Negative Numbers: Obstacles in their Evolution from Intuitive to Intellectual Constructs*

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> *I know people who cannot understand that when you subtract four from zero what is left is zero* — *Blaise Pascal* [b]

> *How much is (+3) + (−3)? Two student opinions* [c]
> *Tim: . is this just +3−3? Well, −3, this must be subtracted from +3, so that we get zero*
> *Teacher: That is one line of reasoning Are there others? Nico?*
> *Nico: That +3 comes out. Because when we take +3+0 we also get +3 and −3 is even less than zero*

Negative numbers: mathematically simple but didactically demanding

Today the negative numbers present no mathematical difficulties. They can be elegantly constructed using the set of natural numbers The method involved is that of embedding a semigroup in a group and is one of the fundamental techniques of mdern algebra The practical computational mastery of the enlarged number system and its relevant use in applications have become obvous and routine on the university level and need hardly be discussed.

The inclusion of the negative numbers in the domain of matters obvious to the professional can easily lead to underestimation of the difficulties associated with the teaching of this topic at the school level Hence this topic can be used very effectively as proof that the defensive thesis that "He who has understood mathematics can teach it effectively" oversimplifies the interplay between professional knowledge and teaching, and describes understanding itself in too onesided a manner. This thesis obscures the many possible facets of understanding as well as the fact that for teaching to generate enthusiasm and insights it must disclose the richness of the phenomenology of the mathematical subject matter

Already in 1923 Lietzmann [10] noted that mastery of the topic "negative numbers" at the abstract level does not necessarily insure that the teacher will devise an ingenious introduction that accords with the thinking and acting of the pupils:

> The abstract core alone does not suffice for the pupil. He wants to imagine something And he is right In much the same way in which he has got used to the world of abstract natural numbers and thinks of it as being multiply and intimately linked to the world of actual reality, [d] so too he wants to link things from reality to the new number concepts and to model the new counting operations in some way The extension is meaningful to him only if it takes him a step further in the mastery of reality. He wants more than the science of mathematics has to offer He is not satisfied with a symbol He wants to link it to concrete images

The interplay between "the abstract core" and the concrete images also marks the previously quoted discussion, especially Nico's argument.

I think what lies behind Nico's argument is the following: addition results in an increase. Addition of zero is a special case and is without effect. Addition of less than nothing is, to put it mildly, obscure When it comes to determining results, the least one can expect of a theory that tries to make such obscure things thinkable is that it will be guided by practical conservation principles: nothing yields nothing This is all the more true of less than nothing

Nico's chain of reasoning is linked to the elementary notions of magnitude consciously made use of by earlier arithmetic teaching. The notions of magnitude that could, so often, be relied upon in the past to assist in calculating now lead to conflict.

When it comes to this conflict, Nico errs in good company The great thinker Blaise Pascal (1623-1662) faced a similar problem: nothing cannot be diminished by subtracting. In a way, this corresponds to the natural idea of extending the subtraction of positive integers by putting "$m - n = 0$ for $m < n$" [8]

The conflict between the negative numbers and the elementary notions of magnitude accompanied the history of these numbers from the very beginning. The result was that for a very long time they were regarded not as numbers but as auxiliary objects The conflict delayed their conceptual penetration and made it considerably more difficult. More than 1500 years elapsed from the time of Diophantus before the rules of signs came to be regarded as obvious In [5], G. Glaeser gives a penetrating description of this process as well as of the attendant intellectual difficulties that had to be overcome. G Schubring has added to this the highly informative investigation [13].

One should bear in mind that the intellectual hurdles that blocked the understanding of this mathematical subject throughout its historical evolution may also block the understanding of present-day students The knowledge of these hurdles may help us to understand, anticipate, and overcome them That is why I propose to describe these

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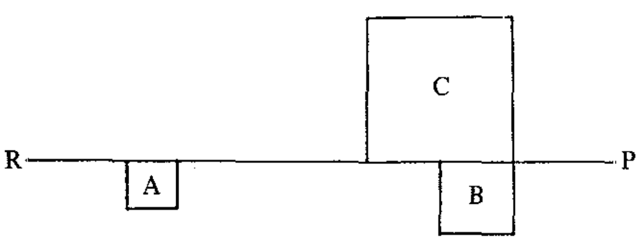
hurdles through examples. In so doing I am following the line of development in G. Glaeser's paper.

Intellectual difficulties in the evolution of negative numbers

The autobiography of the French writer Stendhal (1783-1843; his real name was Henri Beyle) contains a passage that is symptomatic of the indicated difficulties. Between the ages of fourteen and seventeen Stendhal was a pupil at the École Centrale in Grenoble. This was one of the first schools in which pupils were required to study mathematics from the age of thirteen. It seems that neither his reading of Bézout's famous textbook (1772) nor the explanations of his teacher provided the inquisitive adolescent pupil with a satisfactory justification of the rule of signs. We read in [15]:

> I thought that mathematics ruled out all hypocrisy, and, in my youthful ingenuousness, I believed that the same must be true of all sciences which, I was told, used it. Imagine how I felt when I realized that no one could explain to me why minus times minus yields plus (– × – = +)! (This is one of the fundamental rules of so-called *Algebra*.) That this difficulty was not explained to me was bad enough (it leads to truth and so must, undoubtedly, be explainable). What was worse was that it was explained to me by means of reasons that were obviously unclear to those who employed them.
>
> M. Chabert, whom I pressed hard, was embarrassed. He repeated the very lesson that I objected to and I read in his face what he thought: It is a ritual, everyone swallows this explanation. Euler and Lagrange, who certainly knew as much as you do, let it stand. We know that you are a smart fellow (which means: We know that you walked away with the first prize in literature and that you gave brilliant answers to M. Teste-Lebeau and the other members of the department.) It is clear that you want to play the role of an awkward person.
>
> Mathematics takes into consideration only a minor aspect of things (their quantity), but in this matter it offers the advantage of stating just the truth and virtually all the truth. In 1797, when I was fourteen, I imagined that the higher mathematics that I had never learned encompassed *all* or almost all, aspects of things, and that, as I continue to master it, I will reach the point of knowing facts *about all things* that are certain and beyond doubt, facts that I will be able to prove at will.
>
> It took me a long time to conclude that my objection to the theorem: *minus times minus is plus* simply didn't enter M. Chabert's head, that M. Dupuy would invariably answer with a superior smile, and that the mathematical luminaries that I approached with my questions would always poke fun at me. I finally told myself what I tell myself to this day: It *must* be that minus times minus is plus. After all, this rule is used in computing all the time and apparently leads to *true and unassailable* outcomes.
>
> The following figure gave me many a headache:
>
> 
>
>
> Let us assume that RP is the line that separates positive and negative values; whatever is above is positive and whatever is below is negative. How can I bring the square C over to the other side by taking as many copies of the quadrangle B as there are units in the quadrangle A? To use a not-so-apt comparison, made even more inept by M. Chabert's hopelessly drawn out Grenoble accent, let us assume that the negative quantities represent someone's debt. How can this person gain 5 000 000, that is, five million, by multiplying a debt of 10 000 francs by 500 francs?

This impassioned description characterizes the contemporary state of mind *vis-à-vis* the negative numbers. Long before that time, brilliant mathematicians like Euler and Lagrange calculated with these numbers in a virtuoso manner and accepted them because of their usefulness. The other professionals continued to rack their brains about an appropriate interpretation. Specifically, the following hurdles had to be overcome:

(1) There was no notion of a *uniform number line*. The preferred model was that of two distinct oppositely-oriented half lines. This reinforced the stubborn insistence on the qualitative difference between positive and negative numbers. In other words, these numbers were not viewed as "relative numbers."
(2) A related and long-lasting view was that of zero as *absolute zero* with nothing "below" it. This is reminiscent of Pascal's view referred to previously. The transition to *zero as origin* selected arbitrarily on an oriented axis was yet to come.
(3) There was attachment to a *concrete viewpoint,* that is, attempts were made to assign to numbers and to operations on them a "concrete sense."
(4) In particular, one felt the need to introduce a *single model* that would give a satisfactory explanation of all rules of computation with negative numbers. The well-known credit-debit model can "play an explanatory but not a self-explanatory role" [18]. On the other hand, Stendhal's wording of multiplication perpetuates the mystifying nature of the minus times minus rule. [e]
(5) But the key problem was the *elimination of the Aristotelian notion of number* that subordinated the notion of number to that of magnitude (see [13]).

One can help today's pupils to bypass the first two of the difficulties just listed by relying on their knowledge of a thermometer scale. It is just these two difficulties that delayed the introduction of such a scale for so long (see below). These difficulties were removed only by the transition to a theoretical understanding of number (see below). The conceptual understanding of the negative numbers lagged behind their computational mastery for a long time. To quote Glaeser's vivid description, there remained "large islands of incomprehension" [5].

It is at these islands that young Stendhal was stranded. This experience shook his "idolatrous enthusiasm for mathematics" (see [15]) and permanently undermined his respect for the mathematics teachers he had at the time. We read in [15]:

> And is my beloved mathematics just a form of shadow boxing? I did not know how to arrive at truth. Oh, how

> ardently would I then have absorbed any hint on the logic or art of *reaching the truth!* I used my pitiful, slight, intellectual powers to conclude that M. Dupuy might be a liar but that M. Chabert was a self-deceiving philistine incapable of realizing that there are objections he had overlooked
>
> My father and grandfather owned the folio edition of the *Encyclopedia* of Diderot and d'Alembert that costs from seven to eight hundred francs and may have cost much more in their time. It takes a great deal for a provincial to invest that much money in books . I consulted d'Alembert's mathematical articles in the *Encyclopedia* but I was repelled by their arrogant tone and the absence of awe before the truth Also, I understood very little of them. How burning then was my reverence for truth! How deep was my conviction that it was the queen of the world I was to enter!

In the next section we will discuss d'Alembert's article on negative numbers, for, in G Glaeser's view, it gives the most detailed account of the confusion surrounding this topic in the 18th century

Stendhal's despair over the minus-minus rule is more than matched by his "feeling of faintness" brought on by geometry [15]:

> The theorem: minus times minus equals plus causes me a great deal of grief, but one can imagine the gloom of my soul when I approached the *Statics* of Louis Monge, brother of the famous Monge who was to give the test at the Polytechnic At the beginning of geometry we find the statement: *Two lines are called parallel if they never intersect when extended to infinity* At the beginning of his *Statics* Louis Monge, a complete ass, says something like: *We can assume that two parallel lines intersect when extended to infinity* I had the impression that I was reading a completely senseless catechism

Stendhal admits that annoyance very nearly made him into a "scoundrel " In the case of the intersection point of parallels he runs up against the same obstacle as in the case of the minus-minus rule. The problems involved are analogous What is missing is the transition from the concrete to the theoretical viewpoint. And there were no instructors to guide our hero in this matter

If we adopt a viewpoint anchored in the concrete, then two parallels never intersect This is so because they must "maintain their distance" as long as we follow them with our eyes or in our minds. To maintain otherwise is to seem to be talking pure nonsense But the theoretician is free to assume the existence of a fictitious point of intersection outside the space of intuition. His assumption does no harm to any of the theorems of Euclidean geometry and thus does not contradict any of the facts of the space of intuition.

The addition of suitable fictitious objects to the Euclidean plane, namely of points at infinity and a line at infinity on which these points lie, by definition, allows us to make unifying — one is attempted to say generally valid — formulations. One can now say that *any two straight lines intersect.* non-parallel lines intersect, as before, in ordinary, "proper," points, and parallels intersect in "improper" points; a line k intersects the line at infinity g_∞ in an improper point K which is the same for all straight lines parallel to k Euclidean space can be "projectively completed" in a similar manner

Far from being a mere game, the resulting unifying verbal conventions turned out to have substantial consequences. One relevant example is that the projective approach makes it relatively easy to handle the rules of central projections and to develop a unified approach to conic sections It was considerations of perspective that resulted in the shift from Euclidean to projective geometry (see [4])

Attempts to achieve formal unification were also a key reason for the introduction of negative numbers (see [3]). The demand for a unified scheme of solution of all equations of the same type (say, quadratic equations) and the wish for its general validity inclined mathematicians to introduce "fictitious numbers " One of the mathematicians who applied this designation to the new computational objects was M Stifel. Cardano spoke of "fictitious solutions" of an equation [16]. The rules for computing with these objects were decided upon as a result of extension of certain computing rules valid for positive numbers.

Diophantus considered five types of quadratic equations with positive coefficients and treated them separately:

$$
\begin{aligned}
ax^2 &= bx \\
ax^2 &= b \\
ax^2 + bx &= c \\
ax^2 + c &= bx \\
ax^2 &= bx + c.
\end{aligned}
$$

Once negative coefficients are admitted, these five types coalesce to the single normal form

$$ax^2 + bx + c = 0.$$

The rules of computing with negative numbers make it possible to reduce Diophantus' different solution procedures to a single solution scheme. In particular, this scheme yields the positive solutions of the different types of equations that were initially the only solutions of interest. This discovery also had substantial consequences: for example, it made possible a complete description of geometric objects by means of algebraic expressions (see again [3])

The introduction of the new computational objects did not bring with it a new understanding of numbers Mathematicians wanted to subordinate negative numbers and their rules of operation to concrete notions of magnitude rather than leave them in the "realm of the fictitious " This lead to the contradictions eloquently attested by Stendhal. The decisive change of viewpoint in arithmetic was beginning to build up in Stendhal's time and was completed by H Hankel (see below)

As for Stendhal, some readers may be interested to know that he did not remain in a state of open despair. *In the end, it happened by chance that I met an important person and didn't end up being a rascal* [15] The person in question was a mathematically knowledgeable Dominican priest named Gros who gave Stendhal private lessons. The instruction began with quadratic equations and we learn that [15]:

> Being a judicious person, he began by explaining to us these equations, that is, the formation of the square $a^2 + 2ab + b^2$ of $a + b$, the assumption that the first term of the equation was the beginning of a square, the completion of this square, and so on, and so forth. We, or at least I, saw heaven open up. I finally realized why it must be so The solution of equations was no longer an apothecary's prescription that fell out of the skies

In the third or fourth lesson we went over to cubic equations and here Gros was unbelievably new. I feel as if he had brought us speedily to the boundary of knowledge and put us before the difficulty that this knowledge could overcome, before the veil that it could lift For example, he showed us one after another the different ways of solving cubic equations, what the first attempts fo Cardano were possibly like, then the progress and, finally, the modern method

Owing to this instruction, Stendhal completed his studies at the École Centrale of Grenoble after passing the final examination in mathematics with flying colors. In this connection there is no further reference to negative numbers. Perhaps (this is speculation) Stendhal was confirmed in his view that the rule "minus times minus yields plus" is compelling because it leads to results that are true and beyond doubt (see above) — a point of view that we will, for good reasons, come back to.

D'Alembert's explanations

An even more comprehensive proof of the confusion surrounding the issue of negative numbers is found in the article "Négatif" written by d'Alembert (1717-1783) for Diderot's Encyclopedia

> The *negative* magnitudes are the counterpart of the positive ones: The negative begins where the positive ends. *See* Positive
>
> One must admit that it is not a simple matter to accurately outline the idea of negative numbers, and that some capable people have added to the confusion by their inexact pronouncements To say that the negative numbers are below nothing is to assert an unimaginable thing. Those who say that 1 is not comparable with −1 and that the ratio of 1 and −1 is different from the ratio of −1 and 1 are doubly wrong: 1 because in algebraic operations we divide 1 by −1 every day; 2 because the equality of the product of −1 by −1 and of +1 by +1 shows that 1 is to −1 as −1 is to 1
>
> For the sake of precision and simplicity of algebraic operations involving *negative* magnitudes one inclines to the belief that the correct idea to be associated with *negative* magnitudes must be simple and not derived from an artificial metaphysics If one tries to reveal this true concept, then one must note to begin with that the magnitudes called *negative,* erroneously viewed as lying beyond zero, are very often represented by real magnitudes This is the case in geometry, where the *negative* straight lines differ from the positive ones only by their position with respect to a definite straight line in a common point See Curve From this follows the natural conclusion that the *negative* quantities encountered in the calculus are actually real magnitudes; but they are real magnitudes to which one must attribute an idea other than the accepted one By way of an example, suppose that we are looking for the value of a number x which when added to 100 yields 50 According to the rules of algebra; we have $x + 100 = 50$, so that $x = -50$. This shows that the magnitude x is 50 and that instead of being added to 100 it must be subtracted This means that the problem should have been formulated as follows: find a magnitude x which when subtracted from 100 leaves the remainder 50; if the problem had been formulated in this manner, then we would have $100 - x = 50$ and $x = 50$, and the *negative* form of x would cease to exist Thus, in computations, *negative* magnitudes actually stand for positive magnitudes that were guessed to be in the wrong position The sign "−" before a magnitude is a reminder to eliminate and to correct an error made in the assumption, as the example just given demonstrates very clearly See Equation
>
> Note that what we are talking about here is just isolated *negative* magnitudes such as $-a$ or $a - b$ with b greater than a, for in cases in which $a - b$ is positive, that is, when b is less than a the sign causes no difficulties whatever.
>
> In other words, there exists no isolated *negative* magnitude in the real and absolute sense; abstractly, −3 communicates no idea to the mind; however, when I say that one man gave another −3 thaler, this means in understandable parlance, that he took from him 3 thaler. That is why the product of $-a$ and $-b$ is $+ab$: for the symbol "−" that precedes by assumption both a and b proves that the magnitudes a and b are conjoined and combined with others with which they are comparable; for if they are viewed as standing by themselves and being isolated, then the "−" sign that precedes them would communicate nothing clearly graspable to the mind The only reason that the "−" sign appears in the magnitudes $-a$ and $-b$ is that some error is hidden in the assumption of the problem or the calculation: if the problem had been well formulated, then each of the magnitudes $-a$ and $-b$ would turn up with the "+" sign and the product would be $+ab$; for the multiplication of $-a$ by $-b$ signifies that one subtracts the *negative* magnitude $-ab$ times; in terms of the previously given notion of *negative* magnitudes, adding or assigning a *negative* magnitude amounts to subtracting a positive one; for the same reason, subtracting a negative is the same as adding a positive; and so the simple and natural formulation of the problem is not to multiply $-a$ by $-b$ but $+a$ by $+b$, which yields the product $+ab$ In a work of the nature of the present one it is impossible to develop this idea further, but it is so simple that I doubt that it can be replaced by one still clearer and more exact; and I believe that I can guarantee that its application to all problems that are solvable and include *negative* magnitudes will never lead to errors. Be that as it may, the rules of algebraic operations with negative magnitudes are accepted by the whole world and, in general, regarded as exact Also, this idea is always linked with these magnitudes in connection with the *negative* coordinates of a curve and their position with respect to the positive coordinates

For almost a hundred years this article was used as a permanent reference and was often praised for its supposed clarity [5] The very choice of words shows that d'Alembert is unswervingly committed to the idea of magnitude and magnitudes are viewed by him as something positive by nature. Hence the consistent opposition to the notion of "isolated negative magnitude" and the avoidance strategy derived from it. This is expressed in the persistent attempt to reinterpret statements of problems involving negative numbers so that they can be formulated in the domain of positive numbers. This brings d'Alembert into the tradition of "racines fausses" (false roots). This is how Descartes and Fermat designated negative solutions of problems and tried to rework their formulations so as to avoid such "false roots"

The continuing inability to order the real numbers linearly is also apparent. This difficulty was about to be eliminated by the development of oriented geometries by Möbius and Chasles. Until then, two-sided temperature scales, the indispensable modern didactic aid to the introduction of negative numbers, were not common In 1713 Fahrenheit arranged his temperature scale so as to avoid negative numbers, and it took another century before the public got used to speaking of "temperatures below zero."

The drawn-out battle over an appropriate interpretation of the negative numbers becomes easier to understand if one realizes that in everyday life today there is still an application of negative numbers in which they are denied the status of "independent objects of thought" [11] and are likely to be replaced with positive numbers and appropriate interpretations. Thus the temperature of −5 is sometimes given as 5 degrees (not above but) below zero, and, in German, there is even the drastic locution "5 Grad Kälte" (5 degrees of cold) (cf [11])

Overcoming the difficulties by a change of viewpoint

In the 19th century there arose the concept of extension of the number system as the key idea for the understanding of, above all, the negative and the complex numbers [16]. This signified a fundamental change of viewpoint in dealing with numbers. It was launched by the mathematicians Martin Ohm and George Peacock and brought to completion by Hermann Hankel in his "Theorie der complexen Zahlensysteme" (Theory of complex number systems) (1867). The change consisted in the transition from the concrete to the formal viewpoint. Subsequently the concept of number could be introduced in a purely formal manner without consideration of the concept of magnitude; the latter appeared merely as the intuitive substratum of such forms [6]

Explanations relating to this viewpoint

Hankel clarifies his position by considering the connection between formal operation and content interpretation in the case of the calculus of fractions. Fractions can be introduced formally as the solutions of equations of the form $x * a = b$ where a and b are natural numbers. To make unrestricted division possible, one enlarges the domain of numbers by introducing numbers of the form

$$x = b/a$$

These interact with the earlier numbers in accordance with the definiton

$$(b/a) * a = b.$$

The immediate question is that of *actual meaning* (Hankel). What is b/a and what does the definition $(b/a) * a = b$ have to do with the original meaning of multiplication as the repeated addition of certain number units?

To obtain the usual interpretation one subdivides 1 into a equal parts, so that $1/a$ denotes one such part. To these new objects one can apply multiplication in the previous sense and mean by

$$b * (1/a) = b/a$$

the result of adding $1/a$ b times.

It remains to determine the sense of

$$(1/a) * b.$$

When we decide that what is meant here is the operation that subdivides b into a equal parts, then we can give an intuitive proof that

$$(1/a) * b = b * (1/a) = b/a$$

and show that the commutative law holds. Thus to obtain this law, which is obvious for natural numbers, we must modify the original sense of multiplication and, above all, give up the symmetry of the factors. We are guided by the notion of divisible things. But there are untold numbers of things which, by their nature, are not divisible. This makes us consider a suitable segment of reality and interpret it with a purpose in mind. We build in the relations that we regard as desirable on formal grounds.

Now Hankel [6] draws a parallel with negative numbers:

> This circumstance, which advanced the concept of the negative in exactly the same manner — inasmuch as that invertible antithesis does not exist in all domains of physical magnitudes — shows conclusively that the point of view from which we have until now considered negative numbers and fractions is not one of pure theory independent of the nature of the objects to be combined

This being so, he demands that we *free* (ourselves) *from the accidents of reality* and arrives at the following conclusion: *Thus the condition for the construction of a general arithmetic is that it be a purely intellectual mathematics detached from all intuition, a pure science of forms in which what are combined are not quanta or their number images but intellectual objects to which actual objects, or relations of actual objects,* may, *but* need not, *correspond.*

After this there is no point in trying to find in nature practical examples that are metaphors for number systems. The new numbers are not "discovered" but "invented." They are mere symbols (mental images) for which one defines formal relations. The attempt to base calculation with negative numbers on physical situations is given up. This does not mean that Hankel is not concerned with the connection between the extended number system and the real world. This is clear from the demands he makes on the formal operations. They must be defined so as to be free of contradictions. This is a fundamental requirement that applies to every mathematical activity; it holds if the formal rules governing the operations are independent of one another. A definition free of contradictions makes formal arithmetic *logically possible*. But it is not enough for such a system of rules to be merely logically consistent. The danger is that systems put together without giving thought to their content interpretation and without regard for applications will, very likely, be meaningless. To insure interpretable *content,* Hankel [6] sets down a further requirement:

> To avoid the danger of abstruseness we will impose on the operations with mental objects formal rules such that they can accommodate as subordinate the actual operations on visible objects and on the numbers expressing their relations

This is the content of the *Principle of the permanence of formal laws* that is usually described in the literature by means of the following quotation:

> When two forms expressed in terms of the general symbols of arithmetica universalis are equal to one another, then they are to remain equal to one another when the symbols cease to denote ordinary magnitudes; and therefore also the operations acquire some different content

This means: formulas (or, more precisely, definite fundamental formulas) valid in the system of natural numbers are to remain valid in the extended number systems We know that this requirement cannot be realized unrestrictedly. For example, when extending the complex numbers to the quaternions (that belong to the "hypercomplex number systems") we must give up the commutativity of multiplication.

Operation of the principle of permanence: an example

We want to illustrate the operation of the principle of permanence in connection with the introduction of negative numbers Our starting point is the natural numbers and their fundamental rules of computation, namely, the associative and commutative laws of addition and multiplication and the distributive law. These can be obtained empirically or by means of a construction, using, say, the Peano axioms We also make use of the rules of computing with zero

Let n denote a natural number We introduce the solutions of equations $x + n = 0$ as new symbols, $-n$, that is, we always have

$$(-n) + n = 0, \qquad (*)$$

and we require that this equation determines $-n$ uniquely

In this way we obtain the set of negative integers Together with zero and the positive integers they form the domain of the integers. In this domain we define addition and multiplication so that the specified calculation rules continue to hold to the extent to which this is possible without incurring contradictions.

For example, to see how one must define the sum $(-3) + (-4)$ we start out with the defining equations

$$(-3) + 3 = 0, \qquad (1)$$
$$(-4) + 4 = 0, \qquad (2)$$

and add them as if the commutative and associative laws and the neutral nature of zero continued to hold In this way we obtain the equation

$$[(-3) + (-4)] + [3 + 4] = 0.$$

The second bracket contains a sum of natural numbers and its value is 7 In view of the relation (*) and the associated requirement of uniqueness we have

$$(-3) + (-4) = -7$$

To obtain the necessary conditions for the definition of the product $(-3)(-4)$ we also postulate the permanence of the distributive law and of the rule $0 * x = x * 0$, and multiply equations (1) and (2) by 4 and -3, respectively

$$(-3) * 4 + 3 * 4 = 0,$$
$$(-3) * (-4) + (-3) * 4 = 0$$

In view of (*) and the uniqueness requirement we have

$$(-3) * (-4) = 3 * 4 = 12,$$

and, also,

$$(-3) * 4 = -12$$

Quite generally, with m and n denoting natural numbers, we obtain the following rules of signs:

$$(-m) + (-n) = -(m + n),$$
$$(-m) * (-n) = m * n,$$
$$(-m) * n = -(m * n).$$

By requiring that the fundamental rules of calculation with natural numbers hold in the extended system we obtained the necessary conditions for defining the rules of calculation with the new numbers Of course, one must show that the resulting structure is well-defined and satisfies the required laws Thus the permanence principle has a heuristic rather than a proving function.

Connection between form and meaning

Hankel and his supporters, giving up the fruitless search for compellingly clarifying models, extended number systems within the framework of formal mathematics, independently of such basic content notions as quantity and magnitude. This is the new element in their approach. At the very least, what is new is that they elevated this approach into a program. Some mathematicians of earlier generations also inferred the rules of calculation with negative numbers, but their viewpoint was far more naive in the sense that they regarded formal considerations as auxiliary and did not legitimize them as the essential approach.

The separation of the construction of number systems from content considerations did not mean that the extended number systems were detached from content meanings. In fact, the opposite was true Broad possibilities of interpretation and application opened up While the attempt to read off the system of relative numbers and its structure in an indisputable manner from real phenomena failed, these numbers could now be "read into" many areas with tremendous success:

— The coordinatization of geometry was completed and considerably strengthened confidence in negative numbers
— The introduction of the vector concept brought with it further simplifications compared with coordinate geometry; in particular, it was possible to compute with directed distances and one realized that the one-dimensional space of geometric vectors was a model for the rational as well as the real numbers. Chasles' rule, the vector equation $\overline{PQ} + \overline{QR} = \overline{PR}$, that we now disparage as a truism, was made possible by abandoning the notion of zero as an absolute limit
— It now made sense in physics to speak of negative voltages, velocities, forces, and so on In this way useful descriptions and simpler computations were obtained

Compared with earlier conceptions, the relation of number to magnitude was now reversed: magnitude was now subordinated to number. Again, one no longer appealed to reality to justify extensions to the number systems, rather one used numbers to specify and describe real situations.

It is amazing to what extent it is possible to carry through habits resulting from the use of positive numbers:

— Consider a term that describes a halfline in the first quadrant of a coordinate system If we substitute negative as well as positive numbers in it, then we obtain the line determined by the halfline

- The height difference of two places is obtained by subtraction This remains true if we use negative heights
- The midpoint of any interval on the number line can be obtained by using the formal process of obtaining the arithmetic mean

More can be said Using negative numbers one can quickly and effectively solve problems that make sense only for positive numbers and have only positive solutions The following geometric problem is a simple relevant illustration.

> "By putting a rectangle of area $8cm^2$ next to a square we obtain a rectangle of length 6 How long is the square (and the added-on rectangle)?" [6]

The two solutions 2 and 4 can be obtained by trial-and-error. A systematic solution of the quadratic equation

$$x - 6x + 8 = 0,$$

by factoring: $(x - 2)(x - 4)$, say, presupposes knowledge of the multiplication of negative numbers.

The construction of the number system using the permanence principle as a guideline presupposes a minimum of formal laws and requires a maximum of "fits" Hankel emphasizes that there is freedom of choice concerning the laws but the "fits" lead him to the conclusion that laws such as the commutative law have metaphysical significance

Wagenschein formulates this connection more simply He says that the computation rules for negative numbers are "inventions," but they are inventions "that fit" [17], and E Schuberth adds:

> There are elements of freedom in mathematics We can decide in favor of one thing or another Reference to the permanence principle (or another principle) is not a *logical* argument We are free to opt for one or another. But we are not free when it comes to the consequences We achieve harmony if we opt for a certain one (that minus times minus is plus) By making this choice we make the same choice as others in the past and present

Stendhal took a more dispassionate view of the minus-minus rule of multiplication when he said that it must be true if it always yields results that are true and beyond doubt This brought him within reach of the "theoretical viewpoint" [f]

Notes

[a] Title of the German original: L Hefendehl-Hebeker: Die negativen Zahlen zwischen anschaulicher Deutung und gedanklicher Konstruktion — geistige Hindernisse in ihrer Geschichte *Mathematiklehren* 35. August 1989, 6-12

[b] B Pascal *Gedanken.* No 315 in the German translation by W Rüttenauer. Bremen: Carl Schünemann Verlag o J (found following a hint from G Glaeser)

[c] Part of a transcript of a series of lessons given by the author concerning the introduction of negative numbers

[d] "The student is always baffled when he is made aware that there is no 3 in his world Yes, there are 3 benches, 3 tables, the symbol 3, and so on but not the pure number 3" [Lietzmann 1926. p 216. footnote]

[e] To make possible a reasonable wording in a realistic situation we must interpret the factors differently, for example one as debt and the other as time (cf. [7] and [18])

[f] I wish to thank A Kirsch for helpful discussion and criticism and A Shenitzer for taking the initiative in making the translation

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