INDICATORS FOR THE DEVELOPMENT OF NOTICING: HOW DO WE RECOGNIZE THEM?

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Mathematics educators recognize *noticing* as a component of mathematics teaching professional practice. Noticing is understood as a set of processes such as identifying and interpreting relevant aspects in teaching situations to justify what to do next. Researchers have conceptualized this skill in the context of improving the relationship between theory and practice in teacher education, considering noticing as a knowledge-based reasoning process about teaching. While there is consensus about its relevance, there is less consensus on how to recognize and describe its development.

Here, I would like to underline some ideas in relation to noticing and its development. First, that learning in mathematics-teacher-education classrooms is understood as a change in how prospective mathematics teachers notice and talk about teaching situations. Second, the changes in what and how prospective teachers notice and talk about practice are understood as a process of appropriation of scientific knowledge about teaching as a semiotic tool. And finally, that the progressive appropriation of scientific knowledge is reflected in how prospective teachers' practical arguments are improved. I have organized this article as follows. I start by framing the meaning of noticing as a shift of attention. Next, I discuss some ideas about learning from a sociocultural perspective and how we can use the notions of conceptual tools to understand the development of noticing. Then, I describe the meaning of learning as a change in the discourse which can be framed by the notions of eliciting and improving practical arguments.

Noticing

Mason (2002) focuses our attention on the relevance of what is observed in a teaching situation, and argues that only when we notice something, do we give it attention. The process of making familiar what is initially not familiar is linked to the development of awareness about the details of a teaching situation (*e.g.,* in Brent Davis's article in this monograph, he uses the notion of *impasse* to reflect on making the unfamiliar familiar, under the metaphor 'what an expert needs to know to think like a novice'). In the development of awareness, noticing can be considered as three processes: *identifying* (describing and attending to) what is important in a teaching situation; *interpreting* (labeling) the given situation, making connections between specific aspects of the classroom and broader concepts and principles; and, *deciding* (justifying) what to do next (see Figure 1).

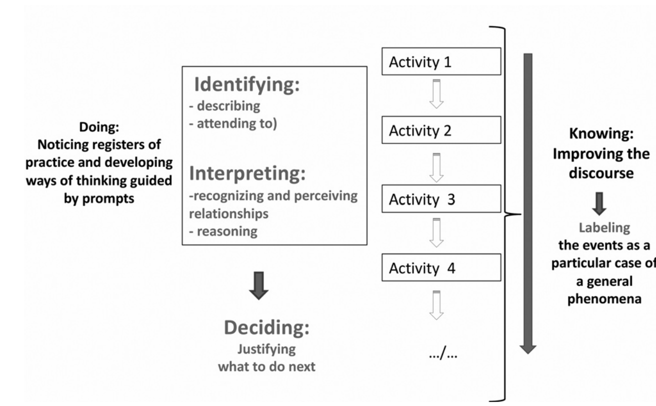

Figure 1. Mental processes and discourse in characterization of noticing.

From this characterization, we can unpack the process of identifying in two mental processes, one of them is *describing*, when prospective teachers talk about everything that they 'see' in the register of practice, and the other is *attending*, when prospective teachers choose (or the teacher educator proposes) some aspects of the register of practice (focal points) to talk about. Regarding the process of interpreting, we can unpack it in three mental processes. One of them is to *recognize relationships* such as becoming aware of sameness and difference or other relationships; the second one is to *perceive properties* when we become aware of particular relationships as an example of a general aspect. And finally, *reasoning* on the basis of agreed properties providing a series of reasons that can be viewed as premises to conclude in an action. Mason (2002) suggests that to reach an agreement when noticing a situation, it is helpful to get agreement about the thing to be analyzed. Therefore, Mason suggests distinguishing between giving an account-of a situation and generating an accounting-for it. So, while an account-of describes as objectively as possible the situation (recognizing that any description is a construction of the observer, from an enactivism perspective), the accounting–for introduces an explanation and theorizing. Account-of and account-for are forms of discourse through which we can infer the mental processes that shape noticing.

Learning: using the notion of tool to understand the development of noticing

We can consider learning as a change in how prospective teachers notice and talk about teaching situations. These

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changes can be understood as a process of appropriation of existing knowledge about teaching. So, noticing can be understood as knowledge-based reasoning that helps prospective teachers identify relevant aspects in a mathematics teaching situation, giving them sense to justify what to do next. In the appropriation of knowledge about practice, prospective teachers can take the knowledge as conjectures to be tested and modified when they are in situations of real teaching. Two ideas emerge from this approach: changes in the discourse about teaching situations are linked to the notion of reconstruction of practical arguments, and the appropriation of scientific knowledge is linked to the notion of semiotic tools.

From a 'learning from practice' perspective, we need to differentiate the role played by both scientific knowledge and the structure of the discourse when prospective teachers talk about practice, that is to say, when prospective teachers interpret the situation (recognizing relationships, perceiving properties and reasoning on the basis of agreed properties) to decide and justify what to do next. Here, *practical reasoning* describes the more general and inclusive activity of thinking, forming intentions and acting, while *practical argument* is the formal elaboration of practical reasoning. Fenstermacher and Richardson (1993) argued that we reason about our actions in relation to what we want to accomplish. To account-for our actions, we might set out our reasons in such a way that the inquirer learns from us what we are trying to accomplish, why we choose to act the way we do, and how the action we take fitted the goal we sought to accomplish. This explanation is a practical argument, "in the sense that it lays out a series of reasons that can be viewed as premises, and connects to a concluding action" (p. 103). The purpose of engaging prospective teachers in the reconstruction of their practical arguments is to encourage and sustain them in the process of improving discourse on their practice. That is to say, helping prospective teachers to transform their practical reasoning into practical arguments by inserting appropriate scientific knowledge into their discourse. In this context, the meaning of practical reasoning can be understood as a first step in the development of noticing, in which prospective teachers think about the events in a teaching situation before inserting in their discourse elements from scientific knowledge.

In the reconstruction of practical arguments, the appropriation of scientific knowledge is linked to the notion of semiotic tools (Wells, 1999). From this perspective, we as teacher educators can organize our work around ideas from a particular domain of mathematics knowledge for teaching. This knowledge is a 'semiotic tool', the internalization of which enables prospective teachers to think powerfully about a whole range of phenomena in mathematics teaching. The change in practical arguments is described as shifts from initial practical reasoning to begin to integrate the scientific knowledge in their discourse to improve their practical arguments.

Knowing and doing in the development of noticing

In a Spanish context, teacher educators provide prospective teachers with activities in which they must notice teaching situations, share and discuss different alternatives, and consider relationships between different evidence and their inferences. When prospective teachers are presented with a sequence of activities in the teacher education program, they engage in doing, thinking about different registers of practice, and developing knowledge-based reasoning. These sequences of activities help prospective teachers to integrate mathematics knowledge and mathematics knowledge for teaching. So, prospective mathematics teachers can identify the relevant aspects and interpret them, labeling them as particular cases of a general phenomenon, and deciding and justifying what to do next. Wells underlines two features of using these registers of practice,

> the first is the creation of a form of argumentation that *integrates the activities of doing and thinking within the same clause.* And the second, which provides the linguistic means for the first, is the progressive use of nominalization to represent, not simply the objects under investigation, but also their attributes and the processes in which they are involved and, finally the mental processes through which the phenomena are interpreted (1999, p. 64, italics added).

In this quote, the word 'nominalization' has the meaning of 'labeling' according to how we are using it to characterize noticing.

The use of practical registers to develop noticing allows prospective teachers to revisit, several times, evidence from practice—sharing what they identify and their interpretations. For instance, when we use primary students' answers to several problems to show differences in their cognitive development, the registers of practice (of student work) that

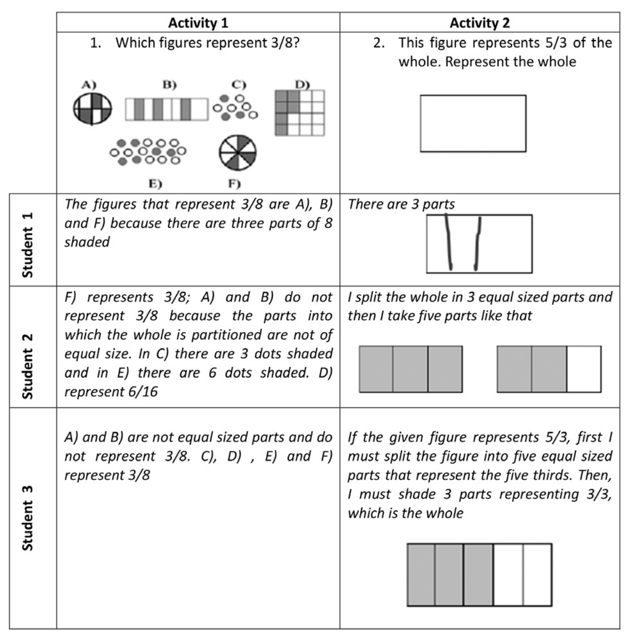

Figure 2. Example of a register of practice—a task with students' answers to several problems showing different students' cognitive development (Ivars, Fernández, Llinares & Choy 2018).

we choose are fine-tuned towards the mathematical knowledge for teaching under consideration (in this case specifically, knowledge of mathematics and students) (Figure 2). Noticing the students' answers means identifying the relevant mathematical elements and generating explanations (inferences about students' understanding of the mathematical concept to decide and justify what to do next).

Prospective teachers are provided with questions to help them to shift from accounts-of to accounting-for the situation. This shift in the attention of prospective teachers is supported by recognizing relationships, perceiving properties and reasoning with them as a means to become aware of particular relationships as instances that could hold in other situations. So, this shift is linked with the noticing that allows prospective teachers to go beyond the details. An example of the guidelines provided in the tasks such as the one presented in Figure 2 is:

> Describe in detail what you think each student did in response to each problem.
>
> Indicate what you learn about students' understanding related to the comprehension of the different mathematics concepts implicated.
>
> If you were a teacher of these students, what would you do next?

This type of task helps prospective teachers to interpret students' answers and to learn to generalize from these interpretations. Thus, prospective teachers have the opportunity to identify and interpret before deciding and justifying what to do next. From this approach, we assume that it is possible to develop noticing when prospective teachers analyze practical registers to identifying relevant elements, interpreting and labeling them as instances of general phenomena to justify what to do next. In this cycle, in which registers of practice are analyzed to develop ways of thinking, we can recognize a connection between doing and knowing when the discourse is improved (Ivars *et al.,* 2018). This approach assumes that teaching entails not only taking action in the classroom, but also developing practices for identifying and interpreting features of mathematics teaching.

The learning situations we use are contexts that prospective teachers can recognize and in which the emphasis is put on the practical use of knowledge to inquire into practice registers. For example, identifying and interpreting students' mathematical thinking, or anticipating possible students' answers in a context of planning a lesson (Llinares, Fernández, Sánchez-Matamoros, 2016). The relationship between doing and knowing illustrates the bi-directional links between experience with practice registers and noticing, when considering the practical register as a particular case of a general phenomena. Here, the successive cycle of different activities in teacher education programs can promote further awareness and reflection in prospective teachers (Figure 3) [1].

The tasks designed by us, as teacher educators, are tools that prospective teachers use and that shape the ways they think about the situation. In the same way, theoretical information can be considered as a semiotic tool in the discourse with others (Wells, 1999). The progressive use of theoretical information as a semiotic tool for identifying relevant aspects and interpreting the registers of practice, labeling events as particular examples of a general phenomenon, can be seen as evidence of the improvement of prospective teachers' discourse and as a manifestations of the appropriation of knowledge. Here, the theoretical information can be considered as an artifact of the teaching culture that serves as a tool for achieving the goals to which their activities are directed (Wells, 1999).

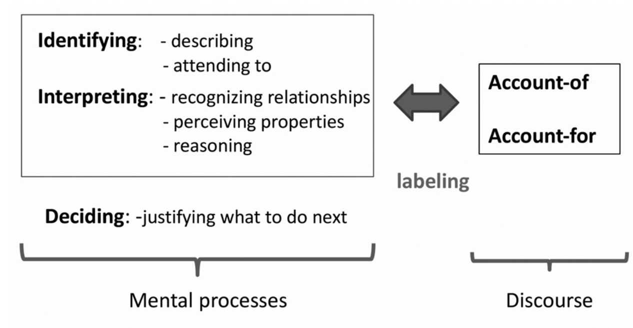

Figure 3. Relationships between Doing and Knowing through cycles of noticing the registers of practice.

What we emphasize here is the activity of knowing through using these different types of tools (the scientific knowledge as theoretical information and the kind of task designed by teacher educators). Here, we consider that the doing (the action) is mediated by semiotic tools, in this case the theoretical information as used in the discourse that, at the same time, is evidence of knowing. So, in this cycle of activities emphasizing the relationships between doing and knowing, prospective teachers have the opportunity to build personal knowledge when they shift from accounts-of the situation to accounting-for it. When these shifts are stable (evidenced / consistent) through different situations they can be considered as evidence of the development of noticing.

The development of noticing as changes in the discourse

We can characterize the development of prospective teachers' noticing as a shift in the attention manifested in their discourse. This approach emphasizes the complementarity of talk and the text as has also been showed in learning environments using blending approaches face-to-face and online modes (Llinares & Valls, 2010). The analysis of registers of practice puts prospective teachers in place of mastering the discourse in which knowledge is constructed and used (Wells, 1999).

The analysis of the prospective teachers' activity in different cycles has allowed us to generate some indicators of the development of noticing (Ivars *et al.,* 2018, 2019). These indicators characterize the shifts of attention evidenced in prospective teachers' discourse by how scientific knowledge (as a semiotic tool) is used. However, we have to recognize that the text produced by prospective teachers does not constitute knowledge, but as Wells (1999) claims "[texts] do mediate the activity of knowing, when engaged with intentionally by those who are equipped to use them. In this respect, they [the texts] can be evaluated as more or less effective as means for achieving the purpose for which they are used" (p. 77). In this sense, knowing and knowledge are in dialectic relation since the discourse is both process and product.

I illustrate these features using data from Ivars' research (2018) in which a group of prospective primary teachers have to notice several primary students' answers to activities with fractions (Figure 2). Particularly, when prospective teachers talk about the students' answers to Activity 2 (*given the picture of a rectangle as a representation of 5/3, represent the whole*). The understanding of the fraction concept that helps to solve this activity is unpacking $5/3$ as 5 times $1/3$ and recognizing that the whole is shaped by 3 times $1/3$. Two characteristics of this activity are relevant. Firstly, that the fraction ($5/3$) is greater than the whole and, secondly the use of a geometrical figure to represent this fraction. The meaning of fraction concept here leans on the schemas of partitioning in parts of equal size (assuming that the shape of parts can be different). The data that constitute the register of practice are primary students' answers to this activity showing different features of students' mathematical thinking about the fraction concept. The two cases described here come from prospective teachers' written answers to this task and illustrate changes in the discourse showing differences in the development of noticing.

Case 1

Regarding the activity

The student must consider three things, the denominator as an iterative unit to form other fractions, that the parts must be equal sized and that the whole must be also equal.

Regarding students' answers

Student 1—The student has not taken into account the denominator as an iterative unit and that the parts of a fraction must be equal sized. The student does not understand that the parts must be of equal size, even if they have a different shape.

Student 2—The student takes the denominator as an iterative unit to form other fractions and considers that the parts of a fraction have to be equal sized. However, he has not taken into account the inverse relationship between the number of the parts in which the whole is divided and the size of each part.

Student 3—The student has taken into account that the parts have to be of equal size, that the more divisions of the whole, make smaller parts , and that the denominator of a fraction is an iterative unit to build other fractions [...] the student has acquired the concept of fraction and its characteristics. He knows how to represent them. Now he should start performing formal algorithms.

Regarding what to do next

Student 1—Learning objective: Student must understand that the parts of a fraction must be of equal size.

Instructional activity: This cake represents $4/4$ [a rectangle is shown]. Represent $2/4$ (considering that all people who eat this cake (4 people) should eat the same).

Student 2—Learning objective: Divide a whole into parts that are increasingly smaller.

Instructional activity: Divide this cake in $1/2$ [a rectangle is shown]. Now divide it in $4/5$. Now in $6/7$. Now in $9/9$.

Student 3—Ana eats $8/9$ of cake and Pedro $5/6$ of another. How much cake have they eaten in total? Use the colored strips [Cuisenaire rods] to solve it.

The practical argument of this prospective teacher seems to be constructed of pieces of knowledge about fractions (about the activity, about the students' understanding and about what to do next), but without having a clear connection with the relevant aspects of the register of practice of the students themselves. It seems that she does not recognize the key element of the Activity 2 (unpacking $5/3$ as 5 times $1/3$). Thus, she generates a discourse without a clear focus but using some concepts and relationships. In the activity, this prospective teacher seems to identify as relevant the 'denominator of a fraction', but without recognizing that the activity already provides a representation of the fraction $5/3$. The general elements associated with Activity 2 are then used to talk about the students' understanding (interpreting) and about what to do next (deciding). We can infer that this prospective teacher takes into account some theoretical ideas but without a clear link to the specific aspect of the situation. The prospective teacher seems to use some concepts at her disposal to describe the students' answers and infer information about their understanding (interpreting). Nevertheless, not identifying the key mathematical element of the activity implies that she generates a discourse on students' understanding which is not specific to the situation. How this prospective teacher talks about the three students shows the key role played by being able to attend to and describe relevant aspects of the situation to generate a coherent discourse as well as the role played by mathematical knowledge. Not attend to the process of unpacking $5/3$ as 5 times 3, when $5/3$ is already represented and is asked to represent the unit could justify her decisions about what to do next when she has to consider this key element (particularly, when she decides what to do next with students 2 and 3 when the specific aspect that should be attended to had not been identified).

Case 2

Regarding the activity

Activity 2 consists of representing the whole from the representation of an improper fraction ($5/3$) in a continuous context. The mathematical elements involved are to identify and iterate one part to construct another fraction ($1 = 3/3 = 3$ times $1/3$); and that the parts must be of equal size (*here you have to divide the given figure into 5 parts, and take a part as an iterative unit*) [italics added].

Regarding students' answers

Student 1—The student confuses the name of the part in relation to the whole with the number of parts in which the figure is divided. With its representation he shows that he does not understand that the parts must be of equal size, and that he must identify and use a part as an iterative unit.

Student 2—Student recognizes that he must divide the whole into equal sized parts, but instead of considering the given figure as $^5/_3$, he considers the given representation as $^3/_3$ (whole) and then he represents $^5/_3$. That is, he performs the problem backwards. This shows that he does not understand that he must identify the fraction ⅓ from the representation of $^5/_3$ to use it as an iterative unit to reconstruct the whole.

Student 3—He divides into 5 congruent parts the given figure to recognize $^5/_3$, and then he identifies $^3/_3$ to show the whole ($^3/_3$) requested. This student understands the need to divide the given figure into five equal sized parts and represents the whole iterating the fraction ⅓. This shows that he understands that a part can be used as an iterative unit (he recognizes and iterates ⅓ to get $^3/_3$).

Regarding what to do next

Student 1—Learning objective: To understand that the parts in which the whole is divided must be of equal size even if their shape is different.

Instructional Activity: Divide a sheet of paper into halves in different ways establishing the focus of the group's discussion in justifying why, although its form is different, the parts are of equal size. The idea to achieve is that the parts are of equal size because if they are repeated twice we always get the whole.

Student 2—Learning objective: Recognize that a part can be divided into other parts in a discrete context and consider a group of parts as a part.

Instructional activity: Indicate ⅔ of the following set of chips: ◯◯◯ ◯◯◯ ◯◯◯.

The student must identify the fraction ⅓ and iterate it twice to get ⅔, and finally, he has to name the 6 chips as ⅔ of the whole. If the exercise is very complex, we can start identifying ⅓ of the 9 chips.

Student 3—Learning objective: Solve simple arithmetic problems with help.

Instructional Activity: Using the color rods, add ⅓ + ½. What rod would you take as the unit?

Students at this understanding level can solve additions and subtractions with different denominators if they have a guide that allows them to identify the whole (unit). With this problem we focus our attention on the need of obtaining a common denominator with the help of the guide.

This prospective teacher identifies the relevant mathematical aspects of the activity which she transforms in focal points to support her accounting-for (particularly, in her report of Students 2 and 3). Being able to attend to the key mathematical element in this activity ("have to divide the given figure into 5 parts, and take a part as an iterative unit") points out the difference with Case 1 and underlines how the accounting-for depends of the features of accounting-of (what is analyzed depends on what is observed). The explanation of a student's answer lays out a series of reasons (the relevant mathematical elements in the students' answers) that are viewed as premises, and connects them to a concluding action, in this case prospective teacher inferences about students' understanding and on what to do next. Therefore, we can take this discourse as more coherent in the use of theoretical elements which can allow us to link the process of appropriation of scientific knowledge to the generation of a network of elements and relations. In this case, the distinction between Case 1 and Case 2 displays differences in knowing and therefore in the development of noticing.

Focusing on the features of the discourse to recognize the development of noticing, we have characterized four levels of development:

Level 1—prospective teachers describe some parts of the register of practice; *e.g.,* a video-clip, or different students' answers (accounting-of).

Level 2—prospective teachers refer to theoretical ideas without linking them to specific aspects of the teaching identified (which in some cases can be considered as a rhetorical use of the labels).

Level 3—prospective teachers identify specific aspects of the register of practice and relate them to certain theoretical points (initial labeling). However, the labeling process and how to justify what to do next, can be unstable (inconsistent) in the different activities (the first steps in an account-for).

Level 4—prospective teachers conceptualize their thinking through a process of theoretical reasoning (conceptualization). Prospective teachers make an integrated use of the theoretical information in identifying and interpreting the key aspects of the teaching situation and in justifying what to do next (account-for).

Levels 2 and 3 try to describe the learning process of prospective teachers when they start to appropriate scientific knowledge, considered as a set of semiotic tools. This appropriation of scientific knowledge sometimes is not stable (consistent) among the different situations analyzed (Llinares & Valls, 2010; Sánchez-Matamoros, Fernández & Llinares, 2019) showing the difficulty in generating relationships between identifying (describing and attending to) and interpreting, and the process of deciding and justifying what to do next. Furthermore, the links between identifying, interpreting and deciding/justifying seem to lean on different domains of knowledge (of mathematics, of mathematics and students, of the instruction, and so on). These features support two ideas related to the development of noticing. Firstly, that it takes time and that teacher education programs only can 'plant a seed', and secondly, it underlines the idea of noticing as a form of knowledge-based reasoning (Amador, 2020; Fernández & Choy, 2020).

In this characterization of the development of noticing, Level 4 depends upon mastering the use of theoretical information (as a semiotic tool) to undertake the activities of identifying and interpreting the register of practice, as a mean to understand acts of teaching and learning in practical situations. That is to say, from a sociocultural perspective of learning, Wells (1999) argues that the object of learning

> is not just the development of the learners' meaning potential, conceived as the construction of discipline-based knowledge, but the *development of the resources of action, speech and thinking* that enable the learner to participate effectively and creatively in further practical, social and intellectual activity (p. 48, italics added).

However, this sociocultural approach to prospective teachers' learning recognizes that some type of reduction of attention to relevant theories is necessary, to allow prospective teachers to concentrate on other things. A reduction in what is noticed in particular aspects of a situation is a characteristic of successful learning in relation to that aspect, a phenomenon that has been seen as necessary in the domain of teaching (Korthagen & Kessels, 1999) [2]. The stage of reduction of attention in the process of learning can be difficult to recognize in prospective teachers since this implies recognizing that theories and theoretical elements play a role in the way a teacher interprets a situation, but not at a conscious level (Korthagen & Kessels, 1999). This is a particular issue in a Spanish context, where teacher educators do not observe prospective teachers teaching.

The four levels of development of noticing have been used as a framework to design and/or modify new tasks in our teacher education program, and as a reference to analyze the productions of prospective teachers. The way of categorizing the development of noticing leans on how prospective teachers appropriate scientific knowledge. One characteristic of the process is that prospective teachers' awareness of details in the relationships and properties of a teaching situation is linked to the amount of detail that they were able to identify before they began to interpret the teaching situation (*e.g.,* just from looking at what a task might demand of students). Another characteristic is that in labeling a particular event as an instance of a general property they should link evidence with interpretation (since sometimes prospective teachers only provide general comments or indicate values and judgments). Here, the labeling can be seen as a process of organizing the situation by theoretical categories and can be understood as a way to relate the particular to the general and to construct knowledge through the analysis of several situations. In this sense, labeling provides prospective teachers with better arguments to justify what to do next.

Some final comments

Focusing on the development of noticing in teacher education, as a particular case of how prospective teachers learned about teaching, suggests two ideas. Firstly, it generates self reflexive opportunities for ourselves as teacher educators, which illustrates our learning as university mathematics teacher educators (Brown, Fernández, Helliwell & Llinares, 2020). Secondly, our approach to teacher education integrates theory and practice. This approach recognizes that during the last decades, mathematics education research has generated an enormous amount of knowledge about learning and teaching mathematics useful to the mathematics teachers and relevant to classroom practices. Here, noticing development is conceptualized as an ongoing process of learning about practical teaching with spaces to reflect, with the guidance of an expert.

Note

[1] This sequence of activities is carried out at the university, and it is a way of bringing some aspects of the practice of teaching mathematics into the classrooms of the university. In this context, in our teacher education program, teacher educators do not see their prospective teachers in school.

[2] From an enactivist perspective this need of level reduction has also been recognized in order for behaviors to become automatic (Coles, 2018).

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