INTERPRETING STUDENTS' NON-STANDARD REASONING: INSIGHTS FOR MATHEMATICS TEACHER EDUCATION

MIGUEL RIBEIRO, MARIA MELLONE, ARNE JAKOBSEN

We are mathematics teacher educators in three different countries with a shared interest in understanding and advancing teachers' knowledge. Miguel works in Brazil, Maria in Italy, and Arne in Norway. We are motivated by a social responsibility for developing prospective teachers' knowledge that we, as educators, deeply feel. Miguel's and Arne's experience working with the research group led by Deborah Ball (at University of Michigan), and Maria's research interest in pupils' early forms of algebraic reasoning, led us to choose the interpretation of pupils' answers to arithmetic and algebra problems as a context for joint research. By "answer" we mean all the work produced by pupils when solving a problem, including any associated reasoning, expressed in natural language, pictorial representations, and algebraic expressions.

In the development of our joint endeavor, we started to explicitly discuss and reflect on our own practice and research experience as a transformative process. In particular, the mathematical discussions that took place between us and our students, focused on interpreting and giving meaning to pupils' answers, were a trigger for reflecting on and developing *our own* mathematical knowledge and awareness. In other words, we started to explore knowledge that educators can gain from discussions with prospective teachers and to reflect on how collaboration with prospective teachers influenced *our* professional growth. Our work reinforces the importance of perceiving the group of educators and prospective teachers as a co-learning community (Jaworski & Goodchild, 2006), where each party contributes with their knowledge as a specialist, collaboratively developing new knowledge in practice.

In this article, inspired by Boylan *et al*.'s (2015) reflections on the praxis of mathematics teacher education, we present a teacher education self-study in which, through the narration of our particular experience, we debate the role and importance of focusing on interpreting students' reasoning as a way of developing mathematics teachers' and teacher educators' specialized knowledge. In the first part of the article, we present the design and implementation of a task for prospective primary school teachers essentially based on interpreting students' answers. We also provide an analysis of prospective teachers' answers to this task. In the second part, we discuss the change of focus that occurred in our joint research experience, from prospective teachers' processes of interpreting students' answers to *our* interpretations of prospective teachers' answers during the mathematical discussion organized after the proposed task. Finally, we argue that these interpretation experiences are powerful sources of mathematical knowledge and awareness, both for prospective teachers and educators.

The design of tasks for teacher education: prospective teachers' interpretation of pupils' answers

Since the inception of our joint research, we have agreed that tasks for prospective teachers need to be framed in a practice-based approach (Smith, 2001). We also agreed that tasks for mathematics teacher education should focus on developing teachers' mathematical knowledge and awareness. Miguel proposed that we conceptualize a task to be used in our teacher education classes, in which prospective teachers would interpret and give meaning to some particular pupils' answers to a mathematical problem. We considered this interpretation work and the knowledge involved to be one of the core features of teachers' practice. Our idea was to use the task in the mathematics education courses for which we were each responsible in our respective countries. After some email exchanges and Skype meetings, a refined version of the task was finalized and, after translation, implemented in the three contexts.

The task is in two parts. In the first part, the prospective teachers are asked to solve the "problem" by themselves. The second part asks them to respond to some pupils' answers to that same initial problem. The original task included seven pupils' answers, although here, for brevity, we provide only two:

> Teacher Maria wants to explore some notions about the concept and the nature of fractions with her pupils. With this aim, she prepared a set of problems, one of which is given below:
>
> > *If we divide five chocolate bars equally among six children, what amount of chocolate would each child get?*
>
> 1. Solve the problem for yourself.
> 2. Now, consider the following pupils' answers to the previous problem (see Figure 1).

For the Learning of Mathematics **36**, 2 (July, 2016)
FLM Publishing Association, Fredericton, New Brunswick, Canada

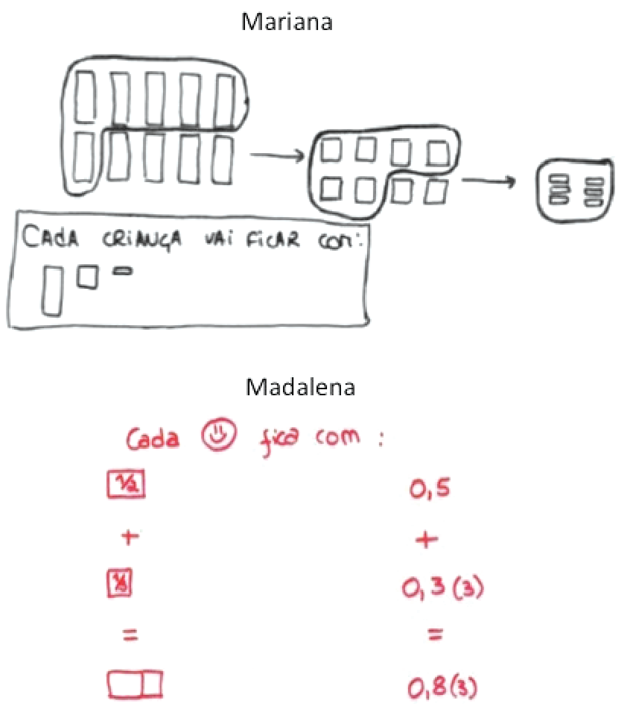

Figure 1. Answers provided by two pupils and included in the task ("each child gets").

i. For each of the pupils' answers, indicate whether you consider it mathematically correct (adequate) or not, and justify the (in)adequacy of the mathematical rationality shown (that you notice).

ii. For the pupils' answers you consider inadequate, define a set of possible questions that you would pose to those pupils in order to support them in making sense of the process and in understanding a correct/adequate answer.

The two answers shown in Figure 1 were collected by Miguel as part of research he developed with primary school teachers and students. These answers were in Portuguese, and during the discussions for preparing the task and selecting the answers to be included, we all agreed to keep the original answers, but to provide additional translation into Italian and Norwegian.

There were several reasons for including the chosen pupils' answers in the task: (i) they were aligned with mathematical topics that could be explored (*e.g.*, fractional representation of rational numbers, sum of fractions with different denominators, recurring decimals); (ii) each answer provided opportunities for navigating through multiple representations (*e.g.*, pictorial, arithmetical, natural language); and (iii) they allowed the (prospective) teachers to make connections within topics and forms of representation (*e.g.*, connections with the history of mathematics, such as Egyptian fractions, or with advanced mathematical topics, like continued fractions).

The goal for the task was to give prospective teachers the opportunity to develop what we have termed *interpretative knowledge* (Ribeiro, Mellone & Jakobsen, 2013). This notion of interpretative knowledge comes from initial results of our research related to the conceptualization of mathematical knowledge for teaching (*e.g.,* Ball, Thames & Phelps, 2008), in which the specificities of the mathematical aspects of teachers' knowledge are taken into account. This interpretative knowledge, being part of mathematical knowledge, is defined as the knowledge that allows teachers to give sense to pupils' non-standard answers (*i.e.,* adequate answers that differ from the ones teachers would give or expect) or to answers containing errors. In this sense, the notion of interpretative knowledge incorporates into the mathematical knowledge for teaching framework the idea that errors and non-standard reasoning (we add) are learning opportunities (Borasi, 1996). Moreover, the content of interpretative knowledge shapes teachers' ability to make informed choices in contingency moments (as defined by Rowland, Huckstep & Thwaites, 2005), in order to respond to and deal with non-planned situations. It corresponds to teachers' knowledge that supports the development of pupils' mathematical knowledge, having as a starting point the pupils' own possible reasoning. This kind of knowledge is certainly linked with the *discipline of noticing* (Mason, 2002). In particular, it encompasses the idea of teachers working "on becoming more sensitive to notice opportunities in the moment, to be methodical without being mechanical" (Mason, 2002, p. 61). It should also allow the shift between hearing and listening that is so crucial in any teacher's practice (Davis, 1997).

Although the three contexts in which we implemented the task were very different (*e.g.*, cultural context, educational systems, teacher education program structures) our aim was not to conduct a comparative study. Our aim was to obtain a broader understanding of the content of prospective teachers' interpretative knowledge, in order to improve our practice as teacher educators. Analysis of the data gave us some insights into both prospective teachers' mathematical knowledge and their interpretation and making sense of students' answers.

Our first insights into the prospective teachers' mathematical knowledge were obtained when they solved the problems by themselves, allowing us to evaluate their common content knowledge—the mathematical knowledge needed in any profession that uses mathematics as a resource (Ball *et al.*, 2008). Complementarily, their interpretation and making sense of pupils' answers allowed us to explore their reasoning when confronted with non-standard approaches to that same initial problem. It also allowed us to better understand the influence of their own solutions (and associated knowledge) on the meaning they ascribe to solutions provided by others.

We found a commonality in the prospective teachers' answers to the problem. Two prototypical solutions were proposed in the three countries: the first one is drawing and dividing each chocolate bar into 6 "equal" pieces (obtaining 30 pieces of chocolate), followed by stating "each child would get 5 pieces of chocolate"; and the second one is to state "each child would get 5/6 of each bar". We can recognize in the first prototypical answer a potential lack of familiarity with fractions and so, a preference to provide an answer using natural numbers. This is problematic given that the task description explicitly states that teacher Maria provided the problem in order to prompt her pupils to work in the context of fractions. The second prototypical answer evidences the problematic management of the unit, or at least a problem with expressing it in natural language, evidence of shortcomings in their common content knowledge.

Although prospective teachers' difficulties in mathematics is a well-known problem already identified in the literature (*e.g.*, Tirosh, 2000), our aim was to understand the possible connection between the content of prospective teachers' mathematical knowledge and their ability to interpret pupils' answers (interpretative knowledge). Our analysis of their comments on the pupils' answers revealed that although most of the prospective teachers tried to interpret pupils' answers, their observations tended to remain at a descriptive and/or evaluative level. This revealed difficulties in interpreting the work of others and elaborating on the processes, reasoning and representations involved. Consequently, their "interpretations" are built upon certain aspects of common content knowledge only.

For example, in all three countries, most prospective teachers indicated that they did not understand Mariana's answer (Figure 1). Some justified this answer by stating "She does not understand fractions—she is just dividing the pieces" or "I don't understand this reasoning. I understand the pupil's aim, but she was neither successful in what she did nor with her reasoning. This solution is incorrect and unclear" [1]. These comments reveal specific beliefs concerning fractions as well as prospective teachers' difficulties in leaving their own space of solutions (given in Part 1 of the task and discussed afterwards) and expanding their reasoning to accommodate alternative approaches to the same problem (Jakobsen, Ribeiro & Mellone, 2014). Indeed, although Mariana's approach can be considered a non-standard one, since it is solely based on pictorial representations and does not involve numbers, her answer was included in the task as it provides a powerful representation for exploring broader mathematical content. In particular, Mariana's answer directly relates to Egyptian fractions, and to other mathematical topics we had not anticipated.

We also noticed that when prospective teachers were confronted with a solution different from their own, most did not question their own knowledge. They failed to perceive the difference in the represented approach to the problem as an opportunity for them to dig deeper into their own skills and knowledge, *i.e.*, as a learning opportunity. Instead, prospective teachers' interpretations and the nature of their feedback focused on showing pupils the prospective teachers' own solution as the one pupils should have submitted as the "correct" answer:

> *PT1:* Mariana's solution cannot be understood, so the first question would be, what does this representation mean? After listening to her answer, I would try to show her my own representation, in order for us to arrive at the solution together.

> *PT2:* I have been teaching for ten years [2] and I find these solutions very confusing. (…) Indeed, I always try to make the visual images as clear as possible and encourage my pupils to do the same. In this case, the reasoning paths are very disorderly and lead to confusion.

In our previous work (Ribeiro *et al.*, 2013), we concluded that such statements reveal prospective teachers' difficulties in leaving their own space of solutions. It could be acceptable in contexts other than teaching, but teachers need to overcome such a biased attitude toward the solutions and reasoning revealed by their pupils. Although PT1 mentioned that she would listen to what Mariana was saying, the option taken after such "listening" ("show her my own representation") can be perceived as listening without actually *hearing* what the student is saying (Davis, 1997), *i.e.*, without giving a mathematical meaning to the student's answer. This also points to the need for teacher education to focus on developing sensitivity, insight, and skills linked to the specificity of teachers' mathematical knowledge. Some may argue that such skills are developed over time and with practice. However, the comment made by PT2 reveals that this is not necessarily the case. Although she is not yet formally a teacher (in terms of the academic degree), she already has ten years of teaching experience, yet still evidently lacks sensitivity and awareness toward alternative mathematical reasoning.

These findings were immensely valuable, as they prompted us to reconsider our own beliefs and practices pertaining to the core aspects of teacher education. It led to our awareness of the need to focus more on developing teachers' interpretative knowledge—a type of knowledge that can enrich their space of solutions—and simultaneously change their "installed" and deeply ingrained beliefs regarding both teachers' and pupils' roles.

Reflections on our own practice as teacher educators

The first phase of our research, summarized above, triggered a kind of disappointment. We felt a need to change our practice as educators, as Maria expressed in one of the emails exchanged during the three years we have been working together:

> I start to feel that I am not developing my work as a mathematics educator in an effective way. For that reason, and looking for ways that can help me improve my own practice, I decided to observe my own teaching during this next course, where we also will be implementing the task. Indeed, Miguel's experience of audio and video recording his classes prompted me to do the same this year. This material will allow us also to analyze and reflect upon our own practice as we continue our collaboration.

This decision was at the beginning of the second phase, where we focused on our own practice when implementing the task. Maria's suggestion was readily accepted, resulting in complementing our focus on prospective teachers' knowledge with a focus on our own practices when implementing the tasks. This complementary focus arose because we felt the need to gain a deeper awareness of our own practice as mathematics educators, with its continued improvement as the ultimate goal. With this in mind, we video recorded the classes of our mathematics education courses in which the chocolate bar task was explored (Sherin, Sherin & Madanes, 2000) and all the prospective teachers' answers were scanned, thus providing data for this new research dimension. These video recordings were conceived not only as data to observe prospective teachers' answers emerging in the discussion of the task, but also as material to analyze and reflect on our own practice.

Our decision to reflect on our own practice underlines the transformative nature of research participation for both prospective teachers and educator-researchers. It highlights, from the teachers' perspective, their "identity renegotiation" and their awareness of the contribution of research to their training. With this new focus, our aim is to start conceiving our courses from an inquiry community perspective, with the goal of promoting the development of a co-learning community (Jaworski & Goodchild, 2006). Indeed, by using such an approach, involving the conceptualization and implementation of tasks similar to the one presented in this article, we, as educators, have been going through a transformative experience, derived from participating in what we consider a co-learning community, first with our trainees (*e.g.,* Wagner, 1997), and second through shared reflections among the three of us.

This transformative experience is, of course, a complex process and involves several considerations. One of the first reflection points pertains to Maria's practice. As previously mentioned, many prospective teachers found understanding pupils' answers challenging during the individual work. After the prospective teachers had solved the task, a discussion was opened to explore pupils' answers together. In particular, the previous analysis of prospective teachers' answers showed their difficulties with Mariana's solution (Figure 1). Hence, this solution was a critical point of mathematical focus during our collective discussions.

In Maria's course, Mariana's answer was projected on the wall and was clearly visible to all participants (around 100 prospective teachers). Maria invited the prospective teachers to comment on Mariana's solution using a microphone. During the discussion, most participants expressed their difficulties in understanding Mariana's reasoning. Nonetheless, as they verbalized their thoughts, and through scaffolding, their understanding of mathematical aspects of Mariana's solution eventually emerged:

Miriam: She basically takes the five bars and divides them into halves; hence, she has ten pieces, and she gives six away, while four remain. Then she divides the four remaining into half again, and then there are eight; and she gives away six, so two remain. The remaining two are divided into three parts, creating six more pieces. Finally, she says that every child will have half of a bar, plus the half of the half of a bar and a third of a bar [pause, voices of PTs who want to intervene].

Maria: Wait a moment, give her time . . .

Miriam: Mmm . . . it's as if . . . a third of the half of the half.

Miriam had not previously understood Mariana's solution (as revealed by the analysis of the response she provided in the written task). However, as she reflected on other participants' comments, considered her reflections during the discussion, and verbalized her reasoning, she managed to understand something that was not clear during the individual work. Most importantly, she was able to recognize in the pieces of Mariana's representation, the particular fractions representing parts of the unit (the chocolate bar) in play.

In the next phase of this class session, prospective teachers were asked to write an arithmetic representation associated with Miriam's explanation. The objective was to allow them to perceive the equivalence between 5/6 and the sum of the particular fractions presented in Mariana's solution:

$$\frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{3}\left(\frac{1}{2}\left(\frac{1}{2}\right)\right)$$

The amazement that most of the prospective teachers felt on discovering the mathematical meaning of Mariana's solution served as a prompt for an interesting discussion. This joint reflection of the process they went through in interpreting Maria's work helped them to appreciate that just because one does not understand something, it does not mean that it is incorrect. This discussion allowed us to explore links between teachers' specialized mathematical knowledge and pedagogical content knowledge, as well as to understand its role and importance in practice. One purpose of the task was to explore and develop prospective teachers' awareness of the mathematical knowledge involved in elaborating constructive feedback, complementing work by Bruno and Santos (2010) on written feedback.

After recognizing Mariana's solution as acceptable, Francesca observed:

Francesca: Yes, I think it is mathematically correct, because there are six children. But if I had seven children, for example, I don't know if this division into equal parts could work. In my view, it was a trial and error process and it succeeded on this occasion. However, I do not know if, with other numbers, it could work.

Maria: So, you are saying that this procedure does not seem to be applicable to other numbers.

Francesca: I do not know. It amazes me, but perhaps it would not work with other numbers.

This comment from Francesca, questioning the possibility of generalizing Mariana's solution, was another useful prompt for reflection and analysis. Such reflection and analysis is grounded in our own beliefs and perspectives about the role of teacher educators in mathematics teacher education. We see research in mathematics education, and its results, as an opportunity to make us stop and think (*e.g.*, Kilpatrick, 1981), allowing possible changes in practice. The task, which aimed at leading prospective teachers to some contingency moments in practice (Rowland *et al.,* 2005), also created a contingency moment for Maria, thus enhancing our own professional development. Maria was not prepared for such a comment and opted not to address it at the time, thus potentially revealing her own weaknesses as mathematician and educator, which was a valuable finding in itself.

The three of us used the video recordings (and the transcriptions) of this session to analyze and discuss various interesting reflections that occurred during the discussion and, in particular, Francesca's issue about the possibility of generalizing Mariana's strategy. The outcome was a deeper awareness of the mathematical potentialities of the task, motivating us to design a strategy to explore it in the following classes. Mariana's solution could be seen as a *progressive parts strategy* (Empson, Junk, Dominguez & Turner, 2006), involving no anticipatory organization of the subdivision. From our perspective, such a solution reflects a peculiar management of subdivisions, with the potential to be generative of a precious mathematical insight. More specifically, Mariana's progressive partitions strategy points to the equivalence between

$$\frac{5}{6} \text{ and } \frac{1}{2}+\frac{1}{4}+\frac{1}{12}.$$

Moreover, it prompted us to explore the possibility of representing uniquely any fraction as a finite sum of decreasing rational numbers, whereby the first is the integer part of the fraction, while each subsequent one is the greatest unitary fraction that is contained in the remaining part, *i.e.*,

$$\frac{n}{m} = i + \frac{1}{q_0} + \frac{1}{q_1} + \ldots + \frac{1}{q_k}$$

something that was well-known in ancient Egyptian mathematics.

Owing to these discussions and reflections (and the knowledge gained through the analysis of transcriptions), we arrived in the following classes much better prepared, as our implementations of the task benefited from the insights we gained. This led to deeper discussions on the mathematics behind pupils' reasoning and representations that seemed, at first sight, straightforward. Such discussions also prompt us, as teacher educators, to develop new complementary elements to be included in our own space of solutions (*e.g.,* Jakobsen *et al.*, 2014).

Some comments and implications for mathematics teacher education

Our intention in this article is to show how working on conceptualizing, implementing, and reflecting upon a mathematical task involving researchers with different experiences, backgrounds, and research interests can contribute to the improvement of both learning opportunities and practice. Of course, as we are the subjects of this research, all our reflections are from an insider perspective (Lampert, 1999).

In this work, we perceived the experience of conceptualizing and implementing the task, as well as the way in which we presented it in this article, from two perspectives. On one hand, we see this endeavor as a contribution to discussions and reflections highlighting the need for more focused work (and research) on the interpretative knowledge required by (prospective) teachers. On the other hand, we also reveal the interpretative knowledge required by educators and, more implicitly, way(s) of promoting this knowledge. In making these assertions, we also argue that the interpretative knowledge required by teachers and by their educators has different nature and content.

In this case, the discussions prompted by Mariana's answer gave us the opportunity to reflect on the development of the interpretative knowledge that both prospective teachers and educators should aim to attain. When conceptualizing the task, and during its implementation, we anticipated a significant number of diverse possibilities and paths for discussion. Fortunately, some unforeseen situations emerged—contingency moments—which we welcomed and approached as learning opportunities. In addition, subsequent reflections and discussions on similar situations we had encountered in our own practice enabled us to develop a broader perspective on the process of teaching teachers, and a deeper and more comprehensive insight into what it requires and entails. Indeed, often when listening to prospective teachers commenting on pupils' answers (both during the implementation of the task and when subsequently analyzing the video recordings), we too had difficulties in interpreting and making sense of some of their reasoning. These difficulties led to some mathematically critical moments (on which we could only briefly comment in this article) that assumed a central role in the development of our own interpretative knowledge. The possibility of sharing these difficulties and dealing with them together further highlighted the importance of our collaboration in facilitating such development. Close collaboration helped us gain a broader appreciation and understanding of the nature of connections within and between topics.

In sum, we would like to highlight the links between the task we conceptualized (type, nature, and focus), the role of research on (prospective) teachers' knowledge and practices, and the learning opportunities we, as facilitators, provided. In order to bring together theory and practice, one must recognize the important role we play as educators in shaping and developing teachers' knowledge, awareness, and ways of perceiving practice. Such a role involves not only our (interpretative) knowledge, but also our beliefs and perceptions of the teaching and learning process. As a result of our shared reflections, we have developed approaches to mathematical topics that we considered relevant for the work of teaching. This relevance takes into account the learners' (pupils' and prospective teachers') potentially different understandings, as well as their different reasoning and representations.

Thus, we argue that if our goal is to enable prospective teachers to give sense to pupils' answers and to provide constructive feedback in contingent moments (*e.g.,* Mellone, 2011), we, as teacher educators, must adopt the same approach in our own mathematics education classes, thereby showing by example this attitude of noticing (Mason, 2002) and listening (Davis, 1997) within our own practice. Although they have different foci, teachers and educators can only develop professionally when recognizing, reflecting, and working on their practice. In particular, we recognize the importance and the need for teacher educators to live and work through transformative experience. To live teaching as transformative experience also means that teacher educators must be willing to admit their vulnerabilities and insecurities and address them. They should not refrain from accepting that they do not always have answers to students' questions. We believe in a total and deep engagement in which trainees and educators experience real and mutual learning together as members of the same inquiry community. It is thus apt to close this paper by citing Radford (2014), who noted, "Teachers and students are in the same boat, producing knowledge and learning together. In their joint labour, they sweat, suffer, and find gratification and fulfilment with each other" (p. 19).

Notes

[1] All quotations from our data in this article are translations.

[2] In Italy it is possible for in-service teachers (precariously employed or employed in private schools) to attend the Master Degree Course for Primary Teachers as continuing education.

References

Ball, D. L., Thames, M. H. & Phelps, G. (2008) Content knowledge for teaching: what makes it special? *Journal of Teacher Education* **59**(5), 389-407.

Borasi, R. (1996) *Reconceiving Mathematics Instruction: A Focus on Errors.* Norwood, NJ: Ablex Publishing.

Boylan, M., Brown, L., Nolan, K., Braathe, H. J., Portaankorva-Koivisto, P. & Coles, A. (2015) Praxis in mathematics teacher education: introduction to a series of short communication. *For the Learning of Mathematics* **3**(1), 39.

Bruno, I. & Santos, L. (2010) Written comments as a form of feedback. *Studies in Educational Evaluation* **36**(3), 111-120.

Davis, B. (1997) Listening for differences: an evolving conception of mathematics teaching. *Journal for Research in Mathematics Education* **28**(3), 355-376.

Empson, S. B., Junk, D., Dominguez, H. & Turner, E. (2006) Fractions as the coordination of multiplicatively related quantities: a cross-sectional study of children's thinking. *Educational Studies in Mathematics* **63**(1), 1-28.

Jakobsen, A., Ribeiro, C. M. & Mellone, M. (2014) Norwegian prospective teachers' MKT when interpreting pupils' productions on a fraction task. *Nordic Studies in Mathematics Education (NOMAD)* **19**(3-4), 135-150.

Jaworski, B. & Goodchild, S. (2006) Inquiry community in an activity theory frame. In Novotná, J., Moraová, H., Krátká, M. & Stehlíková, N. (Eds.) *Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education,* vol. 3, pp. 353-360. Prague, Czech Republic: PME.

Kilpatrick, J. (1981) The reasonable ineffectiveness of research in mathematics education. *For the Learning of Mathematics* **2**(2), 22-29.

Lampert, M. (1999) Knowing teaching from the inside out: implications of inquiry in practice for teacher education. In Griffin, G. A. (Ed.) *The Education of Teachers,* pp. 167-184. Chicago, IL: National Society for the Study of Education.

Mellone, M. (2011) The influence of theoretical tools on teachers' orientation to notice and classroom practice: a case study. *Journal of Mathematics Teacher Education* **14**(4), 269-284.

Mason, J. (2002) *Researching Your Own Practice: The Discipline of Noticing.* Abingdon, UK: Routledge.

Radford, L. (2014) On teachers *and* students: an ethical cultural-historical perspective. In Liljedahl, P., Nicol, C., Oesterle, S. & Allan, D. (Eds.) *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education,* vol. 1, pp. 1-20. Vancouver, BC: PME.

Ribeiro, C. M., Mellone, M. & Jakobsen, A. (2013) Characterizing prospective teachers' knowledge in/for interpreting students' solutions. In Lindmeier, A. M. & Heinze, A. (Eds.) *Proceedings of the 37th Conference of the International Group for the Psychology of Mathematics Education* vol. 4, pp. 89-96. Kiel, Germany: PME.

Rowland, T., Huckstep, P. & Thwaites, A. (2005) Elementary teachers' mathematics subject knowledge: the knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education* **8**(3), 255-281.

Sherin, M. G., Sherin, B. L. & Madanes, R. (2000) Exploring diverse accounts of teacher knowledge. *Journal of Mathematical Behavior* **18**(3), 357-375.

Smith, M. S. (2001) *Practice-Based Professional Development for Teachers of Mathematics.* Reston, VA: NCTM.

Superfine, A. C. & Li, W. (2014) Exploring the mathematical knowledge needed for teaching teachers. *Journal of Teacher Education* **65**(4), 303-314.

Tirosh, D. (2000) Enhancing prospective teachers' knowledge of children's conceptions: the case of division of fractions. *Journal for Research in Mathematics Education* **31**(1), 5-25.

Wagner, J. (1997) The unavoidable intervention of educational research: a framework for reconsidering researcher-practitioner cooperation. *Educational Researcher* **26**(7), 13-22.

---

364. Someone does a calculation in his head. He uses the result, let's say, for building a bridge or a machine.—Do you want to say that it wasn't *really* by a calculation that he arrived at this number? That it has, say, just dropped into his lap, after some sort of reverie? There surely must have been calculation going on, and there was. For he *knows* that, and how, he calculated; and the correct result he got would be inexplicable without calculation.—But what if I said: "It *seems to him* just as if he had calculated. And why should the correct results be explicable? Is it not incomprehensible enough, that without saying a word, without making a note, he was able to CALCULATE?"—

Is calculating in the imagination in some sense less real than calculating on paper? It is *real*—calculating-in-the-head.—Is it similar to calculating on paper?—I don't know whether to call it similar. Is a bit of white paper with black lines on it similar to a human body?

Wittgenstein, L. (1953/2010) *Philosophical Investigations* (4th edition). Chichester, UK: John Wiley.
