (and certainly it does not require the explication of a rule stating that this is the case). Thus, while students could be expected to include a rule like $FF = N$ as one of the rules that governs their computations, they should not be expected to include the existence of inverses as a rule (in our many trials, we have found that students rarely do so.). Further, one can calculate all of the symmetry combinations with only the relations $FF = N$ and $CL^3 = N$ in combination with a dihedral relation (like $FCL = CL^2F$) and avoid using inverses of more complex elements (like FCL) in any explicit way. Thus, the students' mathematical activity presented a new instructional design problem: How can students be supported in transforming their implicit awareness of the identity and inverse properties into the kind of explicit awareness that will be needed in order to define the group concept? Fortunately, the new mathematical reality resulting from the students' activity also provided a new context in which to seek solutions by conducting phenomenological analyses.

One specific episode of our first design experiment ultimately drew our attention to an aspect of the students' mathematical activity that begged to be organized using the inverse axiom. When the pair of participating students first formulated their definition of group, they included the condition that the identity element is unique. When asked if they could prove uniqueness, if this assumption was removed from their definition, the students developed an argument that drew heavily on reasoning involving a generic operation table (Figure 2).

The students imagined what the operation table would look like if the identity element was not unique. This resulted in an equation ($sx = sy$) that begged to be simplified by cancellation. In order to justify cancelling, the students relied on the idea that every element has an inverse. Pure phenomenology also suggests that the task of producing a proof that the identity element of a group is unique is a promising one for developing the inverse axiom, since the way this proof is typically done is by invoking such an axiom. However, from the perspective of didactical phenomenology, this task is not ideal because there is no reason to expect this assumption of uniqueness to emerge from students' activity of developing a system for computing combinations of symmetries (since this is not necessary or useful when doing computations). It does sometimes emerge (as it did for our first pair of students) because students feel it is an interesting fact worth recording.

Fortunately, a closer look at the students' reasoning revealed another assumption that *can* be expected to emerge consistently and similarly begs to be proven using the inverse axiom. The students' argument was based on the idea that a symmetry (the s in Figure 2) could not appear twice in a single row of the operation table. This is a property that students invariably notice and then use to reduce their workload (say by filling the last box in a row with the only remaining symmetry) or to catch errors (say by rechecking their work when a symbol appears twice in a row). Students commonly refer to this property of the table as the *Sudoku property*. (This property typically also includes the statement that each element appears *at least once* in each row. I will not discuss this statement here, but it is also related to the inverse axiom.) Notice that a proof of the Sudoku property would essentially mirror the proof that the identity element is unique. Specifically, an attempt to prove either statement will place the students in a situation where they would like to be able to cancel an element from both sides of an equation. Performing such a step is then typically done using inverses and can be justified by citing a rule explicitly guaranteeing the existence of inverses.

Drawing on this analysis, we designed a task in which students are asked to *prove* that the operation table for the symmetries of a triangle must satisfy the Sudoku property. Specifically, students are asked whether they can efficiently prove this property using the set of rules that they developed for computing combinations. They are told that they may add an additional rule to their list if needed. Here the phenomenon to be organized is the students' set of rules for

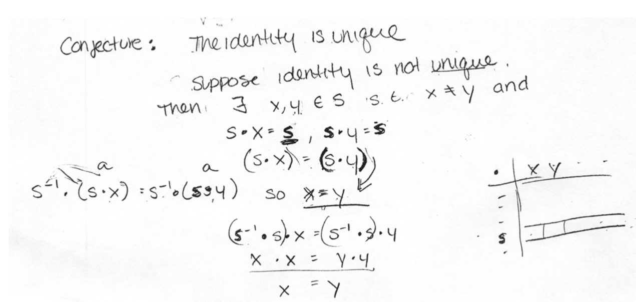


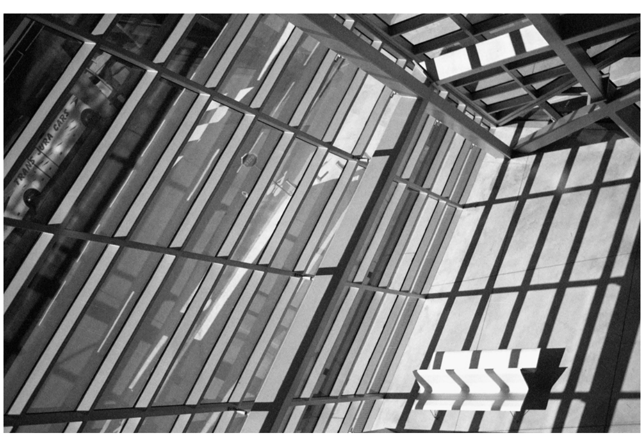
Figure 2. *Students' proof that the identity of a group is unique.*

computing combinations of symmetries, along with their assumptions about how this system of symmetries works (*e.g.,* the operation table satisfies the Sudoku property). It is the specific task of using (and reflexively refining) the rules to prove the Sudoku property that transforms this into a phenomenon that begs to be organized using cancellation (and in turn an explicit statement that each symmetry has an inverse). This further mathematizing of the students' informal activity represents an important shift toward more powerful formal activity because it can support the formulation of formal definition of group. In the language of the emergent models heuristic, it is a key step in the transition of the group concept from a *model-of* students' informal activity to a *model-for* more formal mathematical activity.

Discussion and conclusions

My goal has been to explore the role of didactical phenomenology within RME instructional design theory. I have argued that didactical phenomenology can provide actionable guidance to an instructional designer hoping to support students in reinventing mathematical concepts, especially in conjunction with the emergent models heuristic. First, didactical phenomenology helps the designer create contexts and tasks that can promote the emergence of targeted mathematical ideas as *model-of* students' informal activity. This was illustrated in the group theory context with the initial selection of the context of geometric symmetry and the ongoing analysis of the students' mathematical activity to identify the didactical potential of their composite symbols and their desire to compute combinations. In each case our analysis allowed us to pose tasks that begged to be addressed using informal strategies that anticipated the group concept. Second, didactical phenomenology helps the designer create tasks that can promote the transition of these informal ideas into *models-for* more formal mathematical activity. This was illustrated in the group theory context with the identification of the Sudoku property as an aspect of the students' mathematical activity that begged to be further mathematized using the inverse axiom in order to support students in reinventing and defining the formal group concept. In these two ways, didactical phenomenology can provide momentum within the framework of the emergent models heuristic and thus functions as the engine that drives RME-based instructional design.

I close by emphasizing the point that didactical phenomenology works best in conjunction with engagement with students. Of course, this is not a new idea within the RME tradition. The earlier example from Bakker's (2004) work illustrates the important idea of working with students to evaluate hypotheses based on historical or pure phenomenology in order to develop a didactical phenomenology. Here I further emphasize the importance of learning from students by promoting the idea that the students' mathematical activity creates new mathematical realities that can be productively subjected to (didactical) phenomenological analysis. In this way, the students' evolving strategies and solutions support not only their further mathematical activity but also the designers' activity by continually providing new contexts to explore in order to identify tasks that can provide momentum to the reinvention process.



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