Mathematics Teaching and Mathematics Teachers: A Meta-Case Study [1]

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In his presidential address to the 14th annual meeting of PME [2] in Mexico 1990, Balacheff called for a step beyond a psychological approach "in order to understand the nature of the complexity of mathematics learning in a didactical context" [Balacheff, 1990, p. 2]. He suggested that we must find ways to integrate a social dimension into our methodologies basing his argument on two premises. First, "students have to learn mathematics as social knowledge; they are not free to choose the meaning they construct" and second, "mathematics... requires the confrontation of the student's cognitive model with that of other students, of the teacher, in the context of a given mathematical activity" [Balacheff, 1990, p. 2]. In this paper, I set out to take such a step as suggested by Balacheff — through recognising the teacher as a major influence in the process of learning mathematics and attempting to analyse the ways in which the teacher's role is constructed. Since I have a background in case study research, I have conducted a meta-case study stimulated by Balacheff's challenge. The issue my case study sets out to address is the identification and analysis of the positioning of the teacher within a framework of psychology of mathematics education. The data I have collected and used as a basis for my "findings" are the following: the proceedings of PME conferences since 1979; a literature review in mathematics education; some papers from outside mathematics education in the general area of teacher thinking augmented by particular research within knowledge domains other than mathematics; and finally my own research. As far as this latter data are concerned, I have taken the opportunity to reflect upon my previous research endeavours with a view to pulling out what might be said concerning the issue of the present case study. First I will focus on the proceedings of PME conferences and draw out some trends evident from the papers presented over the last twelve years.

Trends within PME

Balacheff's remarks reflect a changing paradigm of work within the community of PME. There is a clear shift in methodology: from experiments identifying children's errors and their possible "remediation" by "treatments", to case studies within more holistic framework; from theoretical perspectives which are limited to the student/mathematics dyad, to those which encompass this dyad within social settings while incorporating a more conscious problematising of the nature of mathematics itself. These changes have brought a richer more authentic thrust to research in PME yet, I suggest, they have also led to some confusion in focus for investigations within the framework of psychology of mathematics education, together with some concern over the validity and generalisability of research findings.

There are two trends evident in the work of PME: a quantitative increase in research incorporating the teacher as an integral — and crucial — facet of learning mathematics, and a series of qualitative shifts as to how the teacher and the teacher's role are conceptualised.

Recognising the teacher

Of the papers included in the published proceedings of the third PME conference in 1979, all but three focused on student understanding of mathematical concepts. Some papers centred on the diagnostic, identifying the kinds of difficulties students encountered, the strategies used to solve particular sets of problems, and the types of errors that arose. Others considered development in mathematical thought either from the standpoint of identifying barriers to further development, or in terms of advocating aids for promoting understanding. If the teacher was mentioned at all, s/he was discussed purely as a facilitator — to dispense facts and information; to identify errors or misunderstandings; to provide materials or strategies to overcome misconceptions; or to promote further mathematical development. Within this body of papers, Skemp's paper [1979] proved to be an exception, describing the dynamic development of the mental processes of the learner as "necessarily related to the goals of the learner; and if a teacher is involved, *to those of the teacher also*" [p. 197, my emphasis]. Skemp also pointed to the possible "mismatch" between the learning goals of student and teacher.

In 1980, the majority of papers again concentrated on student ability, student understanding and representation of specific mathematical concepts and the "attributes", "attitudes" and "processes" of students in the mathematics classroom. As at the 1979 conference, the teacher's role was described in terms of promoting student progress, either on the basis of identified driving or inhibiting characteristics of the students themselves, or by adopting theoretically-derived methods of instruction. However, the teachers themselves featured a little more prominently, at least in terms of introducing ideas of methods to "connect" with students and overcome problems in the classroom [for example Herscovics and Bergeron, 1980; Bergeron and Herscovics, 1980].

In contrast with the dominant theme of most papers in 1979 and 1980, which may broadly be classified as "what makes learning difficult?", Shroyer [p. 331] asked "What makes teaching difficult?" Referring to the work of researchers who have shown an interest in classifying and quantifying teacher thought [for example, Peterson and Clark, 1978], Shroyer herself analysed "critical moments" whilst teaching mathematics and proposed that differences between teachers reflected differences in "teaching styles,

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instructional goals and expectations" 980, p. 336]. Bishop [1980] also argued that, just as children develop their own problem-solving strategies, so do teachers develop their own personal techniques for dealing with classroom situations. He had "no doubt that the teacher was the key person in mathematics education, and research which ignored this fact stood a good chance itself of being ignored" [p. 343].

Within the context of the present review, the 1981 conference papers were notable only in terms of a marked absence of any substantial new discussion of teacher beliefs and, in 1982, despite an increase in the number of PME papers, there was no corresponding increase in papers devoted to the teacher's role in the classroom. However, between 1982 and 1984 a new strand emerged which recognised the influence of teachers' expectations and attributional interpretations of students' mathematics learning and ability [for example, Hoyles and Bishop, 1982; Zehavi and Bruckheimer, 1983; and Romberg, 1984]. Romberg began to identify a more clearly defined link between beliefs and practice. Taking as a conceptual framework the "Model of Pedagogy" [Romberg *et al*, 1979], he proposed that teacher beliefs about schooling, learning and mathematics are mediated by the specific content to be taught, but also reflected in instructional plans, teacher's actions and students' behaviour and performance. He argued that the character of plans could be inferred from the allocation of time and the emphasis given to various content, and observed that teacher beliefs relating to student ability seemed to determine the different types of tasks students were set in respect of content and difficulty. Thus "low" ability students were given "drill and practice" tasks, high ability students "explorations"

In 1985/86 little new can be reported, and indeed the 10th anniversary conference in 1986 appeared to relegate the teacher once again to the position of a passive conveyor of facts and information, albeit using a variety of diagnostic methods and teaching tools. However, a growing appreciation of the differentiated nature of teacher beliefs can be discerned. Brown [1986] suggested that a conception of mathematics teaching consisted of three components: beliefs about mathematics; appropriate goals and tasks for the mathematics classroom; and the relative responsibilities of teacher and students regarding motivation, discipline and evaluation. Although drawing conclusions from an analysis of only a single teacher, Brown highlighted the possible conflict between a teacher's own conception of mathematics and his/her perception of students' needs and interests, to the point where classroom actions might become inconsistent with the teacher's own expressed beliefs about mathematics teaching. This is the first mention of the influence of context on beliefs and the potential mismatch between beliefs and beliefs-in-practice — here framed to suggest that teachers have "true" beliefs which may not be enacted in practice

In 1987, Oprea and Stonewater took up this issue of inconsistency, noting that the relationship between belief systems and instructional practice was far from simple. They argued for a distinction to be made between how teachers thought about teaching mathematics and how they perceived mathematical content. More generally, Cooney and Grouws [1987] identified a number of research issues and themes for the future: "Studies of teacher knowledge and teacher beliefs and especially how they moderate teaching behaviour and student learning are needed" [Vol. III, p. 423].

In 1988, a rather more theoretical analysis was undertaken by Ernest [1988] who, drawing on previous research [Begle, 1979; Bishop and Nickson, 1983] detailed the complex nature of the claimed relationship between teacher attitudes towards mathematics and effectiveness of teaching suggesting that the picture was complicated by the multi-dimensional nature of attitudes towards mathematics. Nonetheless, Ernest concluded after work with student primary school teachers that it was attitude towards teaching the subject rather than towards the subject *per se* which was the most important factor in "determining" teaching style

In 1990, a number of papers from a variety of perspectives continued to point to the influence of the teachers' beliefs, cognitions, and competence on behaviour in the classroom and assessment practices or how the latter threw light on the former [Dougherty; Flener; Grouws *et al*; Schifter, (all 1990)]. This trend continued into the following year, where once again beliefs were reported as influencing classroom practice, [Jaworski; Even and Markovits; Jurdak, (all 1991)] or framing teachers' expectations of students and their treatment of errors [Boufi and Kafoussi, 1991].

The issue of complexity was taken up by Underhill [1990] who concluded that mathematical conceptions were "watered down" as one moved from the mathematics specialists at school division level through the principal to the classroom teacher. He suggested that this "web of beliefs" influenced the actions and statements of novice teachers. This takes me on to a consideration of factors influencing teacher beliefs and how beliefs might be changes

Changing the teacher

Alongside the trend towards "researching the teacher" in PME, a developing interest can be discerned in the teacher's role in mediating curriculum change. 1987 marked the first time interest in teacher beliefs in relation to curriculum innovation was specifically noted, when teachers were viewed as potential obstacles to innovations, as "something" to take into account and to be changed. For example, Oprea and Stonewater [1987] cited Carpenter *et al.* [1986] who reported that teachers' beliefs affected how they perceived in-service training and new curricula, and therefore influenced the relation of "implementation" to the intentions of the original developers. The same year saw a series of papers collected under the heading "In-service Teacher Training". These tended to argue that it made sense to explore the belief systems of teachers *before* attempting to introduce changes [Dionne; Waxman and Zelman; Jaworski and Gates; Shaughnessy; Kuendiger, (all 1987)]. Again there appears to be an implicit view that teachers have something called "beliefs" in some decontextualised sense which need to be accessed and changed.

In the following year, several papers continued to present evidence concerning how teachers' beliefs mediated their behaviour and the ways curricular innovation was taken up [Klein and Habermann; Jaworski; Simon, (all

1988)]. Positive change was reported by Carpenter and Fennema [1989] who described how curriculum support in the form of research-based knowledge about children's thinking and problem solving was identifiable in teachers' decision-making and instructional techniques, which in turn affected students' learning. These findings contrasted with those of Clark and Peterson [1986] who had contended that teachers did not tend to base instructional decisions on any assessment of children's knowledge.

In a retrospective study, Nolder [1990] examined some of the consequences of curriculum changes on teachers' belief systems and identified some factors influencing decisions about the innovative practice: anticipatory anxiety (e.g. that parents might oppose the new methods and that experimentation might lead to deterioration in examination results); mismatch between teacher's residual ideologies and ideas underpinning curriculum innovations; and concern about the time needed to "implement" new approaches. She also recognised that the social context of teachers' work imposed limitations on classroom practice and curriculum innovation. This paper illustrates the growing appreciation within PME of the teacher's reality and its structuring of the teaching/learning process.

Finally, technology reappeared in a discussion of beliefs (after a gap of 11 years) with Ponte [1990] and Noss, Hoyles and Sutherland [1990] examining conceptions and attitudes of teachers enrolled on computer-based inservice programmes. This discussion was continued by Noss and Hoyles in 1991, who adopted an approach which will be elaborated later in this paper.

Contextualising the teacher

In the late eighties, papers in PME exhibited a shift in research attention away from subject matter structures, teacher beliefs and student achievement, towards a perspective of social interaction and construction, mainly through the report of the Research Agenda Conference on Effective Teaching [Cooney and Grouws, 1987]. Bauersfeld, in particular, argued that a study of the development of mathematics within social interaction in classrooms showed "degeneration" into linear accumulation of tried and tested routines: "The mathematical logic of an ideal teaching-learning process... becomes replaced by the social logic..." [Bauserfeld, 1988, p. 38]. He suggested that the reality of teachers, students and researchers could be seen as a product of constructions by each of the parties involved. Taking an even broader view (perhaps leaving the realm of psychology?), Stigler and Perry [1987] considered teacher beliefs and attitudes in the context of cross-cultural studies of mathematics teaching and learning, claiming that what happened in the classroom was a reflection not only of the culture of the classroom but also of the wider society.

This particular perspective seems little developed within PME, although in the introduction to the proceedings of PME 15 the organisers commented: "More than in the past the general interest of the different activities scheduled in the conference seems focused not only on students' behaviour, but on the figure of the teacher and on the context and social factors intervening in the teaching learning processes" [Furinghetti, 1991, p. iii]. This trend is discernible in the organisation of the PME conference itself, where some of the discussion groups of previous conferences had, by 1991, developed into full blown working groups, including: Social Psychology of Mathematics Education; the Psychology of In-service Education of Mathematics Teachers; Research on the Psychology of Mathematics Teacher Development.

Additionally in 1991, some attention was paid to teacher beliefs and attitudes within a cultural context with an emphasis not so much on identifying existing attitudes, but on the effects of courses aimed at changing attitudes. Bishop and Pompeu Jr [1991], after introducing an ethnomathematical initiative in Brazil, questioned whether a simple shift from one approach to another was sufficient to guarantee better results in teaching. Moreira [1991], in a comparison of attitudes to mathematics and mathematics teaching in English and Portuguese teachers, found quite different attitude profiles in the two countries, attributing these to different educational systems and to different social contexts within which the schools operated. These papers point to the difficulty of identifying generalisable patterns within a paradigm which incorporates a social and cultural dimension, particularly in view of the different theoretical frameworks adopted by the researchers.

Researching the teacher

Evident from the review of the research above is the bewildering range of perspectives adopted. The studies cited are characterised by the range of theoretical perspectives guiding the research, by the variety of investigative tools used, and by the diversity of methodologies employed. Lerman [1992a] has argued that one of the reasons that research on teaching has grown in significance lies in the fact that qualitative research methods have become established over time and correspondingly gained wider acceptance. This is undoubtedly true yet there is no consensus as to appropriate and robust methods.

Within the PME community, the issue of methodology was given serious consideration for the first time when McLeod [1987] emphasised the need to refine frameworks and develop new methodologies. However, there exists a dilemma between, on the one hand, the desire to provide creative methodologies to explore and access teacher beliefs [as advocated, for example, by Shaughnessy, 1987] and, on the other, the problem of interpreting multiple methodology research findings to compare results and identify emergent patterns. Perhaps the concern over methodology was a catalyst for the new innovation in 1990: the plenary symposium, considering the responsibilities of the PME research community?

Summary

An overview of papers presented at PME conferences over the years indicates that increasingly the teacher has attracted attention. There has been a shift in focus; beginning with the recognition of the "formatting" role of teacher interventions and leading to a consideration of teachers' underlying attitudes and beliefs. Different trends under this umbrella can be discerned: studies of the nature of beliefs, and studies seeking to identify their influence on classroom activity and teaching/learning behaviours Alongside these trends, a growing interest is displayed in the part played by social and cultural factors and how these frame attitudes at least in the process of classroom practice. Many papers also admit that much remains to be done in formulating appropriate methodologies and theories.

Thus, in 1992 "teacher research" is a major focus of effort in mathematics education but one that is still emergent: there is no consistent approach or straightforward way to understand and interpret either empirical findings or theoretical approaches A range of methodologies is used but few comparisons made amongst them; there is little evidence of general patterns; each research seems almost to "start again" with a new theory, a new method of data collection. However, my review has thrown up some issues which will serve to organise the rest of my paper:

- the need to disaggregate teacher attitudes taking into account attitudes to teaching mathematics, to learning mathematics, to mathematics itself and to students;
- the need to analyse the "framing" of context on teacher attitudes and take on board the possibility of multiple, even contradictory, belief systems;
- the need to investigate the interaction of teacher beliefs and curriculum innovation;
- the need to consider methodology and ways to achieve in research more coherence, comparability and development.

A search for a theoretical framework

In trying to identify a theoretical framework to research teaching and teachers of mathematics, I consulted a selection of work from outside mathematics education in order to search for similar or contrasting trends and to provoke me to question what I might have previously taken for granted.

Some lessons from outside mathematics education

The changing focus of research evident within PME can also be discerned outside the community of mathematics educators. For example, from the perspective of educational psychology, Berliner has asserted:

> "Studies of the individual learner, dominant paradigm in our field for decades, may no longer be the appropriate conceptual framework for understanding many kinds of learning The educational psychologist of the 21st century needs training in the concepts and methods needed to study groups, to discriminate among and describe environments, and to think systematically or holistically rather than analytically or simplistically about the nature of causality in the learning process (Salomen, in press)" [Berliner, 1991, p. 150]

Similarly, a look at research in reading suggests many parallels with mathematics education: there have been similar shifts in emphasis over the last decade. Early reading research tended to assume a sequence of skill acquisition based on the notion of developmental stages; more recently, emphasis has been on reading as a constructive process whereby readers interpret text according to their own existing understandings Additionally, the mediating role of the reading teacher is now more clearly in focus with a relationship hypothesized between expressed teacher beliefs and theoretical understandings on classroom practice [see for example, Roehler *et al,* 1988]. Concerns over methodologies have also been expressed. Richardson *et al* [1991] reviewed reading research exploring the relationship between theoretical orientation towards reading (phonics, skills, whole language) and teacher behaviours and pointed to the contradictory results obtained which, they suggested, could be interpreted as stemming from problems of measurement. The authors argued that since there was no universal agreement on theoretical orientation or approach to teaching reading, variations in teacher beliefs as well as practice must be expected. In summary, reading research seems to mirror the move in mathematics away from process-product research to constructivism — with all the concomitant methodological advantages and problems.

Moving beyond psychology and specific subject teaching, we find that in his keynote address to the 1985 conference of the International Study Association on Teaching Thinking, Clark reflected on the development of researchers' conceptions of teachers and teaching, the available conceptual frameworks and methodological issues [see Halkes, 1986] Clark argued that the teacher had previously been viewed as a physician-like clinical decision-maker but now tended to be conceptualized as a professional sense-making constructivist, developing and testing personal theories of the world; a change which required new language to mirror and express life in classrooms as seen by teachers and students The need for precise terminology was illustrated in the use of terms such as "teachers' cognitions", "constructs", "subjective imperatives" or "practical knowledge" which, though often used interchangeably, originated from different frameworks Commenting on the conference as a whole, Halkes also noted that, with the growing attention to the individual teacher's perspective and thought processes, a range of non-traditional research methods had been adopted, the diversity of which was inevitably leading to problems of comparability and generalisability.

Clandinin and Connelly [1987], commenting on recent reviews of teacher thinking, theories, beliefs and knowledge claimed that "what is especially interesting about these studies is that... they purport to study "the personal", that is the what, why and wherefore of individual pedagogical action." [p. 487] However the authors questioned what was exactly meant by "the personal" and its relationship to pedagogical action. Clandinin and Connelly's paper attracted considerable interest from other researchers in the field, as evidenced by the publication of four response papers [3] I briefly pick out some arguments of three of them.

Halkes [1988] differentiated between the "particular" perspective-exploring idiosyncrasies of an individual teacher (the focus of Clandinin and Connelly) and a "general" perspective which took account of how different ways of teaching might explain teaching or classroom practices. He also pointed to the need for a third perspective linking teacher behaviour and thinking with the task of teaching as "explicit international behaviour". He argued that such a view would require an analysis independent of the individual teacher and the demands of both general and specific teaching tasks. Roehler *et al* [1988], whilst accepting that the Clandinin and Connelly review provided a useful basis for examining teachers' personal knowledge, argued that the personal knowledge which most influenced teachers' instructional practice was not beliefs or implicit theories, but rather the structures which organized instructional knowledge. They suggested that these knowledge structures were fluid and evolved as the teacher integrated knowledge from new experiences into existing schema, in contrast to beliefs and theories which were usually presented as static and eternal truths. Roehler *et al* also pointed to the difference between the associations of beliefs and knowledge structures to practice — the former influencing what teachers said *outside* the classroom whilst the latter structured their responses *in* classrooms. Olson [1988] similarly argued that making sense of teaching meant interpreting what teachers did and what rules they were following — an approach recognizing the teachers' culture which parallels the work of myself and Noss (see below) The contrast of the general and the particular, the focus on methodological issues and the growing awareness of the social constraints of the classroom mirror our own concerns in mathematics education, to which I will now return.

Beliefs and beliefs-in-practice

Thompson [1992], in a comprehensive review of this area, noted that, since 1980, a flurry of studies in mathematics education have focused on teachers' beliefs about mathematics and mathematics teaching and learning. Drawing on her own work [Thompson, 1984], she identified a consistency in views of mathematics amongst junior high school teachers, which, she argued, suggested a well-integrated system. She also noted a strong, though subtle, relationship between these views and classroom practice — although recognising that this could not be characterised as one of cause-effect. In contrast, other work has identified mismatches between beliefs and practice; for example, the espoused-enacted distinction [Ernest, 1989] and some experimental research within French didactics.

The problem for French researchers as set out by Arsac, Balacheff and Mante [1991] is "one of the reproducibility of didactical situations" in the researching of which a growing recognition of the role of the teacher has been identified. In their first study it became clear that, in practice, the teachers needed "closure" and the production of an acceptable mathematical solution — even though this was not part of the plan agreed with the researchers. In a second experiment, the teacher was given an even more precise formulation of what she was allowed to do and say [4], but once again clear "gaps" between the intended scenario and its implementation were discerned — gaps which were interpreted as stemming from the teacher's personal relationship to mathematics and her ideas about teaching and learning mathematics.

This inconsistency of teachers' beliefs and their practice is also manifested in the findings of Sosniak, Ethington and Varelas [1991] who used the data from the Second International Mathematics Study to explore teacher beliefs. They found that:

> ... eighth-grade mathematics teachers in the US apparently teach their subject matter *without* a theoretically coherent point of view. They hold positions about the aims of instruction in mathematics, the role of the teacher, the nature of learning, and the nature of the subject matter itself which would seem to be logically incompatible [p 127, emphasis in text]

One is led to wonder how to interpret these apparent mismatches. How far are the inconsistencies in findings simply artefacts of the different research tools adopted, for example, questionnaire or case study? Sosniak *et al* themselves recognised that their findings might be influenced by the research instruments they had employed; a point explored further by Glidden [1991] who expressed concern over the lack of concrete referents in the questionnaire items used and the failure to distinguish short and long term goals. Lerman [1992b] suggested that similar "inconsistencies" might merely be in the eye of the beholder, arguing that theory and practice should be "seamless rather than separate" [p. 3]. I tend towards an interpretation suggested by the authors themselves since it rejects (albeit implicitly) any intimation that "teachers are not up to standard". They posited the notion of "distance" of the variables in their study from the activity of teaching — views of mathematics at one extreme and of student activity at the other — and suggested that:

> We have a set of findings regarding teachers' curricular orientations which shift systematically from "progressive" to "traditional" as the teachers move from considering the issue most distant from schooling and classroom instruction to the issue most central to schooling and classroom instruction. [Sosniak *et al*, p 129]

From this perspective, inconsistency merely reflects the different concerns brought to bear in the act of teaching in schools and that of discussing mathematics education. Brown and Cooney [1991] subscribed to this more interactive interpretation of teachers' actions, building on the work of Schön [1983], which recognises the limitation of technical rationality in professional action: "(the teacher's) inquiry is not limited to a deliberation about means which depends on a prior agreement about ends He does not keep means and ends separate, but defines them interactively as he frames a problematic situation" [Schön, 1983, p. 68] Brown and Cooney also pointed to the duality of practice, referring to the work of Shulman [1978] "that when prescribed practice is perceived by teachers to be inconsistent with their existing theories (implicit perhaps) of teaching, that practice is discarded" [p. 113] — that is, when it is a question of trying out a new innovative practice or relinquishing classroom control, the latter must take precedence as teachers need to be able to predict and control classroom events. Decision-making ("reframing" as Schön calls it) occurs in reaction to the data of the classroom. As Desforges and Cockburn [1987] have pointed out with reference to primary classrooms:

> We have worked with many teachers who were well informed on all these matters [children's mathematical thinking, clear objectives, attractive teaching materials — CH] and yet who routinely failed to meet their own aspirations which they shared with mathematical educators. We do not believe that this failure has been due to idle or uninformed teaching. On the contrary, we suspect that the failure lies in the unwillingness of the mathematics education

establishment to take seriously the complexities of the teacher's job. [Desforges and Cockburn, 1987, p. 1-2]

It seems that actions and beliefs are shaped by the conditions of classrooms and teacher decisions stem more from the social practices which frame teaching than the cognitive structures and beliefs of individual teachers. Yet, if we go too far along this road, there is a danger of viewing the teacher as "determined" by the constraints of the role and failing to acknowledge the diversity in both beliefs and practice.

Towards more specificity

In attempting to unravel general tendencies, we run the risk of losing the self-evident diversity amongst teachers and classrooms — across different age ranges, and within different cultures. What *is* a teacher? Is "the teacher" a well-defined "object" of study? Is it the case that the teacher's role and culture is homogeneous across different countries? Almost certainly not — in fact, I have been struck by the question as to how far the change in paradigm identified so far is merely a "western" phenomenon?

Does the teacher of mathematics have the same role and positioning across age phases? Evidence in fact points to the contrary and to a different stance taken towards investigations with primary teachers in comparison with those of secondary or high school teachers. The majority of studies on the former group tend to assume that understanding of mathematics, or at least self-image regarding competence in the subject, are the most influential factors in determining the functioning of the teacher. They therefore tend to concentrate on misconceptions in the teacher's knowledge base or, less frequently, explore teachers' beliefs regarding their perceived lack of competence. In contrast, the majority of studies of teachers of older students are more interested in defining the knowledge and/or beliefs held by teachers regarding mathematics and mathematics teaching and learning. Outside the realm of mathematics, similar differences are identifiable. For example a study of pre-service elementary and secondary science teachers, indicated basic differences between the belief structures and concerns of the two groups [Cronin-Jones and Shaw, 1992] with the former apparently more "simplistic" than the latter: examples of areas of focus unique to the secondary group included concern for the subject and for its assessment.

There is also diversity amongst students — in terms of their perceived mathematical competence (as previously noted) but also in terms of their social position. As Secada [1991] has argued:

> Work on teaching, while attending to teacher beliefs, knowledge, and behaviours in terms of content knowledge, needs to expand to include teacher beliefs, knowledge, and behaviours as a *function of the sorts of students who are in their classrooms* [p. 46-7] [my emphasis]

A failure to acknowledge this diversity in attempts at simplification inevitably rules out consideration of emotional, cultural and social issues, leaving a blandness which does not resonate with life in real classrooms with real teachers and real kids. Finally, there is diversity within mathematics itself. Distinctions have been made between school and academic mathematics, between relational and instrumental mathematics [5]. It might be sensible to disaggregate mathematics still further and in different ways in order to map out a more complete picture of teacher beliefs: we might, for example, usefully classify mathematics according to different sign systems, different modes of expression leading us to distinguish "textbook" mathematics, oral mathematics, Logomathematics etc [6]

Innovation and beliefs

To undertake research which takes account of the diversity amongst teachers, students and mathematics raises daunting methodological problems. One approach which constrains the situation in a way which highlights the issues to be addressed has been to study the teaching/learning setting in the context of an innovation. My meta-case study of teachers in the process of "implementing" an innovation throws up a variety of interesting issues concerning beliefs and how they are revealed.

Perspectives on teachers and change

First, the review of research displays marked differences amongst designers of innovation in terms of their views of teachers: as obstacles to change or as partners in the change process. In fact this distinction resonates with the debate in 1986 between Brophy [1986a,b] and Confrey [1986][7] concerning the ways in which teachers are positioned within the enterprise of teaching: to "meet objectives" or "to construct meanings". One perspective sees teachers as potential "impediments" to curriculum innovation who do not or cannot take on board all the objectives of the reform. For example, Peterson [1991] states:

> One problem is that teachers often see the "surface level" features of the reforms being advocated, such as the changes in instructional practices, without seeing the assumptions and theoretical frames of the persons who have constructed the hoped-for changes in instructional practices including the researcher, the reformer, the textbook writer, or the expert teacher. [Peterson, 1991, p. 3]

Similarly, in relation to a discussion of didactic engineering, Artigue and Perrin-Glorian [1991] suggest that: "the teachers, faced with the products of didactic engineering, will attempt, consciously or not, to reduce the distance between what is proposed and their usual way of functioning" [p. 17]. It is evident from these studies that a "treatment" approach which fails to take account of the teacher and the teachers' work situation as mediators of any innovation is not only doomed to failure but theoretically inadequate.

Olson [1985] proposes an alternative stance towards curriculum change, viewing change as being brought about by teachers as they reflect on and become more aware of their own practices. From this reflexive perspective, teachers' behaviour is viewed as fundamentally sensible, involving the constant resolution of multiple interacting dilemmas and demands as part of their professional life. He rejects viewing teachers as being controlled either by the plans of the system or the constraints of the classroom.

Innovation not only reveals differences amongst the designers of the innovation in terms of their views of teachers but also, because it inevitably perturbs the dynamics of a classroom, makes more apparent the mathematical beliefs and understandings of teachers and students. We have studied this phenomenon in the context of introducing computers into mathematics classrooms [8]. Even if computers were simply "just another innovation" they would adequately serve my purpose as a means to access teacher beliefs. However I claim more, and argue that computers can clarify and amplify the operation and influence of the social norms of the classroom and the belief systems within it. *Seeing how computers are incorporated into practice throws all the issues surrounding beliefs into particularly sharp relief.*

We have long maintained that introducing computers into the teaching/learning process in mathematics, whilst changing the setting, also provides a window [9], a magnifying glass even, on the interaction processes, the mathematical conceptions of the students and the effects of teacher intervention. This exaggerating effect is apparent within kinds of scenarios beyond the scope of this paper; for example, class and gender differentiation within hierarchies of "competence". Others have noticed the same tendency: "Computers can make the things we already do badly in schools even worse, just as they can make many things we already do well infinitely better" [Smith, 1986, p. 206]. As far as teacher beliefs are concerned, I contend it is no coincidence that the first papers I found in the PME proceedings which faced up to the influence of teachers' beliefs, arose in settings where technology was being introduced [JBH du Boulay, 1979; Hutton, 1979]. I will elaborate on this claim in the next sections.

The computer as a window on learning and a mirror on intervention strategies

I first became aware of the power of the metaphor "computer as a magnifying glass" whilst reflecting on data from the Logo Maths Project [Hoyles and Sutherland, 1989]. In this project, we set out to investigate the ways Logo could be used as an aid to students' thinking and learning in mathematics at the secondary age level. Our study pointed to the collaborative efforts of the students in pursuit of long term projects negotiated together. We talked of the effects on attitude to and motivation towards learning mathematics and the shift towards student decision-making. However, unanticipated issues of dominance, particularly in relation to boys and girls, could not be ignored and we became increasingly aware of the influence of the medium on mathematical expression and the fragmented and situated nature of students' mathematical understandings.

More relevant here is that although we set out to intervene "lightly", that is in the context of the students' own work — to suggest ideas to explore or to point to "interesting" mathematical extensions — on analysis of the transcripts we were surprised by the significant structuring role our "subtle" interventions had on student progress and the direction of their work. In the areas we had emphasised, the students made consistent and excellent progress, whilst in others, development was haphazard. In retrospect, the theories of Vygotsky provide a coherent framework for interpreting these findings within the realm of psychology. Initially we had taken a Piagetian approach, expecting that students would construct mathematical knowledge through interaction within our microworlds. We hypothesised that they would build their ideas through interaction and reflection on the results of their actions — a reflective process facilitated by the feedback provided by the computer. However, we came to appreciate how mathematical knowledge emerged through social interaction — with the teacher and other students offering "scaffolding" within the zone of proximal development [Vygotsky, 1978][10].

This interpretation allows us to map out a child's dynamic developmental state and his/her future intellectual growth. It also brings pedagogic intervention to centre stage — as a mediator between the child and his/her experience. Inevitably, the nature and intentions of this intervention, either in general or in particular (to take Clandinin and Connelly's distinction) become legitimate objects of study.

Returning to the Logo Maths project, it is interesting for me now to reflect on our role as participant observers. We were the teachers, organising and guiding the work, so, given the previous analysis, it is interesting to speculate how far our findings were contingent on *our* beliefs about mathematics and mathematics learning. In fact, the metaphor of the computer not only as a window but also as a mirror on beliefs is compelling here. Our interventions and the students' responses bear witness to our commitment to giving students autonomy over their learning and knowledge-building processes — as such we were constructivist, though without the label! But we were not laissez-faire and very evidently had a strong mathematical agenda which, as the research unfolded, became increasingly articulated as we watched the effects of our suggestions in the mirror of the computer screen.

We have argued that group work can play an important intermediary role in the transition from closely-supported work with the computer (as in the Logo Maths Project) to the "real" world of classrooms. We have pointed to "the pivotal importance of small-group work in serving as a bridge between pupil's own meanings and mathematical meaning" [Hoyles and Noss, 1992, p. 54]. However, in mathematics education, there is much rhetoric (at least in the US and UK) that group work is "good" but rather little analysis of the conditions which optimise the potential of group work in mathematics, or indeed little elaboration of appropriate methodologies to research such a question. Again, in retrospect, I see our research on group work with computers as highlighting some crucial aspects of learning mathematics in groups. The major conclusion of our study [Hoyles, Healy, and Pozzi, 1991] was that the social norms and expectations of the student group could *not* be ignored in any analysis of group interactions: a necessary condition for the success of any group setting was that students were sufficiently mature to manage themselves and their resources and were unimpeded by interpersonal antagonism. Computer-use highlighted the tension between group outcome (in this case a computer product) and learning, brought to the surface interpersonal conflicts, emphasised the need to negotiate and share, and drew attention to the absence of connection between students' constructive activity on the computer and their expression orally or on paper.

It might be odd to suggest that a study like this one, where we did not intervene, could have anything to say about the teacher's role and teacher's beliefs. But in a similar way to that described in the context of the Logo Maths

Project, our beliefs were in fact all too apparent — not least in our choice of computer software: the importance of "bumping up" against different perspectives and solution strategies to develop mathematical knowledge together with an understanding of its limitations; the view that learning is not sequential but a function of many encounters; a concern for student decision-making and the provision of opportunities for them to resolve their own dilemmas. What was clear to us, though, during the research was how often we *longed* to intervene as we observed the group interactions — but if we resisted this temptation the children could frequently sort things out for themselves together *given time* [12]. Once again the presence of the computer with which the children could construct, negotiate, debug and validate their mathematical ideas set us free to reflect upon our beliefs as they were acted out in the small group situation. But what about beliefs-in-practice in the "real" world of classroom life? To consider this, I first need to give a brief overview of computer use in schools.

The computer as a window on teacher beliefs and a mirror on innovation

It is clear from a review of the available literature (mainly in the U.S. and the U.K.), that the impact of computers on school life does not match the early claims of the "computer enthusiasts". As Becker put it: "There were "dreams" about computer-using students ... dreams of voice-communicating, intelligent human tutors, dreams of realistic scientific simulations, dreams of young adolescent problem solvers adept at general-purpose programming languages — but alongside these dreams was the truth that computers played a minimal role in real schools" [Becker, 1982, p.6]. In the same vein, Becker later argued: "As we enter the 1990's, it is important to understand how much of that early limited reality still remains and to understand how much of the idea of transforming teaching and learning through computers remains plausible. We need to be aware of the "old habits" and "conventional beliefs" that are common among practising educators and the "institutional constraints" that impede even the best of intentions to improve schooling through technology" [Becker, 1991, p.6]. Once again we have the juxtaposition of beliefs and classroom cultures.

Thus the "average" picture painted is one of stability of attitude to computers with possible incremental rather than major change. Absent from such a review are the, admittedly rare, imaginative usages which have been researched by Sheingold and Hadley [1990]. They reported that amongst "accomplished teachers" — those reputed to be especially adept at incorporating the computer into their classroom practice — the majority had been teaching for over 13 years, worked in schools generously endowed with hardware and had easy access to technical and educational assistance. Given these conditions, teachers reported that using computers had radically changed their conceptions and methods in ways that could be interpreted as "constructivist" — becoming more student-centred in terms of expectations and direction of learning. Prerequisites to reaching such a position were identified as resources, time and perseverance — factors necessary to survive the phase of "disruption" in the face of an innovation and maintain a willingness to reflect upon the new classroom dynamic.

We came to rather similar conclusions after the Microworlds project [Sutherland, Hoyles and Noxx, 1991] where we had set up a course of in-service education for "expert" secondary mathematics teachers. In addition to the time and support required to integrate computers into practice, we also stressed the need for the teachers to experience the power of the medium for expressing *their own* mathematical ideas. It is important to underline here that we recognised from the outset that any study involving innovation with mathematics teachers, must of necessity look at teachers' beliefs as well as their practices — we aimed to map the interrelationships between beliefs and practices. We were interested in how in the interactions of the teachers with the computer activities on our course, and the ways they incorporated them into their practice, provided a window onto their views and beliefs about mathematics teaching — but in a dynamic way, whereby beliefs and course activities were both in the process of change.

I thus extend the idea of the computer as a window and a mirror to this new setting — a window on our teachers' beliefs and a mirror on our own. We set out with a strong commitment to conduct a course focused on mathematics education and not on technology but it was not until the project was under way that what we meant by mathematics education became clearer to us and to our students — particularly with respect to the importance we accorded to group work, social interaction, problem solving and "learner control".

There is still a gap between these general considerations and their practical realisation. We collected interview, observational and descriptive data throughout the course (30 days spread over one year) and wrote case studies of each teacher-participant. Analysis of these revealed a phenomenon of *projection* — the tendency for teachers to attribute their own feelings to their students. Once noticed this serves as a very useful and transparent window on beliefs! The next stage of analysis revealed our notions of teachers and teaching in a rather clearer and more precise way. We constructed *caricatures* of the course participants to provide a synthesis of views, attitudes and practices of a cluster of case studies. The caricatures attempted to draw attention to teacher characteristics and behaviours which *we* deemed crucial by exaggeration of some facets and omission of others [13]. Thus they reflected our ideas about categories by which to gauge mathematics teaching and teachers — they mirrored our beliefs as well as reflecting teacher beliefs in so far as they resonated with any individual viewpoint.

The five caricatures which emerged from our study were: Mary, the frustrated idealist; Rowena, the confident investigator; Denis, the controlling pragmatist; Fiona, the anxious traditionalist and Bob, the curriculum deliverer [Noss, Sutherland and Hoyles, 1991; Noss and Hoyles, 1992]. Each illustrated very different ways of integrating computers into their practice and different foci upon which they reflected during this process. What is very evident is that any curriculum change is complex and subtle. If it is to be anything more than a "technical fix", it must interact with the very essence of teaching and learning mathematics. Thus, as Polin [1991] suggested: "we need to instill a different vision of teacher development in our impatient policy-makers and in our harried teachers, a vision that acknowledges the many years of practice it takes to acquire and integrate a new way of teaching" [Polin, 1991, p.7].

Change must occur from within and cannot be imposed from above — a constructivist message about learning "transposed" to a new arena! I suggest that we might approach innovation from a different perspective: not trying to change beliefs in order to have the "right" effect but rather as a means to throw light on beliefs, beliefs-in-practice, and on the innovation itself. But I go further, and problematise the distinction between beliefs and beliefs-*in-practice* and posit that all beliefs are *situated*. This framework not only influences our approach to innovation but presents a new way to interpret the inconsistencies in research findings on teacher beliefs reviewed earlier.

Situated beliefs

The caricatures described in the previous section were derived from rich and diverse date collected over a period of a year. Each caricature had a personality, beliefs about teaching and mathematics teaching, and consistent, well-established classroom practices with which we (and teachers) could identify. Each caricature held a myriad of beliefs, each constructed within the practice in which they were realised (in this case the classroom, the computer activities, the interviews). This leads me to move from notions of "decontextualised" beliefs which are contrasted with beliefs-in-practice to propose that all beliefs are situated — dialectical constructions, products of activity, context and culture. By analogy with situated cognition and situated knowledge [see, for example, Seely Brown *et al*, 1989], the notion of situated beliefs challenges the separation of what is believed from how beliefs emerge: "situations might be said to co-produce *beliefs* through activities" [Seely Brown *et al*, 1989, p 32, replacing "knowledge" in the original text with "beliefs"). Once the embedded nature of beliefs is recognised, it is self-evident that any individual can hold multiple (even contradictory) beliefs and "mismatch", "transfer" and "inconsistency" are irrelevant considerations and replaced by notions of constraint and scaffolding within settings.

To develop further this notion of situated beliefs, I revisited some research I carried out in the early 1980's, the Mathematics Teaching Project [Hoyles *et al*, 1984]. We had set out to identify the characteristics of "good practice" in secondary school mathematics classrooms and tried to capture the reality of a good mathematics teacher from the point of view of the teacher (through repertory grids and interviews), the students (by questionnaire and interviews) and a classroom observer. Returning to this project now, I am struck by how ambitious we were! Additionally, it seems to me in retrospect, that our design suggested (albeit implicitly) that teacher beliefs were something which could be accessed outside the classroom and then observed as applied to the classroom. We found it difficult to make links between the teacher/student/classroom perspectives or to identify the interplay of beliefs and practice. This could, of course, be because of our inadequacies, or the inadequacies of the date-gathering techniques and conceptual tools available at the time. I feel now though that we were chasing a mirage: given the research paradigm in which we were working, we simply had not appreciated how beliefs of teachers and students were constructed differently in different settings.

Let me take a specific example to illustrate this point. In Hoyles *et al* [1985], we reported a snapshot of a mathematics teacher, Ms X. At the beginning of the paper we described some contextual features of the setting; for example, the class comprised a high ability group of 15 year old girls from a comprehensive school. What is noteworthy from a perspective a decade later, is our failure to see these features as major structuring influences on the vignette. With the benefit of hindsight, the following questions come to mind: How far was Ms X's mathematical perspective constructed by the "high ability" of the group? How far was her emphasis on effort related to her sex and the sex of the students? Was her particular blend of exposition/interaction partly a function of the age and specialism of the students? I would of course now answer these questions in the affirmative!

On re-reading the classroom snapshot, I am also exercised by the remarkable harmony between what the teacher wished to "deliver" and "what the students wanted" — an example of beliefs constructed in practice. Such harmony exemplifies *a* good practice — a different blend of variables in the setting might produce alternative pictures of success. The notion of situated beliefs allows us to cope with this diversity. It does not mean however that we are constrained to consider only individual differences at the level of the teacher; rather we need to seek crucial variables and general patterns at the level of setting.

Concluding remarks

At the beginning of this paper, I set out my agenda to map out the next step forward from that proposed by Balacheff. I am now wondering whether my argument has not led me in another direction. In their quest for scientific status, the French didacticians have pushed us towards a more precise analysis of the processes of interaction between teacher, students and mathematical knowledge [14]. Yet, once teachers are recognised as crucial influences on the learning of mathematics, their role as mediators of the teaching process must also be considered — which inevitably leads to a study of beliefs. By then noting the situatedness of beliefs, I not only point to the enormity of trying to analyse the complexities of the classroom, but also call into question the possibility of "reducing" teaching to a science. Teaching is a human activity which involves the feelings and beliefs of the participants (teachers and students), each of whom has a personal and cultural history colouring their actions. Teachers not only react to situations, they act upon them and shape them — though I wonder as I make this remark, given the discussion earlier, concerning cultural differences, how far this interpretation is a reflection of my own beliefs and my Englishness?

Woven throughout this presentation has been the issue of methodology. I have pointed to the multitude of approaches adopted within my meta-case study of mathematics teachers and to the diversity of research tools used — many borrowed from psychology, sociology, etc.[15]. There is still a need for studies to be more clearly grounded within some framework; be it theoretical, practical or conceptual (a point also made by Eisenhart [1991]). As the importance of teaching the teacher has been recognised within the psychology of mathematics education and the complexities of the research setting acknowledged, case study has become the dominant methodology. This would seem to be entirely appropriate yet there is a danger common to all "fashions" in that, once an approach becomes fashionable, it tends to be adopted as a "default" without due attention paid to the issues addressed. Thus a case study can all too easily degenerate into a mere description of a specific setting, providing little in terms of critical insight or theoretical illumination.

There is still some way to go to clarify the criteria for a "good" case study. How do we judge case studies? Do we have the criteria available to falsify any interpretations? When we write up case studies, is it possible to develop different stories (even unpalatable ones!) from the same date? Perhaps, having largely rejected experimental work and quantitative methods, we can now review them from a new perspective in order to provide complementary or contradictory insights?

At the very least we might agree that a case study should have something noteworthy to say to the research community. It must seek to throw light on the general, through the particular — whilst keeping intact the complexity of the setting. A case study is descriptive and concrete, yet as Papert [1992] has argued:

> Something can be concrete in its form but general enough to be used as a reference marker in other situations ..personal enough to be real and yet show clearly that the main events did not just happen or depend on idiosyncratic conditions. [In Hoyles, C. and Noss, R., 1992, pp.xvi]

As I reflect on my own meta-case study, two further suggestions come to mind which in my view would improve the quality of research within this paradigm: more *development* in case study work with explicit attention paid to discussion of issues and to the interpretation of similarities and differences in findings; more attention paid to *mathematics itself* and its influence on the teaching/learning situation. There is a need to resist the temptation to simplify a case study by treating mathematics merely as a "placeholder" — something which happens to be the knowledge domain of the teaching/learning situation which is neither analysed nor problematised. Even if interesting issues are addressed in case studies, it is also very apparent that case studies are hard to write and harder still to read — their very richness mitigates against their communicability. Data has to be summarised and presented in such a way so as to highlight the central ideas. Different approaches have been devised: critical incidents, evocative episodes, vignettes, gambits [16] and caricatures, to name but a few. It would seem useful if these techniques were further refined and developed to incorporate, for example, cross-cultural or across-age range comparisons. In summary, we need more attention to what we want to say and how we can say it! Perhaps additionally, we need more humility, recognising the limitations of our work and acknowledging the tensions within it? I suggest that these tensions reflect those in schools: between acquisition and construction; collaboration and schooling; the aspirations of mathematical experts and the constraining factors of the classroom [17].

In this paper, I have presented a meta-case study of research into mathematics teaching and mathematics teachers. I have traced the evolution of research on teacher beliefs and their interaction with mathematics learning which has shown evidence of inconsistencies between beliefs and beliefs-in-practice. I argued that this mismatch was thrown into relief when teachers were faced with an innovation, particularly when the innovation involved computers — a point brought home by the adoption of the metaphor of the computer as a window and a mirror on beliefs. The contention that teachers reconstruct their beliefs whilst interacting with an innovation led me to propose the notion of situated beliefs — that all beliefs are, to a certain extent, constructed in settings. Finally, I have addressed the issue of methodology and the need for a more explicit orientation towards general themes and substantial ideas, for more coherence, development and attention paid to the central influence of mathematics. Having set out some general criteria for evaluating case study research, I now wonder if my meta-case study stands up to the test. I set out to address some pervasive issues, to come up with interesting new conjectures and to compare, contrast and build upon prior research. How far I have achieved my goal can only be judged by the reader.

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Notes

[1] This paper is a modified version of a plenary address to the sixteenth Annual Conference of the International Group for the Psychology of Mathematics Education (PME16), New Hampshire, 1992.

[2] PME is the International Group for the Psychology of Mathematics Education

[3] I would welcome similar debates in mathematics education journals

[4] Since this teacher, unlike the former one, had not been a member of the research team.

[5] Thompson [1992] returned to Skemp's [1978] distinction, noticing that he had suggested that these were effectively two different subjects being taught under the same banner of mathematics.

[6] This suggestion presupposes that representation is an essential part of a mathematical concept following Vergnaud [1982], Carraher, Schliemann and Carraher [1988]. Extending these notions from our experiences with Logo, we have coined the term "situated abstraction", a mathematical generalisation articulated in the language of a particular setting [see Hoyles and Noss, 1992, and Hoyles, in press]

[7] The debate is presented in greater detail in Noss & Hoyles [1992, in press].

[8] This work has been conducted at the Institute of Education University of London.

[9] This idea was first introduced by Weir [1986]

[10] "Scaffolding" is a metaphor first coined by Wood *et al* [1976] to describe support provided by teacher or parent to a child which is "just enough" for their progression in an activity. I develop this notion in the context of mathematical education in Hoyles [1991]

[11] We do not use "instructional" software or courseware to deliver pre-specified curriculum objectives which shifts decision-making power away from teachers and students We reject the position of "the neutrality of the tool" and the idea that computers merely implement the curriculum [for further discussion of microworlds and different visions of computer use, see Bullin. 1991, and Hoyles, in press].

[12] We are certain, though, that interventions of the following three types could have assisted progress or at least rendered it more efficient: conceptual — to notice something, recall a point made earlier by a child; practical/managerial — to overcome any disruptions due to dominance, and status hierarchies; syntactic — to assist with the formal structures of the computer language

[13] The caricatures consisted of attitudes and behaviours within a set of dimensions which coalesced as the date analysis proceeded, and which when put together created a recognisable "person".

[14] The influence of this research in the U K. is evident although in our research group, at least, we were always very clear that any microworlds we "produced" were not "scripts", not innovations to be "implemented" — we expected, and found their take-up to be very different in different classrooms.

[15] To list just a few theories: Vygotsky; metacognition; constructivism; humanism; personal construct theory; attitude theory; psychoanalytic theory; social/cultural perspectives; using a bewildering array of tools such as: repertory grids; observational techniques; gambits; analysis of metaphors, videotapes of teaching scenarios, student errors, narratives; attitude scales; belief scales.

[16] These methods have been adopted in many of the researches cited in this text with the exception of gambits which are discussed by Mason and Davis [1987] and Pimm [1987].

[17] As I write this, I note a further tension: the move to situatedness and a holistic, constructivist approach in research, and a parallel development (at least in the U.K.) towards curriculum and assessment mechanisms based on mastery, fragmentation and the separation of outcomes from experiences.

Bibliography

ARSAC, J., BALACHEFF, N., and MANTE, M. [1991] The teachers role and the reproducibility of didactical situations. *Educational Studies in Mathematics*. Vol 23, pp 5-29

ARTIGUE, M., and PERRIN-GLORIAN, M. [1991] Didactic engineering, research and development tool: some theoretical problems linked to this duality, *For the Learning of Mathematics*, 11, 1, February, pp.13-18

BALACHEFF, N [1990] Beyond a psychological approach: the psychology of mathematics education, *For the Learning of Mathematics*, 10, 3, pp.2-8

BAUERSFELD, H [1988] Interaction construction and knowledge: alternative perspectives for mathematics education, In: GROUWS, D.A., COONEY, T.J. and JONES, D. [eds] *Effective mathematics teaching* NCTM, Lawrence Erlbaum, pp 27-46

BECKER, H J. [1982] *Microcomputers in the classroom: dreams and realities*, International Council for Computers in Education

BECKER, H.J. [1991] When powerful tools meet conventional beliefs and institutional constraints, *The Computing Teacher* May

BEGLE, E. [1979] *Critical variables in mathematics education* MAA-NCTM, Washington D.C.

BERGERON J. and HERSCOVICS N [1980] The training of teachers in the use of models of understanding and learning models, *Proceedings of PME4*, Berkeley, California pp 394-399

BERLINER, D C. [1991] Educational psychology and pedagogical expertise: New findings and new opportunities for thinking about training, *Educational Psychologist*, 26(2), pp 145-155

BISHOP A.J. [1980] Classroom conditions for learning mathematics *Proceedings of PME4*, Berkeley, California. pp 338-344

BISHOP, A.J. and NICKSON M. [1983] *A Review of Research in Mathematical Education*. Part B NFER-NELSON Windsor

BISHOP A.J. and POMPEU Jnr., G. [1991] Influences of an ethnomathematical approach on teacher attitudes to mathematics education, *Proceedings of PME15*, Assisi, Italy, Vol. I pp.136-143

BOUFI, A and KAFOUSSI, S. [1991] Teachers conceptions of students mathematical error and conceived treatment of them, *Proceedings of PME15*, Assisi, Italy, Vol. I, pp.176-183

BROPHY, J. [1986a] Teaching and learning mathematics: where research should be going, *Journal for Research in Mathematics Education*, 17, 5, pp.323-346

BROPHY, J. [1986b] Where are the data? — a reply to Confrey, *Journal for Research in Mathematics Education*, 17, 5, pp.361-368

BROWN, C [1986] A study of the socialization to teaching of a beginning secondary mathematics teacher, *Proceedings of PME10*, London, pp 336-341

BROWN, S.I., and COONEY, T.J. [1991] Stalking the dualism between theory and practice, *Zentralbatt für Didaktik der Mathematik*, 91/4, pp.112-117

BUDIN, H R [1991] Technology and the Teacher's Role, *Computers in Schools*, Vol 8., (1/2/3), pp.15-26

CARPENTER, T.P., and FENNEMA, E [1989] Building on the knowledge of students and teachers, *Proceedings of PME13*, Paris, France, Vol. I, pp.34-45

CARPENTER, T.P., FENNEMA, E. and PETERSON, P.L. [1986] Cognitively guided instruction: an alternative paradigm for research on teaching, *Proceedings of PME-NA8*, East Lansing, Michigan pp 225-231

CARRAHER, T.N., SCHLIEMANN, A.D., and CARRAHER, D.W. [1988] Mathematical concepts in everyday life, In: SAXE, G.B and GEARHART, M. [eds] *Children's mathematics* No. 41, San Francisco: Jossey-Bass, pp.71-87

CLANDININ D.J. and CONNELLY, F.M [1987] Teacher's personal knowledge: what counts as "personal" in studies of the personal? *Journal of Curriculum Studies*, 19, 6, pp.487-500

CLARK, C.M. and PETERSON P.L. [1986] Teachers thought processes, In: M.C. WITTROCK [ed] *Handbook of research on teaching* (3rd ed.) (pp.255-296), New York: Macmillan

CONFREY, J [1986] The critique of teacher effectiveness research in mathematics education, *Journal for Research in Mathematics Education*, 17, 5, pp 347-360

COONEY, T.J. and GROUWS D A [1987] Research Agenda Conference on Effective Mathematics Teaching, *Proceedings of PME11*, Montreal, Vol. III, pp.410-424

CRONIN-JONES, L. and SHAW, E.L. [1992] The Influence of Methods Instruction on the Beliefs of Preservice Elementary and Secondary Science Teachers: Preliminary Comparative Analyses, *School Science and Mathematics*, Vol 92(1), January, pp 14-22

DESFORGES, C and COCKBURN, A. [1991] *Understanding the mathematics teacher a study of practice in first schools*. The Falmer Press. London

DIONNE, J.J [1987] Elementary school teachers' perception of mathematics and mathematics teaching and learning, *Proceedings of PME11*, Montreal, Vol. II, pp. 156-162

DOUGHERTY, B. [1990] Influences of teacher cognitive/conceptual levels on problem solving instruction, *Proceedings of PME14*, Mexico, Vol. I, pp.119-126

DU BOULAY, J B H [1979] Teacher Transformations *Proceedings of PME3*, Coventry, England, pp 27-29

EISENHART M. [1991] Conceptual frameworks for research circa 1991: ideas from a cultural anthropologist; implications for mathematics education researchers. *Proceedings of PME-NA13*, pp 202-217

ERNEST, P. [1988] The attitudes and practices of student teachers of primary school mathematics *Proceedings of PME12*, Veszprem, Hungary, Vol. I, pp.288-295

ERNEST, P. [1989] The Knowledge, Beliefs and Attitudes to the Mathematics Teacher: a model, *Journal of Education for Teaching*, Vol. 15, No. 1, pp.13-33

EVEN, R. and MARKOVITS Z. [1991] Teachers pedagogical knowledge: the case of functions. *Proceedings of PME15*, Assisi, Italy, Vol II, pp.40-47

FLENER, F [1990] Can teachers evaluate problem solving ability? *Proceedings of PME 14*, Mexico, Vol. I, pp.127-134

FURINGHETTI, F. [1991] PME-XV Proceedings: Preface, *Proceedings of PME15*. Assisi, Italy, Vol. I, pp ii-iii

GLIDDEN, P L [1991] A mathematics educators' response to 'Teaching mathematics without a coherent point of view", *Journal of Curriculum Studies,* Vol 23., No 2., pp 133-139

GROUWS, D, GOOD, T and DOUGHERTY, B. [1990] Teacher conceptions about problem solving and problem solving instruction *Proceedings of PME14,* Mexico, Vol I, pp 135-142

HALKES, R. [1986] Teacher thinking: a promising new perspective into educational processes Report of the 1983 and 1985 conferences of the International Study Association on Teacher Thinking (ISATT), *Journal of Curriculum Studies,* 18, pp 211-214

HALKES, R. [1988] Perspectives on teacher thinking: Comment on Clandinin and Connelly's study of the "personal" in studies of the personal, *Journal of Curriculum Studies,* 20, pp 155-158

HERSCOVICS, N., and BERGERON, J [1980] The training of teachers in the use of clinical methods, *Proceedings of PME4,* Berkeley, California, pp.384-393

HOYLES, C. [1991] Developing mathematical knowledge through microworlds, In: BISHOP, A J., VAN DORMOLEN, J and MELLIN-OLSEN, S [eds] *Mathematical knowledge: its growth through teaching* Kluwer Academic Publishers, Dordrecht

HOYLES, C [In press] Microworlds/schoolworlds: the transposition of an innovation, In: Dörfler, W., Keitel, C, and Ruthven, K [eds] *Technology and mathematics education,* Springer-Verlag

HOYLES, C., ARMSTRONG, J., SCOTT-HODGETTS, R and TAYLOR, L. [1985] Snapshots of a Mathematics Teacher: some Preliminary Data from the Mathematics Teaching Project, *For the Learning of Mathematics,* 5, 2, June, pp 46-52

HOYLES, C. and BISHOP, A.J. [1982] Affective teachers: an exploratory study, *Proceedings of PME6,* Antwerp. pp 229-232

HOYLES, C, and NOSS, R [1992] A Pedagogy for Mathematical Microworlds, *Educational Studies in Mathematics,* 23, pp 31-57

HOYLES, C, and NOSS R., [eds] [1992] *Learning mathematics and Logo,* Cambridge, Mass., MIT Press

HOYLES, C and SUTHERLAND, R [1989] *Logo mathematics in the classroom* London, Routledge

HOYLES, C., HEALY L., and POZZI, S [1992, in press] Interdependence and Autonomy: Aspects of Groupwork with Computers, In: MANDEL, H., DE CORTE, E., BENNETT S.N., and FRIEDRICH, H F [eds] *Learning and instruction* European Research in an International Context

HUTTON, L [1979] Teacher attitudes and student problem solving after one year use of hand held calculators in the elementary school, *Proceedings of PME3,* Coventry, England pp 117-121

JAWORSKI, B [1988] One mathematics teacher, *Proceedings of PME12,* Veszprem, Hungary, Vol II, pp 425-431

JAWORSKI, B [1991] Some implications of a constructivist philosophy for the teacher of mathematics, *Proceedings of PME15* Assisi Italy, Vol II, pp 213-220

JAWORSKI, B and GATES, P [1987] Use of classroom video for teacher inservice education, *Proceedings of PME11* Montreal Vol II, pp 93-99

JURDAK, M [1991] Teachers' conceptions of mathematics education and the foundations of mathematics, *Proceedings of PME15* Assisi Italy, Vol II, pp 221-228

KLEIN, S and HABERMANN, G [1988] A look at the affective side of mathematics learning in Hungarian secondary schools, *Proceedings of PME12,* Veszprem, Hungary, Vol I, pp 24-53

KUENDIGER, E [1987] The mathematical learning history of preservice teachers, *Proceedings of PME11,* Montreal, Vol III, pp 355-361

LABORDE, C [1992] Guest Editorial, *Educational Studies in Mathematics,* 23, pp.1-4

LERMAN, S [1992a, in press] The Role of the Teacher in Children's Learning of Mathematics, In: BISHOP, A., HART, K, LERMAN S, and NUNES, T. *Significant influences on children's learning of mathematics,* UNESCO.

LERMAN, S [1992b] From Reflection to Research: Some Methodological Issues in Studying teachers' Images of Mathematics, presented at Working Group 21, ICME7, Quebec

MASON, J and DAVIS, J [1987] The use of explicitly introduced vocabulary in helping students to learn and teachers to teach mathematics, *Proceedings of PME11,* Montreal, Vol. III, pp 275-281

McLEOD, D [1987] Beliefs, attitudes and emotions: affective factors in mathematics learning, *Proceedings of PME11,* Montreal, Vol I pp 170-182

MOREIRA, C [1991] Teachers attitudes towards mathematics and mathematics teaching: perspectives across two countries, *Proceedings of PME15,* Assisi, Italy, Vol II pp. 17-24

NOLDER, R [1990] Accommodating curriculum change in mathematics teachers dilemmas, *Proceedings of PME14,* Mexico, Vol I, pp 167-174

NOSS, R and HOYLES, C [1991] Teacher Attitudes and Interactions in Computational Environments, *Proceedings of PME15,* Italy, Vol III, pp.95-101

NOSS, R and HOYLES, C [1992, in press] Bob — A Suitable Case for Treatment? *Journal of Curriculum Studies*

NOSS, R, HOYLES, C and SUTHERLAND, R [1990] Teacher characteristics and attitudes as mediating variables in computer-based mathematics learning, *Proceedings of PME14,* Mexico, Vol. I, pp.175-182

NOSS, R., SUTHERLAND, R and HOYLES, C [1991] *Teacher Attitudes and Interactions, Volume II Final Report of the Microworlds Project,* Institute of Education, University of London

OLSON, J K [1985] Changing our ideas about change *Canadian Journal of Education,* 10, pp 294-307

OLSON, J.K. [1988] Making sense of teaching: cognition versus culture, *Journal of Curriculum Studies,* 20, pp 167-169

OPREA, J M and STONEWATER, J [1987] Mathematics teachers' belief systems and teaching styles: influences on curriculum reform *Proceedings of PME11,* Vol I, pp 156-162

PETERSON, P [1991] Cases of "Constructivist" Mathematics Teaching: Exploring the Thinking of Teachers and Researchers, Paper to the Fourth European Conference for Research on Learning and Instruction, Finland

PETERSON P L and CLARK C M [1978] Teachers' reports of their cognitive processes during teaching *American Educational Research Journal,* Vol. 17, pp.555-565

PIMM, D [1987] Profiles of Accomplished Computer-using Teachers, *The Computing Teacher,* November, pp 5-7

PONTE, J [1990] Teachers' perceived roles of the computer in mathematics, *Proceedings of PME14,* Mexico, Vol. I, pp 183-190

RICHARDSON, V., ANDERS P, TIDWELL, D and LLOYD C [1991] The relationship between teachers beliefs and practices in reading comprehension instruction, *American Educational Research Journal,* 28, pp 559-586

ROEHLER, L., DUFFY, G., HERRMANN, B, CONLEY, M and JOHNSON, J [1988] Knowledge structures as evidence of the personal: bridging the gap from thought to practice *Journal of Curriculum Studies,* 20 pp 159-165

ROMBERG, T [1984] Classroom tasks, instructional episodes and performance in mathematics, *Proceedings of PME8* Sydney, Australia, pp 116-126

ROMBERG, T, SMALL, M and CARNAHAN, R [1979] *Research on teaching from a curricular perspective,* Theoretical paper no 81, Madison Wisconsin Research and Development Centre.

SALOMEN, G [In press] Transcending the qualitative/quantitative debate: The analytic and systemic approaches to educational research *Educational Psychologist*

SCHIFTER, D [1990] Mathematics process as mathematics content: a course for teachers *Proceedings of PME14,* Mexico, Vol I, pp 191-198

SCHÖN, D A. [1983] *The reflective practitioner: how professionals think in action,* New York: Basic Books

SCHÖN, D A [1987] *Educating the reflective practitioner: toward a new design for teaching and learning in the professions* San Francisco: Jossey-Bass

SECADA, W.G. [1991] Diversity, Equity and Cognitivist Research, In: FENNEMA, E., CARPENTER, T P and LAMON, S J [eds] *Integrating research on teaching and learning mathematics,* State University of New York Press, Albany

SEELY-BROWN, J S, COLLINS, A and DUGUID, P [1989] Situated cognition and the culture of learning, *Educational REsearch,* Jan-Feb, pp 32-42

SHAUGHNESSY, J [1987] Research reports on projects with inservice teachers: a reaction *Proceedings of PME11,* Montreal, Vol II, pp 149-162

SHEINGOLD, K and HADLEY, M [1990] *Accomplished teachers; integrating computers into classroom practice,* New York: Center for Technology in Education, Bank St College of Education, September

SHROYER, J [1980] Critical moments in teaching mathematics, *Proceedings of PME4* Berkley, California pp 331-337

SHULMAN, L.E [1978] Recent developments in the study of teaching, *Soviet-American Seminar on Teaching and Learning,* Moscow, December, pp.2-6

SIMON, M. [1988] Formative evaluation of a constructivist mathematics teacher inservice program *Proceedings of PME12,* Veszprem, Hungary, Vol. II, pp.576-583

SKEMP, R.R. [1978] Relational understanding and instrumental understanding, *Arithmetic Teacher,* 26, pp.9-15

SKEMP, R.R. [1979] Goals of learning and qualities of understanding, *Proceedings of PME3,* Coventry, England, pp. 197-202

SMITH, F. [1986] Insult to Intelligence: The Bureaucratic Invasion of our Classroom Computing, *Journal of Educational Computing,* 3, pp.101-105

SOSNIAK, L.A., ETHINGTON, C.A., and VARELAS, M. [1991] Teaching mathematics without a coherent point of view: Findings from the IEA Second International Mathematics Study, *Journal of Curriculum Studies,* Vol. 23., No. 2., pp. 119-131

STIGLER, J. and PERRY, M. [1987] Cross-cultural studies of mathematics teaching and learning: recent findings and new directions. *Proceedings of PME 11,* Montreal, Vol. III, pp 421-422

SUTHERLAND, R., HOYLES, C., and NOSS, R. [1991] *The Microworlds Course: Description and Evaluation, Volume 1, Final Report of the Microworlds Project,* Institute of Education, University of Longon

THOMPSON, A.G. [1984] *Teachers, conceptions of mathematics: three case studies,* Unpublished PhD thesis, University of Georgia

THOMPSON, A.G. [1992] Teachers' Beliefs and Conceptions: A Synthesis of the Research, In: GROUWS, D.A. [ed.] *Handbook of research on mathematics teaching and learning,* New York, Macmillan, pp.127-146

UNDERHILL, R. [1990] A web of beliefs: learning to teach in an environment with conflicting messages. *Proceedings of PME14,* Mexico, Vol. I, pp.207-213

VERGNAUD, G. [1982] Cognitive and developmental psychology and research in mathematical education: some theoretical and methodological issues, *For the Learning of Mathematics,* 3,2, pp.31-41

VYGOTSKY, L.S. [1978] *Mind in society: the development of higher psychological processes,* Cambridge, MA: Harvard University

WAXMAN, B. and ZELMAN, S. [1987] Childrens and teachers mathematical thinking: helping make the connections. *Proceedings of PME11,* Montreal, Vol. ii, pp.142-148

WEIR, S. [1986] *Cultivating minds,* Harper Row

WOOD, D., BRUNER, J. and ROSS, G. [1976] The role of tutoring in problem solving, *Journal of Child Psychology and Psychiatry,* Vol. 17, 89-100

ZEHAVI, N. and BRUCKHEIMER, M. [1983] A case study of teachers thinking and student difficulties, *Proceedings of PME7,* Rehovot, Israel pp.395-401

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References

Australian Education Council [1991] *A national statement on mathematics for Australian schools* Carlton, Victoria: Curriculum Corporation

Brownell, W. [1935] Psychological considerations in the learning and teaching of arithmetic. In *The teaching of arithmetic,*pp. 1-31 Reston, VA: NCTM

Cockroft, W.H [1986] *Mathematics counts* London: Her Majesty's Stationery Office

Crowther [1959] *A report of the Central Advisory Council for Education.* Department of Education and Science London: Her Majesty's Stationery Office

Greeno, J G. [1991] Number sense as situated knowing in a conceptual domain. *Journal for Research in Mathematics*

Hiebert, J. [1984] Children's mathematical learning: The struggle to link form and understanding. *Elementary School Journal, 84,* 497-513

Howden, H [1989] Teaching number sense. *The Arithmetic Teacher,* 36: 6-11

National Council of Teachers of Mathematics [1989] *Curriculum and evaluation standards for school mathematics.* Reston, VA: NCTM

Paulos, J.A. [1988] *Innumeracy: mathematical illiteracy and its consequences* New York: Hill and Wang

Resnick, L.B. [1989] Defining, assessing and teaching number sense In J Sowder & B. Schappelle (Eds.) *Establishing foundations for research on number sense and related topics: report of a conference.* San Diego, CA: San Diego State University, Center for Research in Mathematics and Science Education

Reys, B.J., Barger, R., Dougherty, B., Hope, J., Lembke, L., Markovits, Z., Parnas, A., Reehm, S., Sturdevant, R., Weber, M., & Bruckheimer, M. [1991] *Developing number sense in the middle grades,* Reston, VA: NCTM

Silver, E A. [in press] Teaching estimation and mental computation as situated mathematical processes. In R. Reys & N. Nohda (Eds) *Computational alternatives for the 21st centruy: cross cultural perspectives from Japan and the United States,* Reston, VA: NCTM

Sowder, J T. & Kelin, J. [1991] *Number sense and related topics* NCTM Research Interpretations Project. Reston, VA: NCTM

Sowder, J.T & Schappelle, B P (Eds) [1989] *Establishing foundations for research on number sense and related topics: report of a conference.* San Diego, CA: San Diego State University, Center for Research in Mathematics and Science Education

Willis, S (Ed) [1990] *Being numerate: what counts?* Hawthorn, Victoria: Australian Council for Educational Research